3 page solution of Erdős conjecture on arithmetic progressions:

Erdős conjecture on arithmetic progressions states that "for a set A which contains the elements of positive integers, if $\sum_{n\in A}\frac{1}{n}=\infty$ (i.e. A is a large set as opposed to a small set), A contains arbitrarily long arithmetic progressions."

Let's rephrase this: "If, A does not contain arbitrarily long arithmetic progression of any given length, then $\sum_{n\in A}\frac{1}{n}\neq\infty$

Two options to consider.

1. A contains finite short arithmetic progressions or no arithmetic progression, and another sequence that does not contain any arithmetic progression.
2. A contains infinitely many short arithmetic progressions.

Let A be a set of positive integers. Let B be the subsets of A such that they are, if B isn't empty, finite short arithmetic progressions in A. Let C be the rest of the elements in A.

Then, $\sum_{n\in A}\frac{1}{n}=\sum_{n\in B}\frac{1}{n}+\sum_{n\in C}\frac{1}{n}$

If B is empty, it is very simple and A = C. If B is not empty, then there are finite number of short arithmetic sequences B1, B2, …, BJ for some number J. Because they are short and finite, $\sum_{n\in B}\frac{1}{n}$ converges to a constant either way. The sum of the finite short arithmetic progressions converges because the sum of the infinite biggest short arithmetic progressions converges, and the sum of the finite short arithmetic progressions has to converge by direct comparison test. C is the remainder of the elements in A which does not have any arithmetic sequence. In other words, each element of C has different incremental value from its preceding term.

Now, consider the smallest set C possible such that $\sum_{n\in C}\frac{1}{n}$ is as big as possible. Let B be empty, and let C and A be the same. We want to include as much small numbers as possible in C.

Hence, the elements of C would be: 1, 2, 4, 7, 11, 16, 22, 29, 37, 46, ...

c1 = 1, ci = 1 + (n-1)n/2

We want to know if $\sum_{n \in A} \frac{2}{[(n-1)(n+2)]}, n \to \infty$. I.e. if $\sum_{n \in A} \frac{1}{[(n-1)(n+2)]}, n \to \infty$

Now, to make this simpler, make it larger. Because if the larger one converges, by direct comparison, this one has to converge as well. Consider $\sum_{n \in A} \frac{1}{(n-1)n}, n \to \infty$

But $\sum_{n \in A} \frac{1}{(n-1)n} \leq \frac{1}{n^{1.1}}, n \to \infty$

Hence, A converges when there isn't any arbitrarily long arithmetic progression just like the theorem states whether B is empty or has finite number of short arithmetic progressions.

Now, consider this. A consists of infinitely many short arithmetic progressions. Because we know the sequence which does not contain any arithmetic progression converges, consider A such that it only has infinitely many arithmetic progressions to make it as big as possible.

Let each of these finite short arithmetic progression be represented by b1, b2, ..., bn such that the index represents the numeric incremental in each short arithmetic progression. Such that b1 = c1 (some constant), c1 + 1, c1 + 2, and so on for some finite (not infinite) terms. Let c1, c2, ..., cn be the constants assigned to each short arithmetic progression.

bi = (ci, ci + i, ci + 2i, ...)

Sigma (as n goes infinity in A) =	1/c1	+	1/c2	+	...	+	1/cn	
(there are infinite of	1/(c1+1)	+	1/(c2 +2)	+	...	+	1/(cn + n)	
these progressions)	....							
			(each of the sequence has finite number of elements)					

Now, consider the largest sum possible by finding the lowest b possible because if the largest sum converges, by direct comparison, the smaller ones have to converge as well by the law of direct comparison in convergence test.

We want to get as much of the lower numbers as possible. Let $X_1, X_2, \ldots, X_n$ be the number of elements in each ith short arithmetic progression.

$c_1 < c_2 < \ldots < c_n$, $c_n \to$ infinity and $n \to$ infinity
$X_1 > X_2 > \ldots > X_n$, all of these are finite constant, but $n \to$ infinity
$X_i = X_{i-1} - 1$ (since we want to get more of the lower numbers as much as possible)

$c_1 = c_1$, $c_2 = c_1 + X_1$, $c_3 = c_2 + 2X_2$, … , $c_n = c_{n-1} + (n-1)X_{n-1}$
I.e., $c_1 = c_1$, $c_1 + X_1 = c_2$, $c_2 + (X_1 - 1) \times 2 = c_3$, $c_3 + (X_1 - 2) \times 3 = c_4$ ....
i.e. $c_1 + X_1 = c_2$, $c_2 + 2X_1 - 2 = c_3$, $c_3 + 3X_1 - 6 = c_4$ ....

Consider the summation of the infinite short arithmetic progressions above. That is smaller or equal to the following:

$\le X_1/c_1 + X_1/c_2 + \ldots + X_1/c_n$
$= X_1(1/c_1 + 1/c_2 + \ldots + 1/c_n)$

$\le x_1(1/c_1 + 1/(c_1 + X_n) + 1/(c_2 + 2X_n) + \ldots + 1/(c_{n-1} + (n-1)X_n))$

$\le x_1(1/c_1 + 1/(c_1 + 1) + 1/(c_2 + 2) + \ldots)$

But we know this sequence converges as n goes infinity.

Hence, we have confirmed that if A does not contain any arbitrarily long arithmetic progression, then A is a small set.

Hence, we have proved that if A is a large set ($\sum_{n \in A} \frac{1}{n} = \infty$), then A contains arbitrarily long arithmetic progression.