Kaluza-Cartan Theory And A New Cylinder Condition

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Abstract

Kaluza's 1921 theory of gravity and electromagnetism using a fifth wrapped-up spatial dimension is inspiration for many modern attempts to develop new physical theories. For a number of reasons the theory is incomplete and generally considered untenable. An alternative approach is presented that includes torsion, unifying gravity and electromagnetism in a Kaluza-Cartan theory. Emphasis is placed on admitting important electromagnetic fields not present in Kaluza's original theory, and on the Lorentz force law. This is investigated via a non-Maxwellian kinetic definition of charge related to Maxwellian charge and 5D momentum. Two connections and a new cylinder condition are used. General covariance and global properties are investigated via a reduced non-maximal atlas. Conserved super-energy is used in place of the energy conditions for 5D causality. Explanatory relationships between matter, charge and spin are present.

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1 Introduction

Kaluza's 1921 theory of gravity and electromagnetism [1][2][3][4] using a fifth wrapped-up spatial dimension gives a taste of unification of electromagnetism with gravity in a way that ultimately couldn't be made to work and is generally believed to be untenable. However the underlying aim was particularly promising in terms of explanatory power. The coincidence represented by Kaluza's original theory is tantalising. Hower it may instead be a subset of a more comprehensive and explanatory theory. But how to search for it? Certain requirements are evident: the Lorentz force law [6] must be explained, Maxwell's laws [6] must be present, the Lorentz transformation [6] must define inertial frames, general relativity [6] must be a limit. Of these the Lorentz force law is the most enigmatic and conceptually unsatisfying within current theory. Yes, it comes from the Einstein-Maxwell stress-energy tensor [6], but where does *that* come from? The Lorentz force law is but the relativistic form of Coulomb's law. Surely it should be as fundamental geometrically as the inverse square law of gravity? It is in this vein that search for a variant Kaluza theory makes sense.

The Lorentz force law herein requires a constant scalar field, this places constraints on admissible solutions. The emphasis is then on eliminating the constraint in Kaluza theory that prevents the so-called non-null electromagnetic solutions. Explicit existence proofs are not necessary. It is sufficient to show that the constraint that causes the problems on solutions has been weakened in the new theory. The constraint is the third field equation in [1], and equation (3.2.3) here. When the scalar field is constant this equation becomes one of two equations that characterize the null electromagnetic fields. This equation is as follows, and fields that satisfy this will be called 'nullish':

Definition 1.0.1: 'Nullish' electromagnetic fields satisfy: $F_{ab}F^{ab} = 0$. Null electromagnetic fields have the nullish property plus the following condition, where the star is the Hodge star operator: $F_{ab}(*F^{ab}) = 0$.

Kaluza's original theory [1] prohibits non-nullish solutions (or even near non-nullish solutions) for constant scalar field. Nullishness is too tight to admit important electromagnetic fields, in particular the essential electrostatic fields. That electrostatic or near-electrostatic fields are non-nullish and therefore a problem in any theory that omits them can be seen by comparing definition (1.0.1) with the following well-known fact from special relativity, that is by considering a special relativistic limit:

$$F_{ab}F^{ab} = 2(B \cdot B - E \cdot E) \tag{1.0.2}$$

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At first the objective of the research undertaken here, that is before torsion was finally admitted, was actually to try to discount the need for torsion since its lack of presence is geometrically an obvious assumption in many physical theories. This is analogous to Euclid's fifth postulate in that its assumption is an addition and its removal actually enabled geometric theories like general relativity to be possible. Whilst few would consider it necessary to investigate such an assumption, that was the original program here. The research was therefore based around showing that sufficient missing electromagnetic fields could be obtained (without torsion) from existing Kaluza theory. That program failed and the result was to explicitly allow torsion in this variant theory. It is claimed that the theory presented here is an example of a Kaluza variant theory that better satisfies the requirements of observable classical physics in having a wide range of electromagnetic fields permitted whilst providing a Lorentz force law by construction. This paper therefore shows that such Kaluza variant theories exist and exemplifies a new route in the search for such theories, thus revivifying Kaluza theory from the untenable. The theory presented here can thus be seen as at least an example of such a theory, and has value as such. In addition it can also be considered as a candidate empirical theory, though this is not necessary for it to have the previously mentioned theoretical value.

A new kinetic charge will be defined as the 5th-dimensional component of momentum as in [8]. The Lorentz force law will follow. As momentum the kinetic charge has a divergence law via the (torsionless) Einstein tensor. Maxwellian charge also has a vector potential, see (4.4.1), and thus local conservation, but the kinetic charge being covariant is more fundamental.

2 Conventions

The following conventions are adopted unless otherwise specified. Though unfamiliar in places these are necessary for following the multiple systems used.

Five dimensional metrics, tensors and pseudo-tensors and operators are given the hat symbol. Five dimensional indices, subscripts and superscripts are given capital Roman letters. Lower case indices can either be 4D or generic for definitions depending on context. Index raising is referred to a metric $\hat{g}_{AB}$ if 5-dimensional, and to g_{ab} if 4-dimensional. Terms that might repeat dummy variables or are otherwise in need of clarification use additional brackets. The domain of partial derivatives carries to the end of a term without need for brackets, so for example we have $\partial_a g_{db} A_c + g_{db} g_{ac} = (\partial_a (g_{db} A_c)) + (g_{db} g_{ac})$. Terms that might repeat dummy variables or are otherwise in need of clarification use additional brackets. Square brackets can be used to make dummy variables local in scope. Space-time is given signature (−, +, +, +), Kaluza space (−, +, +, +, +) in keeping with [6]. Under the Wheeler et al [6] nomenclature the sign conventions used here as a default are [+, +, +]. The first dimension (index 0) is time and the 5^{th} dimension (index 4) is the topologically closed Kaluza dimension. Time and distance are geometrized throughout such that c = 1. **G** is the gravitational constant. The scalar field component is labelled ϕ^2. The matrix of g_{cd} can be written as $|g_{cd}|$. The Einstein summation convention may be used without special mention. $\Box$ represents the 4D D'Alembertian [6].

Connection coefficients with torsion will take the form: Γ^c_{ab} or Γ^{abc}. The metric with a torsion tensor defines a unique connection. Therefore two unique connections for a given metric are one with and one without torsion. The unique Levi-Civita connection (ie without torsion) is written as: $\mathit{\Gamma}^c_{ab}$, and the covariant Levi-Civita derivative operator (ie without torsion): $\triangle_a$. So we have:

$$F_{ab} = \triangle_a A_b - \triangle_b A_a = \partial_a A_b - \partial_b A_a \text{ equally } F = dA \tag{2.0.1}$$

In order to distinguish tensors constructed using torsion G_{ab} and R_{ab} (i.e. where the Ricci tensor is defined in terms of Γ^c_{ab}) from those that do not use torsion (ie that are defined in terms of $\mathit{\Gamma}^c_{ab}$), the torsionless case uses cursive: $\mathcal{G}_{ab}$ and $\mathcal{R}_{ab}$. On any given manifold with torsion both these parallel systems of connection coefficients and dependent tensors can be used. That is, the Ricci tensor (with torsion), R_{ab}, and the Ricci tensor, $\mathcal{R}_{ab}$, are both defined and are in general different. Further each of these can have hats on or hats off, giving: $\hat{R}_{AB}$ and $\hat{\mathcal{R}}_{AB}$. It is an extremely confusing part of this work that all four systems can be used at the same time in the same equations! This particularly occurs when the 4D components of a 5D tensor are being used, e.g. looking at $\hat{R}_{ab}$ and $\hat{\mathcal{R}}_{ab}$. Torsion introduces non-obvious conventions in otherwise established definitions. The order of the indices in the connection coefficients matters, and this includes in the Ricci tensor definition and the definition of the connection coefficient symbols themselves:

$$\nabla_a w_b = \partial_a w_b - \Gamma^c_{ab} w_c \tag{2.0.2}$$

Some familiar defining equations consistent with [1] define the Ricci tensor and Einstein tensors in terms of the connection coefficients along usual lines, noting that with torsion the order of indices can not be carelessly interchanged as they can with the symmetric Levi-Civita coefficients:

$$R_{ab} = \partial_c \Gamma^c_{ba} - \partial_b \Gamma^c_{ca} + \Gamma^c_{ba}\Gamma^d_{dc} - \Gamma^c_{da}\Gamma^d_{bc} \tag{2.0.3}$$

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab} = 8\pi \mathbf{G} T_{ab} \tag{2.0.4}$$

For convenience we will define $\alpha = \frac{1}{8\pi\mathbf{G}}$. Analogous definitions can also be used with the Levi-Civita connection to define $\mathcal{R}_{ab}$ and $\mathcal{G}_{ab}$ in the obvious way.

3 Kaluza's Original Theory And Foundational Problems

Kaluza's 1921 theory of gravity and electromagnetism [2][3][4] using a fifth wrapped-up spatial dimension is at the heart of many modern attempts to develop new physical theories [1][5]. From supersymmetry to string theories topologically closed small extra dimensions are used to characterize the various forces of nature. It is therefore inspiration for many modern attempts and developments in theoretical physics. However it has a number of foundational problems and is often considered untenable in itself. This paper looks at these problems from a purely classical perspective.

