The main paradox of KAM-theory for restricted three-body problem (R3BP, celestial mechanics)

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Abstract: Here are presented a key points of criticism of KAM (Kolmogorov-Arnold-Moser) theory in the application of main results to the field of celestial mechanics, especially in the case of restricted three-body problem.

The main paradox of KAM-theory is that an appropriate Hamilton formalism should be valid for the KAM dynamical systems, but Hamilton formalism could not be applied for *restricted* three-body problem (which is proved to have only the Jacobian-type integral of motion, but the integrals of energy, momentum are not invariants).

Besides, we should especially note that there is no analogue of Jacobian-type integral of motion in the case of *photogravitational* restricted three-bodt problem if we take into consideration even a negligible *Yarkovsky* effect.

Key Words: KAM (Kolmogorov-Arnold-Moser) theory, Hamilton formalism, Yarkovsky effect, photogravitational restricted three-body problem, Jacobian-type integral of motion

1. Introduction.

Here are presented a key points of criticism about some *initial* assumptions in KAM-(*Kolmogorov-Arnold-Moser*)-theory [1-2] when the central KAM-theorem is known to be applied for researches of stability of Solar system in terms of *restricted* three-body problem [3], especially if we consider *photogravitational* restricted three-body problem [4] with additional influence of *Yarkovsky* effect of non-gravitational nature [5].

KAM is the theory of stability of dynamical systems which should solve a very specific question in regard to the stability of orbits of so-called "small bodies" in Solar system [1-2], in terms of *restricted* three-body problem: indeed, dynamics of all the planets is assumed to satisfy to restrictions of *restricted* three-body problem (*such as infinitesimal masses, negligible deviations of the main orbital elements, etc.*).

Nevertheless, KAM also is known to assume the appropriate Hamilton formalism in proof of the central KAM-theorem [1-2]: the dynamical system is assumed to be *Hamilton* system as well as all the mathematical operations over such a dynamical system are assumed to be associated with a proper Hamilton system.

According to the Bruns theorem [6-7], there is no other invariants except well-known 10 integrals for three-body problem (*including integral of energy, momentum, etc.*), this is a classical example of Hamilton system. But in case of *restricted* three-body problem, there is no other invariants except only one, Jacobian-type integral of motion [3].

Such a contradiction is the main paradox of KAM-theory: it adopts all the restrictions of *restricted* three-body problem, but nevertheless it proves to use the Hamilton formalism, which assumes the conservation of all other invariants (*the integral of energy, momentum, etc.*).

2. Equations of motion.

Let us consider the system of ODE for photogravitational restricted three-body problem under the influence of Yarkovsky effect, at given initial conditions [5].

We consider three bodies of masses m_1, m_2 and m such that $m_1 > m_2$ and m is an infinitesimal mass. The two primaries m_1 and m_2 are sources of radiation; q_1 and q_2 are factors characterizing the radiation effects of the two primaries respectively.

We assume that m_2 is an *oblate* spheroid. The effect of *oblateness* is denoted by the factor A_2. Let r_i ($i = 1, 2$) be the distances between the centre of mass of the bodies m_1 and m_2 and the centre of mass of body m [5]. The unit of mass is chosen so that the sum of the masses of finite bodies is equal to 1. We suppose that $m_1 = 1 - \mu$ and $m_2 = \mu$, where μ is the ratio of the mass of the smaller primary to the total mass of the primaries and $0 \leq \mu \leq ½$. The unit of distance is taken as the distance between the primaries. The unit of time is chosen so that the gravitational constant is equal to 1.

The three dimensional restricted three-body problem (*we take also into consideration the influence of Yarkovsky effect*), with an *oblate* primary m_2 and both primaries radiating, could be presented in barycentric rotating co-ordinate system by the equations of motion below [5]:

$$\begin{aligned} \ddot{x} - 2n\,\dot{y} &= \frac{\partial \Omega}{\partial x} + Y_x(t)\,, \\ \ddot{y} + 2n\,\dot{x} &= \frac{\partial \Omega}{\partial y} + Y_y(t)\,, \\ \ddot{z} &= \frac{\partial \Omega}{\partial z} + Y_z(t)\,, \end{aligned} \tag{2.1}$$

$$\Omega = \frac{n^2}{2}\left(x^2 + y^2\right) + \frac{q_1(1-\mu)}{r_1} + \frac{q_2\,\mu}{r_2}\left[1 + \frac{A_2}{2r_2^{\,2}} \cdot \left(1 - \frac{3z^2}{r_2^{\,2}}\right)\right], \tag{2.2}$$

- where $Y_x(t)$, $Y_y(t)$, $Y_z(t)$ – are the projecting of *Yarkovsky effect* acceleration $\mathbf{Y}(t)$ onto the appropriate axis *Ox*, *Oy*, *Oz*,

