Exact solution for photogravitational restricted 3-bodies problem

(a case of spiral motion)

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Abstract: Here is presented a new type of exact solution for photogravitational restricted 3-bodies problem (a case of spiral motion).

A key point is that from the Jacobian-type integral we obtain the appropriate specific case of *spiral* motion for photogravitational restricted 3-bodies problem (when orbit of small 3-rd body is assumed to be like *a spiral*).

Besides, we should especially note that there is no analogue of Jacobian-type integral of motion in the case of photogravitational restricted 3-bodies problem if we take into consideration even a small *Yarkovsky* effect.

Key Words: photogravitational restricted three body problem, Jacobian-type integral of motion, Yarkovsky effect, spiral motion

1. Introduction.

Here is presented a new type of exact solution for photogravitational restricted 3-bodies problem [1-3] (*the case of spiral motion*).

According to the Bruns theorem [4], there is no other invariants except well-known 10 integrals for 3-bodies problem (*including integral of energy, momentum, etc.*). But in the case of *restricted* 3-bodies problem, there is no other invariants except only one, Jacobian-type integral of motion [5-6].

A key point is that from the Jacobian-type integral we obtain the appropriate specific case of *spiral* motion for photogravitational restricted 3-bodies problem (when orbit of small 3-rd body is assumed to be like *a spiral*). Besides, we should especially note that there is no analogue of Jacobian-type integral of motion in the case of *photogravitational* restricted 3-bodies problem if we take into consideration even a small *Yarkovsky* effect [3].

The Yarkovsky effect is a force acting on a rotating body in space caused by the anisotropic emission of thermal photons, which carry momentum [7-8]. It is usually considered in relation to meteoroids or small asteroids (*about 10 cm to 10 km in diameter*), as its influence is most significant for these bodies. Such a force is produced by the way an asteroid absorbs energy from the sun and re-radiates it into space as heat by anisotropic way.

Besides, Yarkovsky effect *is not predictable* (*it could be only observed & measured by astronomical methods*); the main reason is unpredictable character of the rotating of small bodies [8], even in the case when there is no any collision between them.

2. Equations of motion.

Let us consider the system of ODE for photogravitational restricted 3-bodies problem under the influence of Yarkovsky effect, at given initial conditions [3].

We consider three bodies of masses m_1, m_2 and m such that $m_1 > m_2$ and m is an infinitesimal mass. The two primaries m_1 and m_2 are sources of radiation; q_1 and q_2 are factors of the radiation effects of the two primaries respectively [2], $\{q_1, q_2\} \in (-\infty, 1]$.

We assume that m_2 is an *oblate* spheroid. The effect of *oblateness* [9] is denoted by the factor A_2. Let r_i ($i = 1, 2$) be the distances between the centre of mass of the bodies m_1 and m_2 and the centre of mass of body m. The unit of mass is chosen so that the sum of the masses of finite bodies is equal to 1. We suppose [2-3] that $m_1 = 1 - \mu$ and $m_2 = \mu$, where μ is the ratio of the mass of the smaller primary to the total mass of the primaries and $0 \leq \mu \leq ½$. The unit of distance is taken as the distance between the primaries. The unit of time is chosen so that the gravitational constant is equal to 1.

The three dimensional restricted 3-bodies problem (*we take also into consideration the influence of Yarkovsky effect*), with an *oblate* primary m_2 and both primaries radiating, could be presented in barycentric rotating co-ordinate system by the equations of motion below [2-3]:

$$\begin{aligned} \ddot{x} - 2n\,\dot{y} &= \frac{\partial \Omega}{\partial x} + Y_x(t) \ , \\ \ddot{y} + 2n\,\dot{x} &= \frac{\partial \Omega}{\partial y} + Y_y(t) \ , \qquad (2.1) \\ \ddot{z} &= \frac{\partial \Omega}{\partial z} + Y_z(t) \ , \end{aligned}$$

$$\Omega = \frac{n^2}{2}\left(x^2 + y^2\right) + \frac{q_1(1-\mu)}{r_1} + \frac{q_2\,\mu}{r_2}\left[1 + \frac{A_2}{2r_2^2}\cdot\left(1 - \frac{3z^2}{r_2^2}\right)\right] , \qquad (2.2)$$

- where $Y_x(t)$, $Y_y(t)$, $Y_z(t)$ – are the projecting of *Yarkovsky effect* acceleration $\mathbf{Y}(t)$ onto the appropriate axis Ox, Oy, Oz,