3.1 The Metric

The original Kaluza theory assumes a (1,4)-Lorentzian Ricci flat manifold to be the 5D metric, split as shown below (and for interest this can be compared to the later ADM formalism [9]). A_a is to be identified with the electromagnetic potential, ϕ^2 is to be a scalar field, and g_{ab} the metric of 4D space-time:

$$\hat{g}_{AB} = \begin{bmatrix} g_{ab} + \phi^2 A_a A_b & \phi^2 A_a \\ \phi^2 A_b & \phi^2 \end{bmatrix} \tag{3.1.1}$$

Note that a scaling factor has been set to $k = 1$ and so is not present, this will be reintroduced later in the text (4.4.1), it is mathematically arbitrary, but physically scales units when units are geometrized. By inverting this metric as a matrix (readily checked by multiplication) we get:

$$\hat{g}^{AB} = |\hat{g}_{AB}|^{-1} = \begin{bmatrix} g^{ab} & -A^a \\ -A^b & \frac{1}{\phi^2} + A_i A^i \end{bmatrix} \tag{3.1.2}$$

Maxwell's law are automatically satisfied, using (2.0.1) to define F with respect to the potential: dF=0 follows from dd = 0. d*F= 4π*J can be set by construction. d*J=0, local conservation of charge follows also by dd=0 on most parts of the manifold, although a more fundamental conservation law is therefore required.

In order to write the metric in this form (3.1.1) and (3.1.2) there is a subtle assumption, that g_{ab}, which will be interpreted as the usual four dimensional space-time metric, is itself non-singular. This will always be the case for moderate or small values of A_x which will here be identified with the electromagnetic 4-vector potential. The raising and lowering of this 4-vector are defined in the obvious way in terms of g_{ab}. The 5D metric can be represented at every point on the Kaluza manifold in terms of this 4D metric g_{ab} (when it is non-singular), the vector potential A_x, and the scalar field ϕ^2. We have also assumed that topology is such as to allow the Hodge star operator and Hodge duality of forms to be well-defined (see [6] p.88). This means that near a point charge source the argument that leads to charge conservation potentially breaks down as the potential may cease to be well-defined. Whereas the kinetic charge that will be defined in the sequel does not have this problem. So two different definitions of charge are to be given: the Maxwellian, and the kinetic charge. It is the kinetic charge that will obey a more general conservation law.

With values of ϕ^2 around 1 and relatively low 5-dimensional metric curvatures we need not concern ourselves with this assumption beyond stating it on the basis that physically these parameters encompass tested theory. Given this proviso A_x is a vector and ϕ^2 is a scalar - with respect to the tensor system defined on any 4-dimensional submanifold (or region of a submanifold) that can take the induced metric g.

3.2 Kaluza's Cylinder Condition And The Original Field Equations

Kaluza's cylinder condition is that all partial derivatives in the 5th dimension i.e. ∂_4 and $\partial_4\partial_4$ etc... of all metric components and of all tensors and their derivatives are zero. A perfect 'cylinder'. This leads to constraints on g_{ab} given in [1] by three equations, the field equations of the original Kaluza theory, where the Einstein-Maxwell stress-energy tensor can be recognised embedded in the first equation. Beware in particular that the conventions are as used by the referenced author and not those used in this paper.

$$G_{ab} = \frac{k^2\phi^2}{2}\left\{\frac{1}{4}g_{ab}F_{cd}F^{cd} - F^c_a F_{bc}\right\} - \frac{1}{\phi}\{\nabla_a(\partial_b\phi) - g_{ab}\Box\phi\} \tag{3.2.1}$$

$$\nabla^a F_{ab} = -3\frac{\partial^a\phi}{\phi}F_{ab} \tag{3.2.2}$$

$$\Box\phi = \frac{k^2\phi^3}{4}F_{ab}F^{ab} \tag{3.2.3}$$

Note that there is both a sign difference and a possible factor difference with respect to Wald's [7] and Wheeler's [6] Einstein-Maxwell equation. The sign difference appears to be due to the mixed use of metric sign conventions in [1]. A k factor is present and this scaling will be investigated. These equations will be referred to as the first, second and third torsionless field equations, or original field equations, respectively. They are valid only in Kaluza vacuum, that is, outside of matter and charge models, and when there is no torsion. That is when $\mathcal{R}_{ab} = 0$ and torsion is vanishing.

3.3 The Foundational Problems

The main issue addressed in this paper is the variety of electromagnetic solutions that are a consequence of Kaluza theory, whilst maintaining the Lorentz force law. This is either not usually considered such a big problem, which is not convincing: a sufficient variety of electromagnetic fields must be available, and the Lorentz force law should be explicitly derivable. Or by others, confusingly, it is considered insurmountable and Kaluza theory written off, which is too negative. Either way this remains the real problem with Kaluza theory as it prevents a convincing geometric unification of gravity and electromagnetism. The missing solutions are the non-nullish solutions and include the important electrostatic fields. So they include some really important fields!

One inadequate and arbitrary fix in standard Kaluza theory is to set the scalar field term large to ensure that the second field equation (3.2.2) is approximately zero despite scalar fluctuations. This approach will not be taken here as it is contrived. The stress-energy tensor under scalar field fluctuations is different from the Einstein-Maxwell tensor [6][7] and the accepted derivation of the Lorentz force law (for electrovacuums [6]) can not be assumed. A variable scalar field as required by the third field equation for non-nullish fields (3.2.3) also implies non-conservation of Maxwell charge via the second field equation (3.2.2), and problems also arise with respect to the Lorentz force law in the case of a variable scalar field. Thus, in this paper, the scalar field will be fixed and the non-nullish solutions will need reintroducing by increasing the available degrees of freedom. This is done via the introduction of torsion. The electromagnetic field devoid of matter and charge sources will then be characterized by $\hat{R}_{AB} = 0$ instead of $\hat{\mathcal{R}}_{AB} = 0$.

Another foundational issue of Kaluza theory is that even with a scalar field it does not have convincing sources of mass or charge built in. The second field equation (3.2.2) has charge sources, but it's unlikely that realistic sources are represented by this equation. The better interpretation is that real matter and charge sources must be defined as being when $\hat{\mathcal{R}}_{AB} \neq 0$ in Kaluza's originl theory.

By identifying electromagnetic fields with $\hat{R}_{AB} = 0$ with torsion we presumably have to identify matter and charge sources now with $\hat{R}_{AB} \neq 0$. However the mass-energy conservation law remains by definition in terms of $\hat{\mathcal{G}}_{AB}$ - i.e. the torsionless Einstein tensor. Further the causality of solutions of the form $\hat{R}_{AB} = 0$ no longer follows, this will lead to additional considerations.

The nature of matter and charge is investigated further in this paper to clarify the consequences of the use of torsion.

Charge will be given an alternative definition: kinetic charge will be defined as the 5th-dimensional component of momentum following a known line of reasoning [8] within Kaluza theory. This will enable a derivation of the Lorentz force law. As momentum the kinetic charge is of necessity locally conserved, provided there are no irregularities in the topology of the Kaluza 5th dimension.

Note that conservation of Maxwellian charge (which will be shown to be identifiable with kinetic charge) is locally guaranteed by the existence of the potential and the exterior derivative, but breaks-down under curvature. The two definitions will be shown to be related, but the kinetic charge deemed more fundamental as it admits a curvature-independent local divergence-free law via the (torsionless) Einstein tensor.

3.4 A Solution?

The Lorentz force law is to be derived from the theory independently of the electrovacuum solutions of general relativity, and the missing non-nullish solutions included at the same time to create a more complete theory. Note that in addition the derivation of the Lorentz force law within general relativity (from an assumed Einstein-Maxwell stress-energy tensor) is not without problems of principle [6], so it is not the Einstein-Maxwell stress-energy tensor that is necessarily here being sought but just those experimental results that classical physics explains using it. A Lorentz force law is here derived from first principles.

The combination of torsion and the 5th spatial dimension justifies the label Kaluza-Cartan theory.

The new definition of charge, the momentum in the fifth dimension, will be introduced and Maxwellian charge (defined with respect to the field as a 2-form) will be shown to coincide at an appropriate limit. The collocation of torsion with electromagnetism is different from other Einstein-Cartan theories where the torsion is limited to within matter models. Here certain specific components of torsion are an essential counterpart of electromagnetism, and other components of torsion could potentially exist outside of matter models.

Restrictions to the geometry and certain symmetries will be handled by reducing the maximal atlas to a reduced Kaluza atlas that automatically handles the restrictions and symmetries without further deferment to general covariance. Physically this represents the idea that in 4D charge is a generally covariant scalar, whereas in 5D charge is dependent on the frame. That this is meaningful stems from the global property of a small wrapped-up fifth spatial dimension with cylinder condition. The Kaluza atlas is a choice of subatlas for which (among other criteria) a global Kaluza direction is defined that satisfies the new cylinder condition. This leads to useful constraints on the connection coefficients for all coordinate systems in the Kaluza atlas. The 5D metric decomposes into a 4D metric and the electromagnetic vector potential and the scalar field.

4 Overview Of Kaluza-Cartan Theory

4.1 Postulates

The following K1-K4 are the core Postulates of the new Kaluza-Cartan theory. For a survey of the torsion mathematics required and for a discussion about the geodesic assumption K4, see Appendix I.

POSTULATE (K1): **Geometry**. A *Kaluza-Cartan manifold* is a 5D smooth Lorentzian manifold with metric torsion connection.

POSTULATE (K2): **Well-behaved**. Kaluza-Cartan space is assumed globally hyperbolic in the sense that there exists a 4D spatial cauchy surface plus time, such that the 4D hypersurface is a simply connected 3D space extended around a 1D loop. And Kaluza-Cartan space is oriented and time-oriented.

POSTULATE (K3) version1: We start with effectively the old and original version: **Cylinder condition (original)**. One spatial dimension is topologically closed and 'small', the Kaluza dimension. This is taken to mean that there are global unit vectors that define this direction, the Kaluza direction. The partial derivatives of all tensors in this Kaluza direction are taken to be zero in some coordinate system. The other spatial dimensions and time dimension are 'large'. 'Large' here simply means that the considerations given to 'small' do not apply. We further add an additional constraint that is novel, **Cylinder condition (additional)**. The covariant derivative $\hat{\nabla}_4$ (with torsion) of all tensors in the Kaluza direction is zero.