- besides, where

$$n^2 = 1 + \frac{3}{2} A_2 \ ,$$

- is the angular velocity of the rotating coordinate system and A_2 - is the *oblateness* coefficient. Here

$$A_2 = \frac{AE^2 - AP^2}{5R^2} \ ,$$

- where *AE* is the equatorial radius, *AP* is the polar radius and *R* is the distance between primaries. Besides, we should note that

$$r_1^2 = (x+\mu)^2 + y^2 + z^2 \ ,$$

$$r_2^2 = (x-1+\mu)^2 + y^2 + z^2 \ ,$$

- are the distances of infinitesimal mass from the primaries [5].

We neglect the relativistic Poynting-Robertson effect which may be treated as a perturbation for cosmic dust (*or for small particles, less than 1 cm in diameter*), see Chernikov [8], as well as we neglect the effect of variable masses of three-bodies [9].

The possible ways of simplifying of equations (2.1):

- if we assume effect of *oblateness* is zero, $A_2 = 0$ ($\Rightarrow n = 1$), it means m_2 is *non-oblate* spheroid (we will consider only such a case below);
- if we assume $q_1 = q_2 = 1$, it means the case of restricted three-body problem.

3. Arnold-diffusion.

The equations of restricted three-body problem are proved to describe the system with *non-Hamilton* formalism. The additional obvious proof could be found in the structure of system (2.1) if we attentively analyze the right part of equations (2.1):

$$\ldots - 2n\dot{y} = \ldots,$$

$$\ldots + 2n\dot{x} = \ldots,$$

- but any components of velocity must be excepted for Hamilton system in the final expressions for balance of momentum [3]. This is axiom for the Hamilton systems: the Hamilton systems are assumed to be the systems *without diffusion*.

That's why Arnold [1] was the 1-st in celestial mechanics who suggested to consider the Hamilton systems *with weak diffusion* which form so-called Arnold web (Fig.1): such a suggestion was very modern, original correction for KAM methodology in regard to restricted three-body problem.

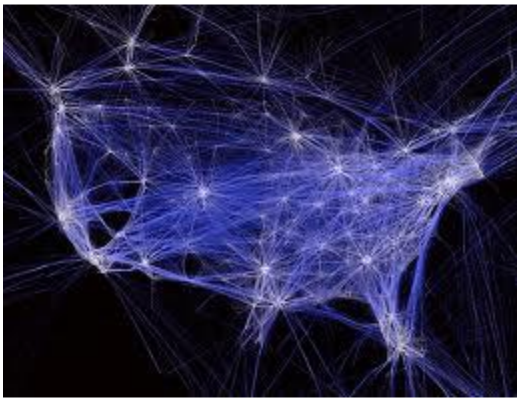

Fig.1. The schematic imagination of an Arnold web.

It means that such a dynamical systems should have a weak *Arnold-diffusion* [1]: the classical invariants of such a system are proved not remaining the same (*the integral of* *energy, momentum, etc.*), but all of them are subjected to a negligible changing (diffusion) during a large time-period. Besides, the restricted three-body problem is proved to have a new, the only stable invariant = Jacobian-type integral of motion [3].

According to [3], we could obtain from the equations of system (2.1) a Jacobian-type integral of motion:

$$(\dot{x})^2 + (\dot{y})^2 + (\dot{z})^2 = 2\Omega(x, y, z) + C \qquad (3.1)$$

- where C is so-called Jacobian constant. As it was proved in [10], such a Jacobian-type integral of motion should not be depending on time for large time-period.

Additionally, we should especially note obvious fact: in the case of *photogravitational* restricted three-body problem with Yarkovsky effect [5] there is no analogue of Jacobian-type integral for ODE system of motion (2.1).

4. Conclusion.

We discussed a key points of criticism of KAM (Kolmogorov-Arnold-Moser) theory in the application to the field of celestial mechanics, especially in the case of restricted three-body problem. The main paradox of KAM-theory is that appropriate Hamilton formalism should be valid for the KAM dynamical systems, but Hamilton formalism could not be applied for *restricted* three-body problem.

Nevertheless, KAM-theory tried to predict the stability for Solar system during a large time-period, despite of the fact that central KAM-theorem adopts all the restrictions of *restricted* three-body problem (which was chosen as a basis for the modelling of Solar system). Such a paradox could be successfully solved if we consider Solar system as dynamical system with Arnold diffusion.

Besides, we should especially note that there is no analogue of Jacobian-type integral of motion in the case of *photogravitational* restricted three-body problem if we take into consideration even a negligible *Yarkovsky* effect.

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