A theorem producing the fine structure constant inverse and the quark and lepton mixing angles

J. S. Markovitch
P.O. Box 2411
West Brattleboro, VT 05303[*]
(Dated: November 13, 2012)

The value 137.036, an excellent approximation of the fine structure constant inverse, is shown to occur naturally in connection with a theorem employing a pair of closely-related functions. It is also shown that the formula producing this approximation contains terms expressible using the sines squared of the experimental quark and lepton mixing angles, implying an underlying relationship between these constants. This formula places the imprecisely measured neutrino mixing angle θ_{13} at close to 8.09°, so that $\sin^2 2\theta_{13} \approx 0.0777$.

PACS numbers: 06.20.Jr

The value 137.036, an excellent approximation of the fine structure constant (FSC) inverse $1/\alpha$ [1], is shown to occur naturally in connection with a theorem employing a pair of closely-related functions [2]. It is also shown that the formula producing this FSC approximation contains terms expressible using the sines squared of the experimental quark and lepton mixing angles, implying an underlying relationship between these constants.

I. TWO FUNCTION DEFINITIONS

We begin by defining the pair of related functions that the theorem will exploit. Let M and N be positive integer constants, so that

$$h(u) = \frac{M^3 - u^3}{N^3} + M^2 - u^3$$
$$j(u) = \left(\frac{M-u}{N}\right)^3 + (M-u)^2$$

where u is a variable such that

$$0 < u \leq 0.1 \tag{1.1}$$

and

$$M \geq 10 \quad . \tag{1.2}$$

II. THE FSC THEOREM

We then specify and prove the theorem making use of these functions.

Theorem 1. (*The FSC Theorem.*) *Let*

$$j(y) = h(x) \tag{2.1}$$

satisfying

$$M = \frac{N^3}{3} + 1 \quad . \tag{2.2}$$

Then at

$$x = \frac{1}{M} \tag{2.3}$$

we get

$$\frac{dy}{dx} \approx \frac{1}{M^3} \quad . \tag{2.4}$$

Proof. Equation (2.1) gives

$$\left(\frac{M-y}{N}\right)^3 + (M-y)^2 = \frac{M^3 - x^3}{N^3} + M^2 - x^3 \quad ,$$

which expands and simplifies to

$$-\frac{3M^2y}{N^3} + \frac{3My^2}{N^3} - \frac{y^3}{N^3} - 2My + y^2 = -\frac{x^3}{N^3} - x^3 \quad ,$$

or

$$3M^2y - 3My^2 + y^3 + 2MN^3y - N^3y^2 = (N^3+1)x^3 \quad .$$

It follows that

$$(3M^2 - 6My + 3y^2 + 2MN^3 - 2N^3y)dy = 3(N^3+1)x^2dx \quad ,$$

so that

$$\frac{dy}{dx} = \frac{3(N^3+1)x^2}{3M^2 - 6My + 3y^2 + 2MN^3 - 2N^3y} \quad .$$

We now want to identify and remove the smallest terms from the denominator. As Eq. (2.2) requires that

$$N^3 = 3M - 3 \quad ,$$

*Electronic address: jsmarkovitch@gmail.com

substituting for N^3 gives

$$\begin{aligned}\frac{dy}{dx} &= \frac{3(3M-3+1)x^2}{3M^2-6My+3y^2+2M(3M-3)-2(3M-3)y} \\ &= \frac{3(3M-2)x^2}{3M^2-6My+3y^2+6M^2-6M-6My+6y} \\ &= \frac{3(3M-2)x^2}{9M^2-12My+3y^2-6M+6y} \\ &= \frac{(3M-2)x^2}{3M^2-4My+y^2-2M+2y} \\ &= \frac{(3M-2)x^2}{3M^2-2M-4My+y^2+2y} \\ &= \frac{(3M-2)x^2}{(3M-2)M-4My+y^2+2y} \quad . \end{aligned} \tag{2.5}$$

But by Eq. (1.1) $y \leq 0.1$ and by Eq. (1.2) $M \geq 10$, so that $4My$, y^2, and $2y$ are small compared to $(3M-2)M$. Hence,

$$\frac{dy}{dx} \approx \frac{3M-2}{3M-2} \times \frac{x^2}{M} \quad .$$

Accordingly,

$$\frac{dy}{dx} \approx \frac{x^2}{M} \quad ,$$

so that at

$$x = \frac{1}{M}$$

we get

$$\frac{dy}{dx} \approx \frac{1}{M^3} \quad . \qquad \square$$

III. THE FINE STRUCTURE CONSTANT INVERSE

Now inspection reveals that $M = 10$ and $N = 3$ are the smallest positive integers fulfilling Eq. (2.2). For this *minimal case* Eq. (2.1) gives

$$\left(\frac{10-y}{3}\right)^3 + (10-y)^2 = \frac{10^3 - x^3}{3^3} + 10^2 - x^3 \quad .$$