We have that the global unit vectors defining the Kaluza direction form unit vectors (index 4) in 'torsion-normal' coordinates via analogous reasoning to the normal coordinates case without torsion in Wald [7] eqn 3.1.17. We can further refine this variation of the original cylinder condition, replacing in full version1 with a more intuitive version2

POSTULATE (K3) version2: **Cylinder condition (new)**. As with version1: one spatial dimension is topologically closed and 'small', the Kaluza dimension. This is taken to mean that there are global unit vectors that define this direction, the Kaluza direction, and that the covariant derivative $\hat{\nabla}_4$ with torsion of all tensors in the Kaluza direction is zero. However, the torsion is constrained to be such that in torsion-normal coordinates the partial derivatives of all tensors in the Kaluza dimension are vanishing.

POSTULATE (K4): **Geodesic Assumption**. That any particle-like model, that is to be identified with a charge, approximately follows 5D auto-parallels.

Definitions 4.1.1: The *Kaluza-Cartan vacuum* is a Ricci flat region of a Kaluza-Cartan manifold with respect to the torsion connection definition of the Ricci tensor. Similarly the *Kaluza vacuum* is a Ricci flat region with respect to the Levi-Civita connection. They are different: $\hat{R}_{AB} = 0$ and $\hat{\mathcal{R}}_{AB} = 0$ respectively. Here they are both defined in terms of the geometry implied by the cylinder condition. Kaluza vacuum will be associated with nullish electromagnetic solutions when there is no torsion, Kaluza-Cartan vacuum will encompass all electromagnetic fields. *Kaluza-Cartan matter* and *Kaluza mass-energy* follow as complements to their vanishing respective Ricci tensors.

Observe that Kaluza-Cartan matter, unlike Kaluza mass-energy, but like common matter, does not have its own divergence law.

LIMIT POSTULATE (B1): There is a Kaluza atlas, see definition (4.2.1), possibly only over a region, such that $\phi^2 = 1$ at every point. The scalar field results from the the decomposition of the Kaluza metric into 4D metric, potential vector and scalar field. It is contained within the metric explicitly in (4.4.1). Thus B1 is a constraint on the 5D metric. (See Appendix II for further issues.)

Additional postulates that can be interpreted as forming conditions necessary for a classical limit now follow. L1-L3 constitute a *weak field limit* that will be applied by way of approximation for the 'classical' limit of behaviour. The deviation from the 5D-Minkowski metric is given by a tensor $\hat{h}_{AB}$. This tensor belongs to a set of small tensors that we might label $O(h)$. Whilst this uses a notation similar to orders of magnitude, and is indeed analogous, the meaning here is a little different. This is the weak field approximation of general relativity using a more flexible notation. Partial derivatives, to whatever order, of metric terms in a particular set $O(x)$ will be in that same set at the weak field limit. In principle we are doing nothing more than following the weak field limit procedure [6] of general relativity. In the weak field approximation of general relativity, terms that consist of two $O(h)$ terms multiplied together get discounted and are treated as vanishing at the limit. We might use the notation $O(h^2)$ to signify such terms. There is the weak field approximation given by discounting $O(h^2)$ terms. But we might also have a less aggressive limit given by, say, discounting $O(h^3)$ terms, and so on. We can talk about weak field limits (plural) that discount $O(h^n)$ terms for $n > 1$, but they are based on the same underlying construction.

LIMIT POSTULATE (L1): The metric can be written as follows in terms of the 5D Minkowski tensor and $\hat{h} \in O(h)$:

$$\hat{g}_{AB} = \hat{\mu}_{AB} + \hat{h}_{AB}$$

Torsion will also be considered a weak field under normal observational conditions, similarly to L1. Torsion is defined in terms of the Christoffel symbols. Christoffel symbols are in part constructed from the partial derivatives of the metric and that part is constrained by L1 to be $O(h)$. The contorsion term being the difference. See [11]. The contorsion (and therefore the torsion) will be treated as $O(h)$ accordingly.

LIMIT POSTULATE (L2): The contorsion (and therefore the torsion) will be an $O(h)$ term at the weak field limits.

One further constraint is required at the weak field limit. Its use will be minimized (both the application of the antisymmetry and the allowance for some small symmetry terms), but it will nevertheless be important. In L3, symmetric parts of the torsion and contorsion tensor (and their derivatives) are treated as particularly 'small' in that they are small relative to any antisymmetric parts of the torsion and contorsion tensor, torsion already assigned to $O(h)$ by L2. The torsion tensor will be given the following limit: It is to be weakly completely antisymmetric - *a weak antisymmetric limit*, this will be so even with respect to L1 and L2. Thus the symmetric parts of the contorsion and torsion tensors will be $O(h^2)$ at the weak field limit. All derivatives thereof follow the same rule:

LIMIT POSTULATE (L3): The symmetric parts of the contorsion and torsion tensors will be $O(h^2)$ at the weak field limits.

It is claimed that such a limit may be approached without loss of generality of the solutions from a physical perspective. In other words at the L1-L2 weak field limit equation (5.1.10) is compatible with the weak antisymmetric limit L3, and poses no constraint due to the product of the potential and field also being discounted at the weak field limit via L1.

SUPER-ENERGY POSTULATE (SE1): That the conserved super-energy hypothesis (see Appendix I for details) applies fully to Kaluza-Cartan space.

This both ensures causality and provides a well-defined conservation law for 5D Kaluza-Cartan space that can potentially be used in place of the energy conditions currently used in general relativity.

4.2 The Cylinder Condition And Charts

The cylinder condition by construction allows for an atlas of charts wherein the Kaluza dimension is naturally presented by the fourth index. The atlases that are compliant are restricted. This means that the cylinder condition can be represented by a subatlas of the maximal atlas. The set of local coordinate transformations that are compliant with this atlas (called a Kaluza atlas) is non-maximal by construction. A further reduction in how the atlas might be interpreted is also implied by setting c=1, and constant **G**. The existence of a single unit for space and time can be assumed, and this must be scaled in unison for all dimensions. Consistently with cgs units we can choose either centimetres or seconds. This would leave velocities (and other geometrically unitless quantities) unchanged in absolute magnitude. This doesn't prevent reflection of an axis however, and indeed reflection of the Kaluza dimension is here equivalent to a (kinetic) charge inversion. However, given orientability and an orientation we can remove even this ambiguity. We can further reduce a Kaluza atlas by removing boosts in the Kaluza dimension. Space-time is taken to be a subframe within a 5D frame within a Kaluza subatlas of a region wherein uncharged matter can be given a rest frame via a 4D Lorentz transformation. Boosting uncharged matter along the Kaluza axis will give it kinetic charge.

The Kaluza atlas represents the 4D view that kinetic charge is 4D covariant. The justification for this assertion will be given later. Rotations into the Kaluza axis can likewise be omitted. This results in additional constraints on the connection coefficients associated with charts of this subatlas, and enables certain geometrical objects to be more easily interpreted in space-time. The use of this subatlas does not prevent the theory being generally covariant, but simplifies the way in which we look at the Kaluza space through a 4D physical limit.

Definition 4.2.1: A*Kaluza atlas* is:

(i) A subatlas (possibly just over a region) of the maximal atlas of Kaluza-Cartan space where boosts and rotations into the Kaluza dimension (as defined by the cylinder condition K3) are explicitly omitted.

(ii) All partial derivatives in the Kaluza direction are vanishing.

(iii) Inversion in the Kaluza direction and rescalings can also be omitted so as to establish units and orientation.

(iv) For each point on the Kaluza atlas a chart exists with 'torsion-normal' coordinates (see Appendix) where index 4 is the Kaluza dimension.

4.3 Kinetic Charge

Kinetic charge is defined as the 5D momentum component in terms of the 5D Kaluza rest mass of a hypothesised particle: ie (i) its rest mass in the 5D Lorentz manifold (m_{k0}) and (ii) its proper Kaluza velocity ($dx_4/d\tau^*$) with respect to a frame in the maximal atlas that follows the particle. And equally it can be defined in terms of (i) the relativistic rest mass (m_0), relative to a projected frame where the particle is stationary in space-time, but where non-charged particles are stationary in the Kaluza dimension, and in terms of (ii) coordinate Kaluza velocity (dx_4/dt_0):

Definition 4.3.1: kinetic charge (scalar): $Q^* = m_{k0}dx_4/d\tau^* = m_0 dx_4/dt_0$

This makes sense because mass can be written in fundamental units (i.e. in distance and time). And the velocities in question defined relative to particular frames. It is not a generally covariant definition but it is nevertheless mathematically meaningful. This kinetic charge can be treated in 4D space-time, and the Kaluza atlas, as a scalar: the first equation above is covariant with respect to the Kaluza atlas. It can be generalized to a 4-vector, and it is also conserved as shown. In general relativity at the special relativistic Minkowski limit the conservation of momenergy can be given in terms of the stress-energy tensor as follows [9], $j \neq 0$:

$$\frac{\partial \hat{T}^{00}}{\partial t} + \frac{\partial \hat{T}^{i0}}{\partial x_i} = 0 \text{ and } \frac{\partial \hat{T}^{0j}}{\partial t} + \frac{\partial \hat{T}^{ij}}{\partial x_i} = 0 \tag{4.3.2}$$

This is approximately true at a weak field limit and can be applied equally to Kaluza theory, via the (torsionless) connection. We have a description of conservation of (torsionless) momentum in the 5th dimension as follows:

$$\frac{\partial \hat{T}^{04}}{\partial t} + \frac{\partial \hat{T}^{i4}}{\partial x_i} = 0 \tag{4.3.3}$$

We also have i=4 vanishing by the cylinder condition. Thus the conservation of kinetic charge becomes (when generalized to different space-time frames) the property of a 4-vector current, which we know to be locally conserved:

$$\partial_0 \hat{T}^{04} + \partial_1 \hat{T}^{14} + \partial_2 \hat{T}^{24} + \partial_3 \hat{T}^{34} = 0 \tag{4.3.4}$$

As in relativity this can be generalized to a new definition that is valid even when there is curvature. Nevertheless the original kinetic charge definition (4.3.1) has meaning in all Kaluza atlas frames as a scalar. Kinetic charge current is the 4-vector, induced from 5D Kaluza-Cartan space as follows (using the Kaluza atlas to ensure it is well-defined as a 4-vector):

$$J^{*a} = -\alpha \hat{\mathcal{G}}^{a4} \tag{4.3.5}$$

Using Wheeler et al [6] p.131, and selecting the correct space-time (or Kaluza atlas) frame, we have:

$$Q^* = J^*_a (1,0,0,0)^a \tag{4.3.6}$$

So we have a scalar, then a vector representation of relativisitic invariant charge current, and finally a 2-tensor unification with conserved (torsionless) mass-energy via the (torsionless) Einstein tensor. It follows that the vanishing of the divergence of kinetic charge in 4D is only approximate, in 5D not.