- besides, where

$$n^2 = 1 + \frac{3}{2} A_2 ,$$

- is the angular velocity of the rotating coordinate system and A_2 - is the *oblateness* coefficient. Here

$$A_2 = \frac{AE^2 - AP^2}{5R^2} ,$$

- where AE is the equatorial radius, AP is the polar radius and R is the distance between primaries. Besides, we should note that

$$r_1^2 = (x + \mu)^2 + y^2 + z^2 ,$$

$$r_2^2 = (x - 1 + \mu)^2 + y^2 + z^2 ,$$

- are the distances of infinitesimal mass from the primaries [2].

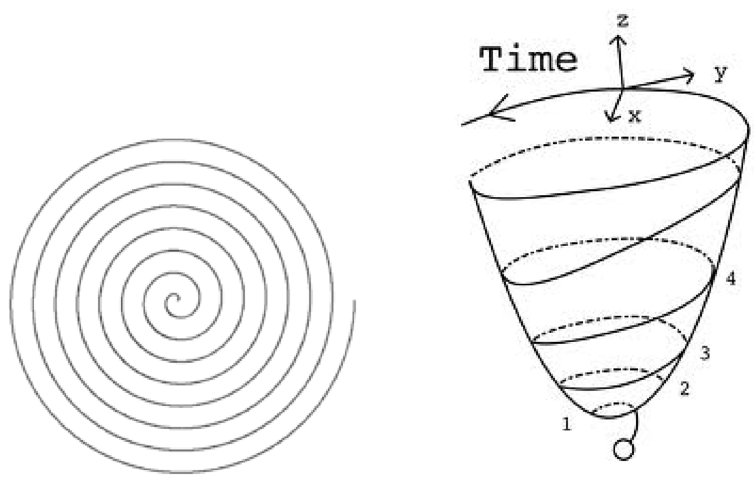
We neglect the relativistic Poynting-Robertson effect which may be treated as a perturbation for cosmic dust (*or for small particles, less than 1 cm in diameter*), see Chernikov [10], as well as we neglect the effect of variable masses of 3-bodies [11].

The possible ways of simplifying of equations (2.1):

- if we assume effect of *oblateness* is zero, $A_2 = 0$ ($\Rightarrow n = 1$), it means m_2 is *non-oblate* spheroid (we will consider only such a case below);
- if we assume $q_1 = q_2 = 1$, it means a case of restricted 3-bodies problem [5].

3. Exact solution (a case of spiral motion).

Regarding the orbit of small 3-rd body, let us assume such an orbit to be like *spiral* (Pic.1). Also let us assume Yarkovsky effect is zero.



Pic.1. Types of spiral motion.

Besides, let us remind that we could obtain from the equations of system (2.1) a Jacobian-type integral of motion [5-6]:

$$(\dot{x})^2 + (\dot{y})^2 + (\dot{z})^2 = 2\Omega(x, y, z) + C \qquad (3.1)$$

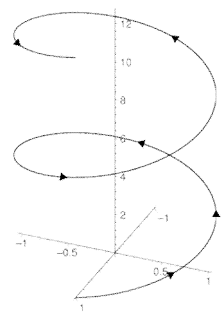
- where C is so-called Jacobian constant.

As per assumption above, it means that components of solution $\{x_i\} = \{x(t), y(t), z(t)\}$ ($i = 1, 2, 3$) should be presented as below:

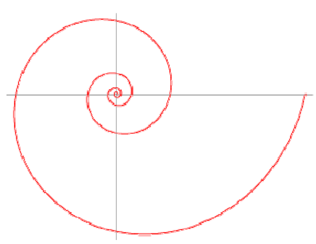
$$x = \xi(t) \cdot \cos t, \quad y = \xi(t) \cdot \sin t, \quad z = z(t),$$

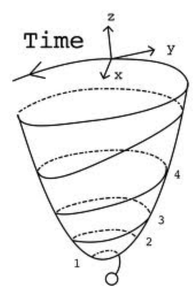
- where, for example:

1) If $\xi(t) = a{\cdot}t$, $z(t) = b{\cdot}t$ - we should obtain the spiral of *screw line* type:



2) If $\xi(t) = a{\cdot}exp(b{\cdot}t)$, $z(t) = c{\cdot}t$ - we should obtain the *3-D logarithmic* spiral:





- here $\{a,b,c\}$ are supposed to be the arbitrary positive real constants.