Then, by Theorem 1, at

$$x = \frac{1}{M} = 0.1$$

we get

$$\frac{dy}{dx} \approx \frac{1}{M^3} = 0.001 \quad ,$$

where Eq. (2.1) gives

$$y \approx 0.000\,033\,333\,408\,73 \quad .$$

Moreover, substituting $M = 10$ into Eq. (2.5) gives

$$\begin{aligned}\frac{dy}{dx} &= \frac{28x^2}{(28-y)(10-y)} \\ &\approx 0.001\,000\,004\,524 \quad , \end{aligned}$$

which shows the approximation's excellent accuracy.

The key point, however, is that this solution to Eq. (2.1) *simultaneously* produces

$$\begin{aligned} h(x) &= \frac{M^3 - x^3}{N^3} + M^2 - x^3 \\ &= \frac{10^3 - 0.1^3}{3^3} + 10^2 - 0.1^3 \\ &= 137.036 \quad , \end{aligned}$$

the fine structure constant inverse approximation promised at the outset, fit to within seven parts per billion [1]. Hence, Theorem 1 will be termed *The FSC Theorem*, and Eq. (2.1) *The FSC Equation.* In this way this excellent approximation occurs as the natural and uniquely minimal result of the analysis of the above pair of closely-related functions, showing that 137.036 is relevant to pure mathematics independent of its role as a constant well known to physicists.

IV. THE QUARK AND LEPTON MIXING ANGLES

But the above solution also gives

$$\begin{aligned} j(y) &= \left(\frac{M-y}{3}\right)^3 + (M-y)^2 \\ &\approx \left(\frac{10}{3} - \frac{0.000\,033\,3334}{3}\right)^3 + (10 - 0.000\,033\,3334)^2 \\ &= 137.036 \quad , \end{aligned}$$

whose four terms can be replicated, within the limits of experimental error, by the sines squared of the six quark and lepton mixing angles [3, 4]:

$$\begin{aligned} M/3 &= \quad 1/\sin^2 L12 \\ y/3 &= \quad \sin^2 Q13 \\ M &= \sin^2 L23 \times 1/\sin^2 Q12 \\ y &= \sin^2 Q23 \times \quad \sin^2 L13 \quad . \end{aligned}$$

Accordingly,

$$10/3 = 1/\sin^2 L12 \tag{4.1a}$$

$$0.000\,033\,3334/3 \approx \sin^2 Q13 \tag{4.1b}$$

$$10 = \sin^2 L23 \times 1/\sin^2 Q12 \tag{4.1c}$$

$$0.000\,033\,3334 \approx \sin^2 Q23 \times \sin^2 L13 \quad , \tag{4.1d}$$

TABLE I: Quark mixing data compared against predictions.

	$\lvert V_{us}\rvert$	$\lvert V_{ub}\rvert$
Prediction	0.2236	0.003 333
2012 [a]	$0.225\,34^{+0.000\,65}_{-0.000\,65}$	$0.003\,51^{+0.000\,15}_{-0.000\,14}$
Error in SD	2.7	1.3

[a]Ref. [3]. Particle Data Group 1σ global fit.

TABLE II: Lepton mixing data compared against predictions.

	$\sin^2 L12$	$\sin^2 L23$	$\sin^2 L13$
Prediction	0.3	0.5	0.0198[a]
2012 (Aug.)[b]	$0.320^{+0.016}_{-0.017}$	$0.427^{+0.034\,c}_{-0.027}$	$0.0246^{+0.0029}_{-0.0028}$
Error in SD	1.2	2.1	1.8

[a]This prediction rests on the assumption that $Q23 = 2.35°$, which derives from [3]. See [5] for a mixing model predicting all six mixing angles, where $Q23 \approx 2.367\,442°$, making $L13 \approx 8.034\,394°$ and $\sin^2 L13 \approx 0.019\,53$.

[b]Ref. [4]. A 1σ global fit that assumes the normal hierarchy.