Definition 4.3.7: *Kinetic charge current* is defined to be the 4-vector $J^{*a} = -\alpha \hat{\mathcal{G}}^{a4}$, with respect to the Kaluza atlas. Note the divergence of the (torsionless) Einstein tensor:

$$\hat{\triangle}_A \hat{\mathcal{G}}^{AB} = 0 \text{ and } \hat{\triangle}_A \hat{\mathcal{G}}^{A4} = 0 \approx \hat{\triangle}_a \hat{\mathcal{G}}^{a4}$$

4.4 Two Types Of Geometrized Charge

The metric components used in [1] will be used here as the Kaluza-Cartan metric. The vector potential and electromagnetic fields formed via the metric are sourced in Maxwell charge Q_M. Maxwell's law are automatically satisfied, using (2.0.1) to define F with respect to the potential: dF=0 follows from dd = 0. d*F= 4π*J can be set by construction. d*J=0.

A_a is to be identified with the electromagnetic potential, ϕ^2 is to be a scalar field, and g_{ab} the metric of 4D space-time:

Definition 4.4.1: The 5D Kaluza-Cartan metric.

$$\hat{g}_{AB} = \begin{bmatrix} g_{ab} + k^2\phi^2 A_a A_b & k\phi^2 A_a \\ k\phi^2 A_b & \phi^2 \end{bmatrix}$$

$$\hat{g}^{AB} = \begin{bmatrix} g^{ab} & -kA^a \\ -kA^b & \frac{1}{\phi^2} + k^2 A_i A^i \end{bmatrix} \tag{4.4.1}$$

This gives (without torsion [1]) nullish solutions under the original Kaluza cylinder condition and constant scalar field, such that $G_{ab} = -\frac{k^2}{2} F_{ac}F^c_b$. Compare this with [7] where we have $G_{ab} = 2F_{ac}F^c_b$ in geometrized units for ostensibly the same fields. The units need to be agreed between the two schemes. We would need to set either $k = 2$ or $k = -2$ for compatibility of results and formulas. And this is particularly important as we wish to derive the Lorentz force law with the same units as [7]. N.B. the sign change introduced by [1] - where it appears that the Einstein tensor was defined relative to $(+, -, -, -)$, despite the 5D metric tensor being given in a form that can only be $(-, +, +, +, +)$, which is confusing. This makes no fundamental difference, but must be noted. It is a confusion seemingly introduced by accident in [1]. The use of conventions in this type of work are excruciatingly tricky.

The geometrized units, Wald [7] p470-471, define units of mass in terms of fundamental units. This leads to an expression for kinetic charge in terms of Kaluza momentum when $k = 2$ and $\mathbf{G} = 1$. $\mathbf{G}$ and k are not independent however. If we fix one, the other is fixed too: A consequence of requiring the Lorentz force law written in familiar form and compatibility with the units used in [7]. The relation between $\mathbf{G}$ and k is given in equation (6.5.5) via the derivation of the Lorentz force law. Simple compatibility with Wald [7] results where $k = 2$ and $\mathbf{G} = 1$. The sign of k is also fixed by (6.1.4). The result of dimensional analysis gives kinetic charge, Q^*, in terms of 5D momentum:

$$Q^* = \frac{c}{\sqrt{\mathbf{G}}} P_4 \tag{4.4.2}$$

This calculation and the consistency of the construction with special relativity are detailed further in Appendix 1.

5 The Field Equations

5.1 The New Cylinder Condition And Scalar Field

Here we look at how the new Kaluza-Cartan cylinder condition affects the connection coefficients of any coordinate system within the Kaluza atlas (using $k = 1$). The Appendix (see section containing 10.1.1 and related) contains a reference for connection coefficients working both with and without the torsion component.

The following requires the selection of coordinates (the Kaluza atlas) that set the partial derivatives in the Kaluza dimension to zero and from the relationship between these two and the Christoffel symbols given in Wald [7] p33 eqn (3.1.14) as applied to a number of test vectors. Note that there is no symmetry of the (with torsion) connection coefficients suggested here. That is, these terms are forced zero by the fact that both the partial derivatives and the covariant derivatives in the Kaluza direction are zero. Cf equation (2.0.2), where the consequences of setting both the partial derivatives and the covariant derivative to zero can be seen on the connection coefficients.

$$0 = 2\hat{\Gamma}^A_{4c} = \sum_d \hat{g}^{Ad}(\partial_4 g_{cd} + \partial_4\phi^2 A_c A_d + \partial_c\phi^2 A_d - \partial_d\phi^2 A_c) + \hat{g}^{A4}\partial_c\phi^2 - 2\hat{K}_{4c}{}^A \tag{5.1.1}$$

$$0 = 2\hat{\Gamma}^A_{44} = 2\sum_d \hat{g}^{Ad}\partial_4\phi^2 A_d - \sum_d \hat{g}^{Ad}\partial_d\phi^2 + \hat{g}^{A4}\partial_4\phi^2 - 2\hat{K}_{44}^A \tag{5.1.2}$$

$$2\hat{K}_{4c}{}^A = \hat{g}^{Ad}(\partial_c\phi^2 A_d - \partial_d\phi^2 A_c) + \hat{g}^{A4}\partial_c\phi^2 \tag{5.1.3}$$

$$2\hat{K}_{44}{}^A = -\hat{g}^{Ad}\partial_d\phi^2 \tag{5.1.4}$$

Inspecting these equations, and given that $\hat{K}_{A(BC)} = 0$ (9.1.4), and applying A=c without summing, we have a constraint on the scalar field in terms of the vector potential. This constraint is used in Appendix II to show how postulate B1 may be natural. The result, applying B1, is as follows (using $k = 1$):

$$2\hat{K}_{4c}{}^A = \hat{g}^{Ad}(\partial_c A_d - \partial_d A_c) \tag{5.1.5}$$

$$2\hat{K}_{44}{}^A = 0 \tag{5.1.6}$$

This gives the contorsion a very clear interpretation in terms of the electromagnetic field.

$$\hat{K}_{4c}{}^a = \frac{1}{2}F^a_c \tag{5.1.7}$$

$$\hat{K}_{4c}{}^4 = -\frac{1}{2}A^d F_{cd} \tag{5.1.8}$$

We also have from (5.1.1) the following:

$$\hat{\Gamma}^4_{4c} + \hat{K}_{4c}{}^4 - \hat{K}_{c4}{}^4 = \hat{\Gamma}^4_{4c} + \hat{K}_{4c}{}^4 = \hat{\Gamma}^4_{c4} \tag{5.1.9}$$

In the case of complete antisymmetry of torsion/contorsion, again using (5.1.1), this specialises to:

$$\hat{\Gamma}^4_{4c} = 0 = \hat{K}_{4c}{}^4 = -\hat{K}_{c4}{}^4 = \hat{K}_{c4}{}^4 = A^d F_{cd} \tag{5.1.10}$$

$$\hat{\Gamma}^4_{4c} + \hat{K}_{4c}{}^4 - \hat{K}_{c4}{}^4 = \hat{\Gamma}^4_{c4} = 0 \tag{5.1.11}$$

In particular (5.1.10) presents too tight a constraint on electromagnetism as it employs more degrees of freedom than any gauge conditions. For this reason non-completely antisymmetric torsion is allowed, yet constrained at the weak field limits by L3. Using (5.1.9) for the last equation of (5.1.13), in the general case we have:

$$\hat{K}_{[4c]}{}^{4} + \hat{K}_{(4c)}{}^{4} = \hat{K}_{4c}{}^{4} = -\frac{1}{2}A^{d}F_{cd}$$
$$\hat{K}_{c4}{}^{4} = 0 = -\hat{K}_{[4c]}{}^{4} + \hat{K}_{(c4)}{}^{4} \tag{5.1.12}$$

$$\hat{K}_{[4c]}{}^{4} = \hat{K}_{(4c)}{}^{4} = \frac{1}{2}\hat{K}_{4c}{}^{4} = -\frac{1}{4}A^{d}F_{cd}$$
$$\hat{K}_{c4}^{\ 4} = \hat{\Gamma}^{4}_{4c} = 0$$
$$\hat{K}_{4c}^{\ 4} = \hat{\Gamma}^{4}_{c4} = -\frac{1}{2}A^{d}F_{cd} \tag{5.1.13}$$

We can see from this section why postulate L3 is necessary, that is, why non-completely antisymmetric torsion terms must be allowed.

5.2 The First Field Equation With Torsion, $k = 1$

The first field equation in this theory is somewhat complicated (5.2.3), but an analysis here will show that Kaluza-Cartan theory and the original Kaluza theory share a limit for certain nullish solutions.

