Another proof of the I. Pătraşcu's theorem

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In [1] professor Ion Pătraşcu proves the following theorem:

The Brocard's point of an isosceles triangle is the intersection of the medians and the perpendicular bisectors constructed from the vertexes of the triangle's base, and reciprocal.

We'll provide below a different proof of this theorem than the proof given in [1] and [2].

We'll recall the following definitions.

Definition 1

The symmetric cevian of the triangle's median in rapport to the bisector constructed from the same vertex is called the triangle's symmedian.

Definition 2

The point Ω from the plane of triangles ABC with the property $\widehat{\Omega BA} \equiv \widehat{\Omega AC} \equiv \widehat{\Omega CB}$ is called the Brocard's point of the given triangle.

Observation

In a random triangle there exist two Brocard's points.

The proof of the I. Pătraşcu's theorem

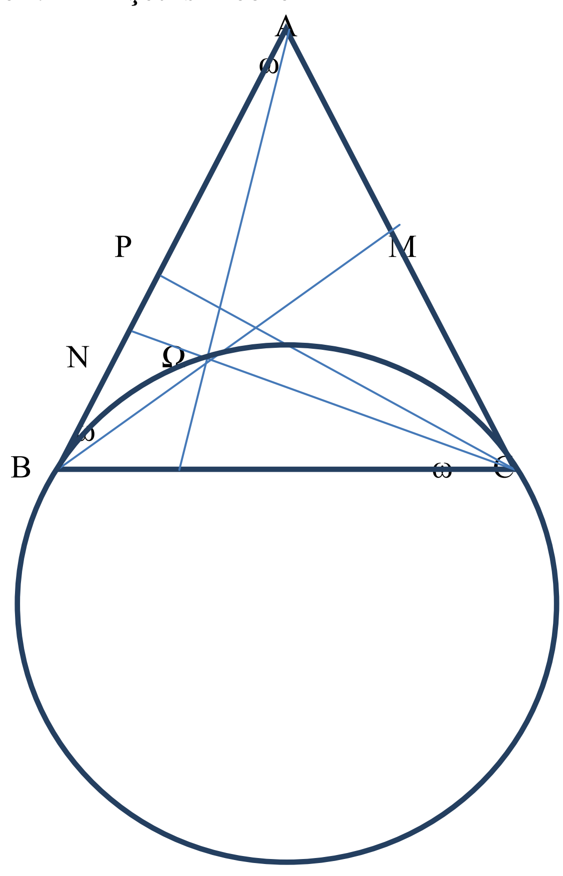

Fig.1

Let ABC an isosceles triangle $AB = AC$ and Ω the Brocard's point, therefore

$$\widehat{\Omega BA} \equiv \widehat{\Omega AC} \equiv \widehat{\Omega CB} = \omega$$

We'll construct the circumscribed circle to the triangle $B\Omega C$ (see figure 1)

Having $\widehat{\Omega BA} \equiv \widehat{\Omega CB}$ and $\widehat{\Omega CA} \equiv \widehat{\Omega BC}$, it results that this circle is tangent in B respectively in C to the sides AB respectively AC.

We note M the intersection point of the line $B\Omega$ with AC and with N the intersection point of the lines $C\Omega$ and AB.

From the similarity of the triangles ABM, ΩAM we obtain

$$MB \cdot M\Omega = AM^2 \tag{1}$$

Considering the power of the point M in rapport to the constructed circle, we obtain

$$MB \cdot M\Omega = MC^2 \tag{2}$$

From the relations (1) and (2) it results that $AM = MC$, therefore, BM is a median.

If CP is the median from C of the triangle, then from the congruency of the triangles ABM, ACP we find that $\sphericalangle ACP \equiv \sphericalangle ABM = \omega$. It results that the cevian CN is a symmedian and the direct theorem is proved.

We'll prove the reciprocal of this theorem.

In the triangle ABC is known that the median BM and the symmedian CN intersect in the Brocard's point Ω. We'll construct the circumscribed circle to the triangle $B\Omega C$. We observe that because

$$\widehat{\Omega BA} \equiv \widehat{\Omega CB} \tag{3}$$

this circle is tangent in B to the side AB. From the similarity of the triangles $ABM, \Omega AM$ it results

$$AM^2 = MB \cdot M\Omega$$

But $AM = MC$, it results that $MC^2 = M\Omega \cdot M\Omega$. This relation shows that the line AC is tangent in C to the circumscribed circle to the triangle $B\Omega C$, therefore

$$\widehat{\Omega CA} \equiv \widehat{\Omega BC} \tag{4}$$

By adding up relations (3) and (4) side by side, we obtain $\sphericalangle ABC \equiv \sphericalangle ACB$, consequently, the triangle ABC is an isosceles triangle.

References

[1] Ion Pătraşcu – O teoremă relativă la punctual lui Brocard – Gazeta Matematică, anul LXXXIX, nr. 9/1984.

[2] Ion Pătraşcu – Asupra unei teoreme relative la punctual lui Brocard – Revista Gamma, nr. 1-2 (1988), Braşov.