Title: Solution for Hadwiger Conjecture of Graph Theory

Author: Wilber Valgusbitkevyt

Abstract: Reverse Modus Ponens followed by set theory using lines followed by considering the maximum number of colours that can be used using graph homomorphism.

Proof: Hadwiger Conjecture states that if a graph G uses k or more colours in proper colouring, then one can find k disjoint connected subgraphs of G such that each subgraph is connected to each other. Let's use reverse modus ponens on this. This is true only if the following is true: if one cannot find k disjoint connected subgraphs of G such that each subgraph is connected to each other, G uses less than k colours. In other words, if a graph G can be divided into i subgroups but not into i + 1 subgroups, G uses i colours or less (in other words, less than i + 1 colours).

Consider the followings sets, each of them representing a subgroup of G: G1, G2, G3, …, Gi. Each of these contains arbitrary number of vertexes in them. Each vertex is connected by at least one line. Now, consider the following sets again. There may be more lines which are irrelevant. These are important, necessary, and relevant.

G1 = {l11, …, l1i}/{l11}, G2 = {l11, …, l1i}/{l22}, G3 = {l11, …, l1i}/{l33}, …, Gi = {l11, …, l1i}/{lii}

lij means it is a line that connects from a vertex in Gi to a vertex in Gj. /{lii} means to exclude that element from the set. The graph has i subgraphs.

Now, consider this: G1 = {l11, …, l1i+1}/{l11}, G2 = {l11, …, l1i+1}/{l22}, G3 = {l11, …, l1i+1}/{l33}, …, Gi+1 = {l11, …, l1i+1}/{li+1i+1}. If this were possible, then the graph can have i+1 subgraphs. However, we are observing a graph that cannot have i+1 subgraphs. Hence, at least one of the elements must be missing.

Let the subgraph with the missing element be Gi+1. Merge this with some graph, let's say Gi. This means that Gi cannot have another set of vertices connected to G1, …, Gi-1 or cannot have the vertices in Gi+1 to connect to a vertex in Gi. This is the graph that cannot have i + 1 subgraphs.

Now, let's consider the graph that can have i + 1 subgraphs. There is a subgroup Gi such that it contains two sets of vertices connected to G1, …, Gi-1. Also, at least one of the vertices in the Gi+1 should connect to a vertex in Gi.

Always put the extra vertexes in Gi.

The vertices in each line should have different colours.

Consider a graph that cannot have i + 1 subgraphs. When does this require the most colours? When there are as few vertices as possible and as much lines as possible, we require the most colours for such a given graph. If such graph can be coloured with i colours, then Hadwiger Conjecture is proven true.

Consider a graph such that it has i + 1 subgraphs, and lose a line. The subgraphs are arranged, and can be arranged, by grouping all the extra vertexes to be in Gi+1.

Since there are i subgroups but not i + 1 subgroups, there cannot be more than i colours used in each group. Consider a graph G that has i subgroups but not i + 1 subgroups. Now, consider an arbitrary number j that is bigger than i. Let's suppose a subgroup A needs at least j colours in it. If so, we can divide the group A into j subgroups, and include each group in one of the j groups in A by the vertices they are connected to. Now, we have j groups, and j is bigger than i. But we cannot have i + 1 groups. If we can have j groups, then we can have i + 1 groups because you can merge several groups into one group to have less subgraphs. Hence, this is a contradiction. Hence, each subgraph cannot use more than i colours.

Now, we have a graph G such that it has i subgraphs, and each subgraph has only i colours in them. Let's consider two subgraphs of G: subgraph B and subgraph C. Let's say that subgraph B uses i colours in it. Let's say a vertex, let's call it V, in subgraph C is connected to all vertexes of i colours in B. Then, we would need i + 1 colours for G. However, if so, let each of the i vertices in B be a subgroup like the previous example. Now, let V be another subgroup. We know this is connected to all of these i different colours (this is why we would need i + 1 colours in the first place as opposed to just i), hence this can be a subgroup. Now, we have i + 1 subgroups. This is a contradiction. Hence, a graph G that has I subgraphs but not i + 1 subgraphs can be coloured with I colours. Hence, in our original sentence, a graph G that cannot have k disjoint connected subgraphs of G such that each subgraph is connected to each other, G uses less than k colours. Since this reverse modus ponens is true, by modus ponens, if a graph G uses k or more colours in proper colouring, then one can find k disjoint connected subgraphs of G such that each subgraph is connected to each other.