Erdős–Faber–Lovász conjecture states that if k complete graphs, each having exactly k vertices, have the property that every pair of complete graphs has at most one shared vertex, then the union of the graphs can be colored with k colors.

Consider Case 1: All pairs of complete graphs have one shared vertex in each pair.
Consider Case 2: All pairs except 1 pair have one shared vertex in each pair. One graph is disjoint from at least one graph, i.e. one pair is disjoint with no shared vertex and not a pair.

We can colour the Case 2 the same as Case 1, the only difference is that the vertex is ripped apart. This works for 2 or more cases. Draw a Case 1 graph with Case 1 colourings, then take the shared vertex you don't want to share, and rip it apart.

Hence, we just have to consider Case 1.

Let the k graphs be G1, G2, ..., Gk.
Let the k shared vertices be S1, S2, ..., Sk.
Let P(Si) denote the number of graphs sharing the vertex Si.
Let G(Si) denote the graphs sharing the vertex Si. Set theory is used here.

For instance, G(Si) = {G1, G2, ...}

Now, consider this: either 1 vertex is shared or more than k-1 vertices are shared.

Suppose only k - i vertices are shared such that 1 < k - i < k.

Since we are sharing k - i vertices, k - i - 1 vertices should have at least 1 graph for each of them. Otherwise, they wouldn't be "shared vertices".

Hence, we are left with k - (k - i - 1) graphs for 1 shared vertex as the largest upper bound, which is i + 1. Hence, 2 <= P(Sj) <= (i + 1) in this case for 1 <= j <= k - i.

Because we are looking for graphs so that each pair has only, at least and at most, one shared vertex, Si cannot be connected to more than one graph of the graphs that are connected to Sj. For instance, let's say G(Sj) = {Ga, Gb, ...}. If Si is connected to both Ga and Gb, then Ga and Gb have both Si and Sj as shared vertices. This means the graphs connected to Si cannot be connected to more than one graph of the graphs that are connected to Sj. This means it is case 2.

Hence, either 1 vertex is shared or more than k-1 vertices are shared.

Case 1: There is only one shared vertex for k graphs. This is easy. Have the same k graphs coloured the same way sharing the same one vertex.

Case 2: We have k or more shared vertices.

Suppose it is k. We have S1, ..., Sk.

Colour S1, ..., Sk in k different colours. For any Gi, regardless of the number of shared vertices it has, it has all different colours for the coloured vertices in it. The vertices that are not coloured yet are not shared by other graphs, and you can colour them whatever you want. Each Gi has k vertices, and each Gi is allowed to use k colours. Just colour the non-shared vertices with whatever remaining colours you want. Hence, this can be coloured by only k colours.

Suppose it is k + i.

This is the same. All you have to do is just colour the first k shared vertices in k different colours, and colour the remaining shared vertices with whatever the colour you want. In this case, for some Gi, they have must more than 2 shared vertices. Just colour the shared vertices within Gi differently, and you can colour the remaining vertices with the remaining colours whatever you want.

Hence, using the set theory, we have proven that if k complete graphs, each having exactly k vertices, have the property that every pair of complete graphs has at most one shared vertex, then the union of the graphs can be cloured with k colours.