Looking at the Ricci tensor, but here with torsion (using equations 10.1.7 repeatedly, and the cylinder condition as required):

$$\hat{R}_{ab} = \partial_C\hat{\Gamma}^{C}_{ba} - \partial_b\hat{\Gamma}^{C}_{Ca} + \hat{\Gamma}^{C}_{ba}\hat{\Gamma}^{D}_{DC} - \hat{\Gamma}^{C}_{Da}\hat{\Gamma}^{D}_{bC}$$
$$\hat{R}_{ab} = \partial_c\hat{\Gamma}^{c}_{ba} - \partial_b\hat{\Gamma}^{c}_{ca} + \hat{\Gamma}^{C}_{ba}\hat{\Gamma}^{D}_{DC} - \hat{\Gamma}^{C}_{Da}\hat{\Gamma}^{D}_{bC}$$
$$\hat{R}_{ab} = \partial_c\hat{\Gamma}^{c}_{ba} - \partial_b\hat{\Gamma}^{c}_{ca} + \hat{\Gamma}^{C}_{ba}\hat{\Gamma}^{d}_{dC} - \hat{\Gamma}^{C}_{da}\hat{\Gamma}^{d}_{bC} \tag{5.2.1}$$

Doing the same for the without torsion definitions (using equations 10.1.6 repeatedly, and the cylinder condition as required):

$$\hat{\mathcal{R}}_{ab} = \partial_C\hat{\digamma}^{C}_{ba} - \partial_b\hat{\digamma}^{C}_{Ca} + \hat{\digamma}^{C}_{ba}\hat{\digamma}^{D}_{DC} - \hat{\digamma}^{C}_{Da}\hat{\digamma}^{D}_{bC}$$
$$\hat{\mathcal{R}}_{ab} = \partial_c\hat{\digamma}^{c}_{ba} - \partial_b\hat{\digamma}^{c}_{ca} + \frac{1}{2}\partial_b(A^{d}F_{ad}) + \hat{\digamma}^{c}_{ba}\hat{\digamma}^{D}_{Dc} - \hat{\digamma}^{C}_{Da}\hat{\digamma}^{D}_{bC} \tag{5.2.2}$$

In the original Kaluza theory the Ricci curvature of the 5D space is set to 0. The first field equation (3.2.1) comes from looking at the Ricci curvature of the space-time that results. An advantage of this is the conservation law (4.3.7). We show that this identification of electromagnetism with the Kaluza vacuum is not possible if we wish to reproduce electromagnetism sufficiently, even with the presence of torsion. Setting $\hat{\mathcal{R}}_{ab} = 0$ (as would be required, that is, identifying electromagnetic fields with the Kaluza vacuum) allows:

$$\mathcal{R}_{ab} = \mathcal{R}_{ab} - \hat{\mathcal{R}}_{ab}$$

$$= \partial_c \Gamma^c_{\ ba} - \partial_b \Gamma^c_{\ ca} - \partial_c \hat{\Gamma}^c_{\ ba} + \partial_b \hat{\Gamma}^c_{\ ca} - \frac{1}{2}\partial_b (A^d F_{ad}) + \frac{1}{2}\partial_b (A^d F_{ad} + A_a F_c{}^c)$$

$$+\Gamma^c_{\ ba}\Gamma^d_{\ dc} - \Gamma^c_{\ da}\Gamma^d_{\ bc} - \hat{\Gamma}^c_{\ ba}\hat{\Gamma}^D_{\ Dc} + \hat{\Gamma}^C_{\ Da}\hat{\Gamma}^D_{\ bC}$$

$$= -\frac{1}{2}\partial_c (A_b F_a{}^c + A_a F_b{}^c) + \Gamma^c_{\ ba}\Gamma^d_{\ dc} - \Gamma^c_{\ da}\Gamma^d_{\ bc} - \hat{\Gamma}^c_{\ ba}\hat{\Gamma}^D_{\ Dc} + \hat{\Gamma}^C_{\ Da}\hat{\Gamma}^D_{\ bC}$$

$$= -\frac{1}{2}\partial_c (A_b F_a{}^c + A_a F_b{}^c) + \Gamma^c_{\ ba}\Gamma^d_{\ dc} - \Gamma^c_{\ da}\Gamma^d_{\ bc}$$

$$-\hat{\Gamma}^c_{\ ba}\hat{\Gamma}^D_{\ Dc} + \hat{\Gamma}^c_{\ Da}\hat{\Gamma}^D_{\ bc} + \hat{\Gamma}^4_{\ Da}\hat{\Gamma}^D_{\ b4}$$

$$= -\frac{1}{2}\partial_c (A_b F_a{}^c + A_a F_b{}^c) + \Gamma^c_{\ ba}\Gamma^d_{\ dc} - \Gamma^c_{\ da}\Gamma^d_{\ bc}$$

$$-(\Gamma^c_{\ ba}+\frac{1}{2}(A_b F_a{}^c+A_a F_b{}^c))(\Gamma^d_{\ dc}+\frac{1}{2}(A_d F_c{}^d+A_c F_d{}^d))-(\Gamma^c_{\ ba}+\frac{1}{2}(A_b F_a{}^c+A_a F_b{}^c))(-\frac{1}{2}A^d F_{cd})$$

$$+(\Gamma^c_{\ da}+\frac{1}{2}(A_d F_a{}^c+A_a F_d{}^c))(\Gamma^d_{\ bc}+\frac{1}{2}(A_b F_c{}^d+A_c F_b{}^d))+(\frac{1}{2}F_a{}^c)(-A_d\Gamma^d_{\ bc}+\frac{1}{2}(\partial_b A_c+\partial_c A_b))$$

$$+(-A_c\Gamma^c_{\ da} + \frac{1}{2}(\partial_d A_a + \partial_a A_d))(\frac{1}{2}F_b{}^d) + (-\frac{1}{2}A^d F_{ad})(-\frac{1}{2}A^c F_{bc})$$

$$= -\frac{1}{2}\partial_c (A_b F_a{}^c + A_a F_b{}^c)$$

$$-\frac{1}{2}(A_b F_a{}^c + A_a F_b{}^c)\Gamma^d_{\ dc}$$

$$+\frac{1}{2}\Gamma^c_{\ da}(A_b F_c{}^d+A_c F_b{}^d)+\frac{1}{2}(A_d F_a{}^c+A_a F_b{}^c)\Gamma^d_{\ bc}+\frac{1}{2}(A_d F_a{}^c+A_a F_d{}^c)\frac{1}{2}(A_b F_c{}^d+A_c F_b{}^d)$$

$$+\frac{1}{2}F_a{}^c(-A_d\Gamma^d_{\ bc} + \frac{1}{2}(\partial_b A_c + \partial_c A_b))$$

$$+(-A_c\Gamma^c_{\ da} + \frac{1}{2}(\partial_d A_a + \partial_a A_d))\frac{1}{2}F_b{}^d + \frac{1}{4}A^d F_{ad} A^c F_{bc}$$

$$= -\frac{1}{2}A_b \partial_c F_a{}^c - \frac{1}{2}A_a \partial_c F_b{}^c - \frac{1}{2}(\partial_c A_b) F_a{}^c - \frac{1}{2}(\partial_c A_a) F_b{}^c - \frac{1}{2}(A_b F_a{}^c + A_a F_b{}^c)\Gamma^d_{\ dc}$$

$$+\frac{1}{2}\Gamma^c_{\ da} A_b F_c{}^d + \frac{1}{2}A_a F_b{}^c \Gamma^d_{\ bc} + \frac{1}{4}(A_d F_a{}^c + A_a F_d{}^c)(A_b F_c{}^d + A_c F_b{}^d)$$

$$+\frac{1}{4}F_a{}^c(\partial_b A_c + \partial_c A_b) + \frac{1}{4}(\partial_d A_a + \partial_a A_d)F_b{}^d + \frac{1}{4}A^d F_{ad} A^c F_{bc}$$

$$= -\frac{1}{2}A_b \partial_c F_a{}^c - \frac{1}{2}A_a \partial_c F_b{}^c$$

$$-\frac{1}{2}(\partial_c A_b) F_a{}^c - \frac{1}{2}(\partial_c A_a) F_b{}^c + \frac{1}{4}F_a{}^c(\partial_b A_c + \partial_c A_b) + \frac{1}{4}(\partial_d A_a + \partial_a A_d)F_b{}^d$$

$$-\frac{1}{2}(A_b F_a{}^c + A_a F_b{}^c)\Gamma^d_{\ dc} + \frac{1}{2}\Gamma^c_{\ da} A_b F_c{}^d + \frac{1}{2}A_a F_b{}^c \Gamma^d_{\ bc}$$

$$
\begin{aligned}
&+\frac{1}{4}(A_d F_a{}^c + A_a F_d{}^c)(A_b F_c{}^d + A_c F_b{}^d) + \frac{1}{4}A^d F_{ad} A^c F_{bc} \\
&= -\frac{1}{2}A_b \partial_c F_a{}^c - \frac{1}{2}A_a \partial_c F_b{}^c + \frac{1}{2}F_{ac}F_b{}^c \\
&-\frac{1}{2}(A_b F_a{}^c + A_a F_b{}^c)\varGamma^d_{dc} + \frac{1}{2}\varGamma^c_{da} A_b F_c{}^d + \frac{1}{2}A_a F_b{}^c \varGamma^d_{bc} \\
&+\frac{1}{4}(A_d F_a{}^c + A_a F_d{}^c)(A_b F_c{}^d + A_c F_b{}^d) + \frac{1}{4}A^d F_{ad} A^c F_{bc} \qquad (5.2.3)
\end{aligned}
$$

The electrovacuum terms for a nullish electromagnetic field can be seen embedded in this equation as the third term, this shows that we are not producing a completely new theory from Kaluza's original theory. Kaluza-Cartan theory has a limit in common with Kaluza theory. Taking $O(h^3)$ L1-L3 weak field clarifies this. Only the first three terms of (5.2.3) survive, of which the first two are charge terms and the latter is the stress-energy of nullish solutions. However, if the charge terms are ignored then there is a lack in the above equation of likely significant terms to provide any other type of solution, non-nullish electromagnetic fields in particular. It is therefore too restrictive when the scalar field is constant just like Kaluza's original theory.