Thus if we substitute the representation above for the components of solution $\{x_i\}$ = $\{x(t), y(t), z(t)\}$, we should obtain from (3.1) the proper equation below

$$(\dot{\xi}(t)\cdot\cos t - \xi(t)\cdot\sin t)^2 + (\dot{\xi}(t)\cdot\sin t + \xi(t)\cdot\cos t)^2 + (\dot{z})^2 \;=\; 2\Omega(x,y,z) + C\,,$$

$$\Rightarrow \qquad \dot{\xi}^2(t) + \xi^2(t) \;+\; (\dot{z})^2 \;=\; 2\Omega(x,y,z) + C \qquad\qquad (3.2)$$

- where the expression for $\Omega\,(t)$ in (2.2) should be simplified in the case of *non-oblateness* $A_2 = 0$ ($n = 1$):

$$\Omega(t) \;=\; \frac{\xi^2(t)}{2} + \frac{q_1(1-\mu)}{r_1} + \frac{q_2\mu}{r_2}\,, \qquad (3.3)$$

$$r_1^{\,2} \;=\; (\xi(t)\cdot\cos t + \mu)^2 + (\xi(t)\cdot\sin t)^2 + z(t)^2\,,$$

$$r_2^{\,2} \;=\; (\xi(t)\cdot\cos t - 1 + \mu)^2 + (\xi(t)\cdot\sin t)^2 + z(t)^2\,.$$

So, taking into consideration the expression (3.3) for $\Omega\,(t)$, we obtain from (3.2)

$$\dot{\xi}^2(t) + (\dot{z})^2 \;=\; \frac{2q_1(1-\mu)}{r_1} + \frac{2q_2\mu}{r_2} + C \qquad (3.4)$$

$$r_1^{\,2} \;=\; (\xi(t)\cdot\cos t + \mu)^2 + (\xi(t)\cdot\sin t)^2 + z(t)^2\,,$$

$$r_2^{\,2} \;=\; (\xi(t)\cdot\cos t - 1 + \mu)^2 + (\xi(t)\cdot\sin t)^2 + z(t)^2\,.$$

Besides, we should note from (3.4) that the proper restriction below should be valid:

$$\frac{2q_1(1-\mu)}{r_1} + \frac{2q_2\mu}{r_2} + C \;\geq\; 0$$

- here $\{q_1, q_2\} \in (-\infty, 1]$.

There are a two possibilities to solve the equation (3.4):

- 1) first, we assume z (t) to be given as a proper function of parameter t, then we should find a solution of ODE of the 1-st kind for $\xi(t)$;

- 2) or the 2-nd, we assume $\xi(t)$ to be given as a proper function of parameter t, then we should find a solution of ODE of the 1-st kind for z (t).

Let us demonstrate the 2-nd way of above:

$$(\dot{z})^2 = \frac{2q_1(1-\mu)}{\sqrt{z(t)^2 + r_1^2(x,y)}} + \frac{2q_2\mu}{\sqrt{z(t)^2 + r_2^2(x,y)}} + f \qquad (3.5)$$

$$r_1^2(x,y) = (\xi(t)\cdot\cos t + \mu)^2 + (\xi(t)\cdot\sin t)^2,$$
$$r_2^2(x,y) = (\xi(t)\cdot\cos t - 1 + \mu)^2 + (\xi(t)\cdot\sin t)^2,$$
$$f = C - \dot{\xi}^2(t) \ .$$

Equation (3.5) could be simplified if we consider *a quasi-planar case* of orbital motion:

$$(\dot{z})^2 = \frac{2q_1(1-\mu)}{r_1(x,y)\cdot\sqrt{1+\dfrac{z(t)^2}{r_1^2(x,y)}}} + \frac{2q_2\mu}{r_2(x,y)\cdot\sqrt{1+\dfrac{z(t)^2}{r_2^2(x,y)}}} + f$$

$$\left\{ \frac{z(t)}{r_1} \to 0, \ \frac{z(t)}{r_2} \to 0 \ \ \forall\, t \in [0, +\infty] \right\} \ \Rightarrow$$

$$(\dot{z})^2 = \frac{2q_1(1-\mu)}{r_1(x,y)} + \frac{2q_2\mu}{r_2(x,y)} + f\,, \quad \left\{ f = C - \dot{\xi}^2(t) \right\}$$