[c]Ref. [4]. One of two minima, the other being $0.613^{+0.022}_{-0.040}$. Maximal mixing (i.e., $L23 = 45°$) is disfavored at ~90% C.L.

so that

$$\begin{aligned} j(y) &= \left(\frac{1}{\sin^2 L12} - \sin^2 Q13\right)^3 \\ &+ \left(\frac{1}{\sin^2 Q12} \times \sin^2 L23 - \sin^2 L13 \times \sin^2 Q23\right)^2 \\ &= 137.036 \quad . \end{aligned}$$

Now, if we assume that

$$\sin^2 L23 = 0.5$$

then

$$\sin^2 Q12 = 0.05 \quad .$$

Moreover, given that Eq. (4.1d) implies that

$$\sin^2 L13 \approx \frac{0.000\,033\,3334}{\sin^2 Q23} \quad , \tag{4.2}$$

while we know that $Q23$ measures roughly 2.35 degrees [3]. Accordingly,

$$\begin{aligned} \sin^2 L13 &\approx 0.0198 \\ L13 &\approx 8.09° \\ \sin^2 2L13 &\approx 0.0777 \quad . \end{aligned}$$

The fit of the above predictions in the quark sector can be seen in Table I, where $Q12$ and $Q13$ are primarily responsible for $\lvert V_{us}\rvert$ and $\lvert V_{ub}\rvert$, respectively. The largest of these errors is 2.7 SD, for $\lvert V_{us}\rvert$.

The fit of the above predictions in the leptonic sector can be seen in Table II. The largest of these errors is 2.1 SD, for $\sin^2 L23$.

And, finally, here are all of the predicted leptonic mixing angles [4]

$$\begin{aligned} L23 &= 45° \\ L13 &\approx 8.09° \\ L12 &\approx 33.210\,911° \end{aligned}$$

and the predicted CKM mixing angles [3]

$$\begin{aligned} Q13 &\approx 0.190\,987° \\ Q12 &\approx 12.920\,966° \quad . \end{aligned}$$

(Note: In 2007 the author's first mixing model set $L13$ equal to, not ~8° as above, but ~1/73[rd] of a degree [6], with the value for $\sin^2 L13$ equaling ~0.000 033 333 *multiplied* by $\sin^2 Q23$, instead of ~0.000 033 333 *divided* by $\sin^2 Q23$, as in Eq. (4.2). See [5] for a version of the 2007 model incorporating this important change, which gives all six mixing angles, including $Q23 \approx 2.367\,442°$ and $L13 \approx 8.034\,394°$.)

V. CONCLUSION

Consider that Theorem 1 is purely mathematical: it possesses neither constants nor equations chosen for physical reasons, which means that whatever physical predictivity it achieves has not been superimposed by a succession of expedient choices. And yet its minimal case gives values that reproduce four of the six mixing angles, as well as eight decimal digits of the precisely-measured fine structure constant. For this theorem to generate closely either the above angles or the fine structure constant would, by itself, be powerful evidence that it has physical significance; as it is, it closely reproduces both.

Moreover, [7] offers an alternative to Theorem 1 that employs even simpler initial assumptions, while still managing to produce the same empirical constants, a remarkable result given the greater economy of this more fundamental approach.

And, finally, the author has shown that rotation matrices can be used to impose three additional, entirely independent constraints on the mixing angles, where these constraints mesh neatly with the framework described in this article and make possible a model fitting all six mixing angles [5]. Such convergence of independent approaches is likewise unlikely to occur purely coincidentally.

[1] P. J. Mohr, B. N. Taylor, and D.B. Newell (2011), "The 2010 CODATA Recommended Values of the Fundamental Physical Constants" (Web Version 6.0). This database was developed by J. Baker, M. Douma, and S. Kotochigova. Available: http://physics.nist.gov/constants [Friday, 22-Jul-2011 10:04:27 EDT]. National Institute of Standards and Technology, Gaithersburg, MD 20899.

[2] J. S. Markovitch, "The fine structure constant derived from the broken symmetry of two simple algebraic identities," (2012) http://vixra.org/abs/1102.0012 .

[3] A. Ceccucci, Z. Ligeti, and Y. Sakai, "The CKM Quark-Mixing Matrix" in J. Beringer *et al.* (Particle Data Group), Phys. Rev. D **86**, 010001 (2012).

[4] D. V. Forero, M. Tortola and J. W. F. Valle, "Global status of neutrino oscillation parameters after Neutrino-2012," arXiv:1205.4018 [hep-ph].

[5] J. S. Markovitch, "Underlying symmetry among the quark and lepton mixing angles (Five year update)" (2012) http://www.vixra.org/abs/1211.0049 .

[6] J. S. Markovitch, "Underlying symmetry of the quark and lepton mixing angles" (2007) http://lss.fnal.gov/archive/other/jm-ph-2007-72.pdf.

[7] J. S. Markovitch, "A rewriting system applied to the simplest algebraic identities" (2012) http://www.vixra.org/abs/1211.0029.