For this reason we can try an alternative formulation of electromagnetism in order to obtain a fuller range of geometries via the torsion bearing Ricci tensor instead: $\hat{R}_{AB} = 0$. That is, by identifying electromagnetism with the Kaluza-Cartan vacuum instead of the Kaluza vacuum. Using (5.2.1) gives:

$$
\begin{aligned}
\mathcal{R}_{ab} = \mathcal{R}_{ab} - \hat{R}_{ab} &= \partial_c \varGamma^c_{ba} - \partial_b \varGamma^c_{ca} + \varGamma^c_{ba}\varGamma^d_{dc} - \varGamma^c_{da}\varGamma^d_{bc} \\
&-\partial_c \hat{\Gamma}^c_{ba} + \partial_b \hat{\Gamma}^c_{ca} - \hat{\Gamma}^C_{ba}\hat{\Gamma}^d_{dC} + \hat{\Gamma}^C_{da}\hat{\Gamma}^d_{bC} \qquad (5.2.4)
\end{aligned}
$$

Detailing each term here without a specific point to make is not profitable, is lengthy, and shall not be undertaken. There are however clearly more degrees of freedom than before, and this is the main requirement. There is a limit in common for both formulations of electromagnetism when there is no appreciable torsion (noting that nullish solutions under the first formulation need no such torsion). More generally allowing torsion terms allows for non-nullish electromagnetic fields. Similarly other formulations of electromagnetism are likely to provide the required degrees of freedom. However in all cases it is necessary to show that prospective electromagnetic solutions also obey the Lorentz force law as the general relativistic Einstein-Maxwell equation will not be satisfied in general. This will be done later. Here the fairly simple principle has been shown via the first field equation that releasing the Kaluza constraint on vanishing (torsionless) Ricci curvature is an effective way to obtain the required missing solutions. It is further required that Maxwell's laws without sources be approximately satisfied. This will be studied via the second field equation.

5.3 The Second Field Equation With Torsion

Rederivation of the second field equation under the cylinder condition:

$$
\begin{aligned}
\hat{\mathcal{R}}_{a4} &= \partial_C \hat{\Gamma}^C_{\ 4a} - \partial_4 \hat{\Gamma}^C_{\ Ca} + \hat{\Gamma}^C_{\ 4a}\hat{\Gamma}^D_{\ DC} - \hat{\Gamma}^C_{\ Da}\hat{\Gamma}^D_{\ 4C} \\
&= \partial_c \hat{\Gamma}^c_{\ 4a} + \hat{\Gamma}^c_{\ 4a}\hat{\Gamma}^D_{\ Dc} - \hat{\Gamma}^c_{\ Da}\hat{\Gamma}^D_{\ 4c} = \partial_c \hat{\Gamma}^c_{\ 4a} + \hat{\Gamma}^c_{\ 4a}\hat{\Gamma}^d_{\ dc} - \hat{\Gamma}^c_{\ da}\hat{\Gamma}^d_{\ 4c} \\
&= \frac{1}{2}\partial_c F^c_a + \frac{1}{2}F^c_a \Gamma^d_{\ dc} + \frac{1}{4}F^c_a A^d F_{cd} - \frac{1}{2}(\Gamma^c_{\ da} + \frac{1}{2}(A_d F^c_a + A_a F^c_d))F^d_c
\end{aligned}
$$

Looking at this at an $O(h^2)$ L1-L3 weak field limit (re-inserting general k):

$$\hat{\mathcal{R}}_{a4} \to \frac{k}{2}\partial_c F^c_a \tag{5.3.1}$$

This couldn't be a clearer (albeit approximate at the $O(h^2)$ weak field limit) conception of Maxwell charge. This coincides with the Einstein (without torsion) tensor at the same limit, thus providing an alternative conception of the conservation of Maxwell charge locally (cf 6.1.1):

$$\hat{\mathcal{G}}_{a4} \to \hat{\mathcal{R}}_{a4} \to \frac{k}{2}\partial_c F^c_a \tag{5.3.2}$$

On the other hand, by definition (and the cylinder condition, and 10.1.7), we immediately get:

$$\hat{R}_{a4} = 0 \tag{5.3.3}$$

Whereas $\hat{R}_{4b}$ simplifies at the $O(h^2)$ weak field limit to:

$$\hat{R}_{4b} \to \frac{1}{2}\partial_c F^c_b - \partial_c \hat{K}_{b4}{}^c + \partial_b \hat{K}_{c4}{}^c \tag{5.3.4}$$

This is also approximately conserved Maxwell charge (re-inserting general k) given at the $O(h^2)$ L1-L3 weak field limit. Using L3 and equation (5.1.7):

$$\hat{R}_{4b} \to k\partial_c F^c_b \tag{5.3.5}$$

This means that the Kaluza-Cartan vacuum may not have stray charges in it of any significance, which is a required quality of a sourceless electromagnetic field. Any low significance charge source, further, necessarily implies antisymmetric components of the Kaluza-Cartan Ricci tensor: $\frac{1}{2}(\hat{R}_{4a} - \hat{R}_{a4})$, which at the completely antisymmetric (and also weak field) limit implies also no spin sources by (9.1.17). The Kaluza-Cartan vacuum can not contain significant spin sources.

5.4 The Third Field Equation With Torsion, $k = 1$

This section shows how torsion releases the constraint of the third torsionless field equation (3.2.3), thus allowing non-nullish solutions. The constraint that the Ricci tensor be zero leads to no non-nullish solutions in the original Kaluza theory. This is caused by setting $\hat{R}_{44} = 0$ in that theory and observing the terms. The result is that (when the scalar field is constant) $0 = F_{cd}F^{cd}$ in the original Kaluza theory. The same issue arises here:

We have:

$$\begin{aligned}\hat{\mathcal{R}}_{44} &= \partial_C \hat{\digamma}^C_{\ 44} - \partial_4 \hat{\digamma}^C_{\ C4} + \hat{\digamma}^C_{\ 44}\hat{\digamma}^D_{\ DC} - \hat{\digamma}^C_{\ D4}\hat{\digamma}^D_{\ 4C} \\ &= 0 - 0 + 0 - \hat{\digamma}^C_{\ D4}\hat{\digamma}^D_{\ 4C} = -\hat{\digamma}^c_{\ d4}\hat{\digamma}^d_{\ 4c} \\ &= -\frac{1}{4}F^c_d F^d_c \end{aligned} \tag{5.4.1}$$

The result is that whilst we can have non-nullish solutions, we can only have them outside of a Kaluza vacuum, for example in a Kaluza-Cartan vacuum.

By definition (and the cylinder condition, and 10.1.7), we immediately get:

$$\hat{R}_{44} = 0 \tag{5.4.2}$$

There is no reason in general for equation (5.4.1) to be 0, and so non-nullish solutions are generally available in the presence of torsion providing we are not constrained to the Kaluza vacuum as with Kaluza's original theory.

6 The Lorentz Force Law

Toth [8] derives a Lorentz-like force law where there is a static scalar field and Kaluza's cylinder condition applies in the original Kaluza theory. The resulting 'charge' is the momentum term in the fifth dimension and it was not apparent how this related to the Maxwell current. Here we make use of the Geodesic Assumption K4. First the identification of kinetic charge and Maxwell charge is investigated.

6.1 Kinetic Charge

Now to investigate the relationship between kinetic charge and Maxwell charge. For this we need the $O(h^2)$ weak field limit defined by L1 (cf equation 5.3.2) and discounting $O(h^2)$ terms:

$$\hat{\mathcal{G}}^{a4} = \hat{\mathcal{R}}^{a4} - \frac{1}{2}\hat{g}^{a4}\hat{\mathcal{R}} = \hat{\mathcal{R}}^{a4} - \frac{1}{2}(-A^a)\hat{\mathcal{R}} \to \hat{\mathcal{R}}^{a4}$$

$$\hat{\mathcal{R}}^{a4} = \partial_C \hat{\digamma}^{C4a} - \partial^4 \hat{\digamma}^{C\ a}_{\ C} + \hat{\digamma}^{Cba}\hat{\digamma}^D_{\ DC} - \hat{\digamma}^{C\ a}_{\ D}\hat{\digamma}^{Db}_{\ \ C}$$

$$\hat{\mathcal{G}}^{a4} \to \hat{\mathcal{R}}^{a4} = \partial_c \hat{\digamma}^{c4a} \tag{6.1.1}$$

Putting k back in, and by using Appendix equation (10.2.1) for the Christoffel symbol, we get:

$$\hat{\mathcal{R}}^{a4} \to \frac{1}{2}\partial_c k F^{ac} \tag{6.1.2}$$

And so by (4.3.7),

$$J_a^* \to -\frac{\alpha k}{2} \partial_c F_a{}^c \tag{6.1.3}$$

So kinetic and Maxwell charges are related by a simple formula. The right hand side being Maxwell's charge current (see p.81 of [6]), and has the correct sign to identify a positive kinetic charge Q^* with a positive Maxwell charge source $4\pi Q_M$, whenever $\alpha k > 0$. In the appropriate space-time frame, and Kaluza atlas frame, using (4.3.6), and approaching the $O(h^2)$ limit given by L1:

$$4\pi Q_M \to +\frac{2}{\alpha k} Q^* \tag{6.1.4}$$

This correlates the two definitions of charge at the required limit.