- where the solution for the last equation is presented below, $\{q_1, q_2\} \in (-\infty, 1]$:

$$z(t) = \int \left(\sqrt{\frac{2\,q_1(1-\mu)}{r_1(x,y)} + \frac{2\,q_2\,\mu}{r_2(x,y)} + C - \dot{\xi}^2(t)} \right) dt\,, \qquad (3.6)$$

- but

$$\frac{2\,q_1(1-\mu)}{r_1(x,y)} + \frac{2\,q_2\,\mu}{r_2(x,y)} + C \geq \dot{\xi}^2(t)\ .$$

Expression (3.6) defines the evolution of variable z (t), which is assumed to be depending on the given function $\xi(t)$.

Besides, due to the appropriate *restrictions* of meanings of variables in *restricted* 3-bodies problem, from inequality above we should note that the proper restriction below should be valid for all meanings of parameter $t \geq 0$:

$$\dot{\xi}(t) \leq (C_\infty)^{\frac{1}{2}}\,, \quad \Rightarrow \quad \xi(t) \leq (C_\infty)^{\frac{1}{2}} \cdot t + C_0\,,$$

$$\{\,C_\infty\,,\ C_0\,\} = const.$$

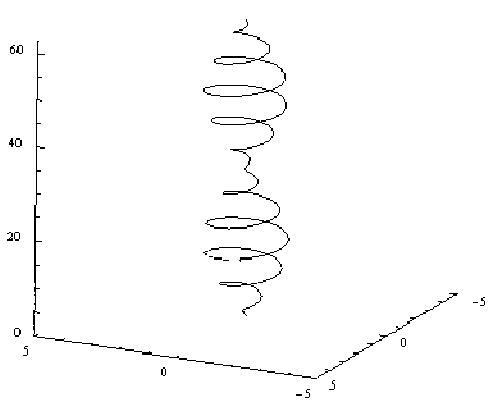
It means a proper restriction to the type of spiral for orbital motion of small 3-rd body, which could be possible for choosing as the exact solution of equations (2.1) or (3.1).

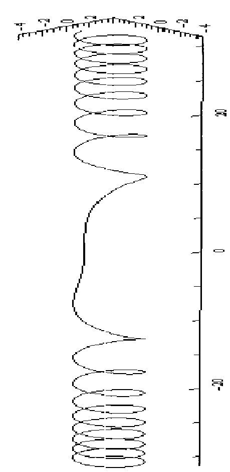
For example, let us demonstrate a calculation for the case of spiral motion of *screw line* type, $\xi(t) = \sqrt{C} \cdot t,\ C \geq 0$; for simplicity, let us consider the case $\mu = ½\ (m_1 = m_2)$:

$$z(t) = \int \sqrt{\frac{q_1}{\sqrt{\xi^2(t) + \frac{1}{4} + \xi(t)\cdot\cos t}} + \frac{q_2}{\sqrt{\xi^2(t) + \frac{1}{4} - \xi(t)\cdot\cos t}}}\, dt \cong \int \sqrt{\frac{q_1 + q_2}{\xi(t)}}\, dt\,,$$

$$z(t) = \int \sqrt{\frac{q_1 + q_2}{C^{\frac{1}{2}} \cdot t}}\, dt\,, \quad \Rightarrow \quad z(t) = \frac{2(q_1 + q_2)^{\frac{1}{2}}}{C^{\frac{1}{4}}} \cdot \sqrt{t}\,, \quad (q_1 + q_2) \geq 0\ .$$

It mean a case of orbital motion as below, in regard to variable z (*t*) (*where the beginning of the orbital motion is located at the top of Pic.2 below*):





Pic.2. Types of spiral motion (exact solutions).

Additionally, we should especially note well-known fact: in the case of *photogravitational* restricted 3-bodies problem with Yarkovsky effect [3] there is no analogue of Jacobian-type integral [5-6] for ODE system of motion (2.1).

4. Conclusion.

We obtained a new type of exact solution for photogravitational restricted 3-bodies problem [1-3] (*the case of spiral motion*).

According to the Bruns theorem [4], there is no other invariants except well-known 10 integrals for 3-bodies problem (*including integral of energy, momentum, etc.*). But in the case of *restricted* 3-bodies problem, there is no other invariants except only one, Jacobian-type integral of motion [5-6].

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