6.2 A Lorentz-Like Force Law

The Christoffel symbols and the geodesic equation are the symmetric ones defined in the presence of completely antisymmetric torsion. We will here initially use $k = 1$, a general k can be added in later.

$$\begin{aligned}
&\hat{\Gamma}^c_{(4b)} = \tfrac{1}{2} g^{cd}(\delta_4 \hat{g}_{bd} + \delta_b \hat{g}_{4d} - \delta_d \hat{g}_{4b}) + \tfrac{1}{2}\hat{g}^{c4}(\delta_4 \hat{g}_{b4} + \delta_b \hat{g}_{44} - \delta_4 \hat{g}_{4b}) = \\
&\tfrac{1}{2} g^{cd}[\delta_b(\phi^2 A_d) - \delta_d(\phi^2 A_b)] + \tfrac{1}{2} g^{cd}\delta_4 \hat{g}_{bd} + \tfrac{1}{2}\hat{g}^{c4}\delta_b \hat{g}_{44} = \\
&\tfrac{1}{2}\phi^2 g^{cd}[\delta_b A_d - \delta_d A_b] + \tfrac{1}{2} g^{cd} A_d \delta_b \phi^2 - \tfrac{1}{2} g^{cd} A_b \delta_d \phi^2 + \tfrac{1}{2} g^{cd}\delta_4 \hat{g}_{bd} + \tfrac{1}{2}\hat{g}^{c4}\delta_b \phi^2 = \\
&\tfrac{1}{2}\phi^2 F_b^c + \tfrac{1}{2} g^{cd} A_d \delta_b \phi^2 - \tfrac{1}{2} g^{cd} A_b \delta_d \phi^2 + \tfrac{1}{2} g^{cd}\delta_4 \hat{g}_{bd} + \tfrac{1}{2}\hat{g}^{c4}\delta_b \phi^2 = \\
&\tfrac{1}{2}\phi^2 F_b^c - \tfrac{1}{2} g^{cd} A_b \delta_d \phi^2 + \tfrac{1}{2} g^{cd}\delta_4 \hat{g}_{bd} = \tfrac{1}{2}\phi^2 F_b^c \quad - \tfrac{1}{2} g^{\mathrm{cd}} A_b \delta_d \phi^2
\end{aligned} \tag{6.2.1}$$

$$\hat{\Gamma}^c_{44} = \tfrac{1}{2}\hat{g}^{\mathrm{cD}}(\delta_4 \hat{g}_{4D} + \delta_4 \hat{g}_{4D} - \delta_D \hat{g}_{44}) = \text{-}\ \tfrac{1}{2} g^{\mathrm{cd}}\delta_d \phi^2 \tag{6.2.2}$$

$$\begin{aligned}
&\hat{\Gamma}^c_{(ab)} = \tfrac{1}{2} g^{cd}(\delta_a g_{db} + \delta_b g_{da} - \delta_d g_{ab}) \\
&+\tfrac{1}{2} g^{cd}(\delta_a(\phi^2 A_d A_b) + \delta_b(\phi^2 A_a A_d) - \delta_d(\phi^2 A_a A_b)) + \tfrac{1}{2}\hat{g}^{c4}(\delta_a \hat{g}_{4b} + \delta_b \hat{g}_{4a} - \delta_4 \hat{g}_{ab}) \\
&= \Gamma^c_{(ab)} \quad + \tfrac{1}{2} g^{cd}(\delta_a(\phi^2 A_d A_b) + \delta_b(\phi^2 A_a A_d) - \delta_d(\phi^2 A_a A_b)) \\
&\qquad -A^c(\delta_a \phi^2 A_b + \delta_b \phi^2 A_a)
\end{aligned} \tag{6.2.3}$$

So, for any coordinate system within the maximal atlas:

$$\begin{aligned}
0 &= \tfrac{d^2 x^a}{d\tau^2} + \hat{\Gamma}^a_{(BC)} \tfrac{dx^B}{d\tau} \tfrac{dx^C}{d\tau} \\
&= \tfrac{d^2 x^a}{d\tau^2} + \hat{\Gamma}^a_{(bc)} \tfrac{dx^b}{d\tau} \tfrac{dx^c}{d\tau} + \hat{\Gamma}^a_{(4c)} \tfrac{dx^4}{d\tau} \tfrac{dx^c}{d\tau} + \hat{\Gamma}^a_{(b4)} \tfrac{dx^b}{d\tau} \tfrac{dx^4}{d\tau} + \hat{\Gamma}^a_{44} \tfrac{dx^4}{d\tau} \tfrac{dx^4}{d\tau} \\
&= \tfrac{d^2 x^a}{d\tau^2} + \hat{\Gamma}^a_{(bc)} \tfrac{dx^b}{d\tau} \tfrac{dx^c}{d\tau} + (\phi^2 F_b^a - g^{ad} A_b \delta_d \phi^2) \tfrac{dx^b}{d\tau} \tfrac{dx^4}{d\tau} - \tfrac{1}{2} g^{\mathrm{ad}}\delta_d \phi^2 \tfrac{dx^4}{d\tau} \tfrac{dx^4}{d\tau}
\end{aligned} \tag{6.2.4}$$

Taking $\phi^2 = 1$ and the charge-to-mass ratio to be:

$$Q'/m_{k0} = \frac{dx^4}{d\tau} \tag{6.2.5}$$

We derive a Lorentz-like force law:

$$\frac{d^2x^a}{d\tau^2} + \hat{\Gamma}^a_{(bc)}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = -(Q'/m_{k0})F^a_b\frac{dx^b}{d\tau} \tag{6.2.6}$$

Putting arbitrary k and variable ϕ back in we have:

$$\frac{d^2x^a}{d\tau^2} + \hat{\Gamma}^a_{(bc)}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = -k(Q'/m_{k0})(\phi^2 F^a_b - g^{ad}A_b\delta_d\phi^2)\frac{dx^b}{d\tau} - \frac{1}{2}g^{\mathrm{ad}}\delta_d\phi^2\frac{dx^4}{d\tau}\frac{dx^4}{d\tau} \tag{6.2.7}$$

6.3 Constant Kinetic Charge

Having derived a Lorentz-like force law we look also at the momentum of the charge in the Kaluza dimension. We look at this acceleration as with the Lorentz force law. We have, with torsion (and $k = 1$):

$$0 = \frac{d^2x^4}{d\tau^2} + \hat{\Gamma}^4_{(BC)}\frac{dx^B}{d\tau}\frac{dx^C}{d\tau}$$

$$= \frac{d^2x^4}{d\tau^2} + \hat{\Gamma}^4_{(bc)}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} + \hat{\Gamma}^4_{(4c)}\frac{dx^4}{d\tau}\frac{dx^c}{d\tau} + \hat{\Gamma}^4_{(b4)}\frac{dx^b}{d\tau}\frac{dx^4}{d\tau} + \hat{\Gamma}^4_{44}\frac{dx^4}{d\tau}\frac{dx^4}{d\tau}$$

$$= \frac{d^2x^4}{d\tau^2} + \hat{\Gamma}^4_{(bc)}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} + 2\hat{\Gamma}^4_{(4c)}\frac{dx^4}{d\tau}\frac{dx^c}{d\tau} + \frac{1}{2}A^d\delta_d\phi^2\frac{dx^4}{d\tau}\frac{dx^4}{d\tau} \tag{6.3.1}$$

6.4 Unitary Scalar Field And Torsion

Both equations above (6.2.7) and (6.3.1) have a term that wrecks havoc to any similarity with the Lorentz force law proper, the terms at the end. Both terms can however be eliminated by setting the scalar field to 1. This is postulate B1.

The two equations under B1 become (for all k):

$$\frac{d^2x^a}{d\tau^2} + \hat{\Gamma}^a_{(bc)}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = -k(Q'/m_{k0})F^a_b\frac{dx^b}{d\tau} \tag{6.4.1}$$

$$\frac{d^2x^4}{d\tau^2} + \hat{\Gamma}^4_{(bc)}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = -k^2(Q'/m_{k0})A_cF^c_b\frac{dx^b}{d\tau} \tag{6.4.2}$$

This certainly looks more hopeful. The more extreme terms have disappeared, the general appearance is similar to the Lorentz force law proper. The right hand side of (6.4.2) is small, but in any case the well-behaved nature of charge follows from local momentum conservation (divergence of the torsionless Einstein tensor) and the consequential constraints on charge models.

6.5 The Lorentz Force Law

It is necessary to confirm that equation (6.4.1) not only looks like the Lorentz force law formally, but is indeed the Lorentz force law. Multiplying both sides of (6.4.1) by $\frac{d\tau}{d\tau'}\frac{d\tau}{d\tau'}$, where τ' is an alternative affine coordinate frame, gives:

$$\frac{d^2x^a}{d\tau'^2} + \hat{\Gamma}^a_{(bc)}\frac{dx^b}{d\tau'}\frac{dx^c}{d\tau'} = -k\frac{d\tau}{d\tau'}(Q'/m_{k0})F^a_b\frac{dx^b}{d\tau'} \tag{6.5.1}$$

Given $Q^* = Q'\frac{d\tau}{d\tau^*}$ and therefore $\frac{m_{k0}}{m_0}Q^* = Q'\frac{d\tau}{dt_0}$ by definition, we can set the frame such that $\tau' = t_0$ via the projected 4D space-time frame of the charge. And the Lorentz force is derived:

$$\frac{d^2x^a}{d\tau'^2} + \hat{\Gamma}^a_{(bc)}\frac{dx^b}{d\tau'}\frac{dx^c}{d\tau'} = -k(Q^*/m_0)F^a_b\frac{dx^b}{d\tau'} \tag{6.5.2}$$

In order to ensure the correct Lorentz force law using the conventions of Wald [7] p69, this can be rewritten as follows, using the antisymmetry of $F^a_b = -F^a{}_b$:

$$= k(Q^*/m_0)F^a{}_b\frac{dx^b}{d\tau'} \tag{6.5.3}$$

Using (6.1.4) as its L1 weak field limit is approached, this can be rewritten again in terms of the Maxwell charge:

$$\to k(\frac{\alpha k}{2}(4\pi Q_M)/m_0)F^a{}_b\frac{dx^b}{d\tau'} \tag{6.5.4}$$

The result is that we must relate $\mathbf{G}$ and k to obtain the Lorentz force law in acceptable terms:

$$\frac{d^2x^a}{d\tau'^2} + \hat{\Gamma}^a_{(bc)}\frac{dx^b}{d\tau'}\frac{dx^c}{d\tau'} \to (Q_M/m_0)F^a_b\frac{dx^b}{d\tau'}$$

$$k = 2\sqrt{\mathbf{G}} \tag{6.5.5}$$

This shows that the Lorentz force law proper can be derived given (6.1.4) and the required limit.

7 Analysis Of The Electromagnetic Field; Super-energy

In this theory the sourceless electromagnetic field has been identified with vanishing Ricci curvature, where the Ricci curvature, in contradistinction from the original Kaluza theory, is defined in terms of the torsion tensor. This is the Kaluza-Cartan vacuum as opposed to the Kaluza vacuum. The Kaluza-Cartan vacuum has tight restrictions on the presence of charge and spin sources: it approximately follows the sourceless Maxwells laws, in that respect quite similar to the original Kaluza theory, but has more variety of solutions and better admits electrostatic or near electrostatic fields.

In this variant theory all particles following auto-parallels obey the Lorentz force law. It is assumed that particle-like charge sources follow auto-parallels. The derivation of the Lorentz force law works for all auto-parallel-following paths whether charged or not however. The derivation is quite general. In the absence of non-completely anti-symmetric torsion this is essentially geodesic motion. In any case the two are herein required to be close via the weak field limit postulates.

The fundamental conservation/divergence-free law for mass-energy belongs to the (torsionless) Einstein tensor. Noting that what we really mean by conservation also requires energy conditions. The complement of the Kaluza-Cartan vacuum is therefore called Kaluza-Cartan matter (rather than mass-energy). Matter and fields are able to transfer Kaluza mass-energy (the complement of the Kaluza vacuum) to and from each other. Unlike (torsionless) mass-energy, the divergence law for Kaluza-Cartan matter depends on the torsion tensor as seen by combining (9.1.13) and (9.1.15). It is only vanishing at the completely antisymmetric limit. We thus have no *a priori* guarantee that a Kaluza-Cartan vacuum (when not also a Kaluza vacuum) might not just evolve so as to cease being a Kaluza-Cartan vacuum even when there are energy conditions imposed.

By imposing the conserved super-energy hypothesis, SE1, causality is imposed. The result is that on Kaluza vacuums well-behaved characteristics globally are ensured, but more importantly causality is also imposed on Kaluza-Cartan vacuums. This imposes the time evolution of Maxwell's laws at least locally and approximately, and any divergence from the Lorentz force law in extremis will at least be manifest deterministically.

Kinetic charge is also fundamental in its conservation under definition (4.3.7), though perhaps not in the way that charge is usually understood. Here it is a part of (torsionless) 5D mass-energy conservation. The correlation with Maxwell charge comes from the weak field limit L1. See identity (6.1.4).

With respect to spin, spin current obeys the fundamental divergence law (9.1.13). This is then also a fundamental quantity in Kaluza-Cartan theory, and complementary to (torsionless) mass-energy in that sense. It is divergence-free relative to the 5D torsion connection.

In all matters of divergence and conservation we must note the well-behaved postulates such as K2. Otherwise topology can be manipulated to create unphysical results.

Maxwell charge requires spin, at least at a local $O(h^2)$ L1-L3 weak field and completely antisymmetric limit. This follows from (9.1.17) and (5.3.5). By definition of kinetic charge, components of 5D (torsionless) mass-energy are also required. A matter model defined by Kaluza-Cartan matter can have charge, but stray charges in a Kaluza-Cartan vacuum region are limited in significance by the weak field assumptions. Further a minimum component of Kaluza-Cartan matter and (torsionless) conserved mass-energy is required to form a charge model, in addition to the (with torsion) divergence-free spin. The weak field assumptions therefore keep a certain amount of matter and spin assigned and bound to any charge model. And even if such diverge far from these limits, they must be reassigned and rebound upon return.

Since Maxwell charge is not necessarily conserved in this theory but merely identified with the divergence-free (with respect to the torsionless connection) kinetic charge at the weak field limit, it is the spin that becomes the fundamental quantity (9.1.13) that gives matter models their character. (Torsionless) mass-energy is also important but does not distinguish matter from fields, a distinction that is needed: what we mean by matter, intuitively, whilst at first glance Kaluza-Cartan matter, is perhaps more fundamentally characterized by its spin content.

Approximate conservation or divergence laws arise at the weak field anti-symmetric limit: (9.1.15), and that implied by applying this in turn to (9.1.17). The result is the appearance of Maxwell charge as a significant term in (9.1.17), via (5.3.5) and (5.3.3) - approximately divergence-free (relative to the torsion connection). Components of the spin current/charge also get identified at this limit with the Maxwell current/charge.

Matter models (here omitting the Kaluza-Cartan prefix) are more broadly any region where a significant (in the sense that it can not be discounted by L1-L3) amount of charge, spin or matter is present. The presence of charge ensures the presence of spin, both ensuring the presence of matter at this limit. The problem of spins cancelling out is a non-issue as there is an additional dimension whose spin components must be more obviously cumulative with increased charge, at least until the L3 limit ceases to be valid. Consistent with observation, matter does not necessarily imply the presence of spin or charge, but in itself cannot be distinguished as fully separate over time from adjacent fields. The quantity is not in itself necessarily conserved or divergence-free, and mass-energy is free to pass from Kaluza-Cartan vacuum to Kaluza-Cartan matter and vice versa. Spin is therefore more fundamental in matter-charge-spin particle models and in characterizing them as distinct from any surrounding fields. This can be interpreted as shedding light on the nature of matter and an explanation as to why local matter is such a hard thing to define in general relativity.

8 Conclusion

Kaluza's 1921 theory of gravity and electromagnetism using a fifth wrapped-up spatial dimension is inspiration for many modern attempts to develop new physical theories. However for a number of reasons it is generally considered untenable.

A cylinder condition based on torsion was imposed as with Kaluza's original theory. A number of other constraints and definitions were provided. The result was the appearance of missing and needed electromagnetic fields (in particular essential electrostatic fields) and a new definition of charge in terms of momentum in the fifth dimension. The entire theory was in effect derived from the need for these missing solutions to be present: they include the electrostatic fields. Conservation of charge was generalized into the vanishing divergence of the torsionless Einstein tensor. The new definition of kinetic charge and the

Maxwellian charge coincide at an appropriate limit. In order to obtain the missing electromagnetic fields it was necessary to generalize Kaluza's original theory to the vanishing of the Ricci tensor defined in terms of torsion.

Classical electrodynamics is rederived in the spirit of Kaluza's original theory but more fully. Gravity and electromagnetism are unified in a way not fully achieved by general relativity, Einstein-Cartan theory or Kaluza's original theory. The collocation of torsion with electromagnetism is different from many other Einstein-Cartan theories where the torsion is often bound to matter models. Here certain specific components of torsion are an essential part of electromagnetism. The scalar field (which is present in the original Kaluza theory), on the other hand, was fixed constant. This was to ensure the Lorentz force law.

Some interesting results such as that spin and charge can exist only in the presence of matter (to any significance) follow. An issue is raised as to when the limit postulates that characerterize the classical general relativistic limit break-down and how that would relate to real phenomena and experiment.

One outstanding issue is that realistic charge models are not possible without involving imaginary numbers (imaginary proper velocities in the Kaluza dimension and imaginary Kaluza rest mass). Obviously to deal properly with this the Kaluza-Cartan space or the Kaluza-Cartan theory would have to be adapted further in some way. But on the other hand limiting the theory solely to the 4D resultant space-time region or manifold, and applying a realistic charge model by hand need cause no such problems provided the net result makes sense, provided all the 4D numbers are real. Barring this failure to provide realistic charge models, which poses challenges to the 5D theory, the postulates currently required are straight forward. It is in a certain sense a simple theory. In effect all we have is a 5D manifold with a cylinder condition on one spatial dimension and torsion with an approximately completely antisymmetric metric torsion tensor limit, with certain well-defined weak fields and limits. Interpretation of many of the postulates can be made in physically appealing terms. However, many of the consequences are really very complicated.

Super-energy was here introduced to resolve problems of causality, time evolution and stability. This replaces the need to worry too much about energy conditions as in general relativity, but is not unique in its contribution to the theory: alternatives could perhaps be applied.

Why go to all this effort to unify electromagnetism and gravitation and to make electromagnetism fully geometric? Because experimental differences could be detectable given sufficient technology on the one hand, and, on the other, simply because such an attempt at unification might be right or lead in the right direction. Such an attempt may also widen the search. This theory differs from both general relativity and Einstein-Cartan theory and may be empirically testable. Also the expected ω-consistency of Einstein-Cartan theory together with the derivation of a Lorentz force law via the Kaluza part of the theory gives a theoretical motivation, as does the fact that the other approaches beyond general relativity have not fulfilled their promise in terms of approaching unification. Attempting to extend and unify classical theory prior to a unification with quantum mechanics may even be a necessary step in a future unification whether Kaluza-Cartan theory turns out to be the right way or not. It may be that current attempts are more difficult than necessary as the problem may not yet have been framed correctly.

It is often asserted that the true explanation for gravitational theory and space-time curvatures will most likely, by reductionist logic, emerge out of its constituent quantum phenomena. Such an approach has merit, but is overly optimistic, and does not optimize the search [23]. Before constituent quantum parts can be properly defined and subdivided the larger scale whole must have been present initially to then be so divided. Something of the context is evidently missing from quantum mechanics, general relativity or both on account of the difficulty of squaring the two. The dividing and putting together of parts assumes a context, and a context assumes a whole [22]. Implicitly reductionism assumes contextual knowledge. There is paradoxically an implicit non-reductionist assumption within reductionism. Generally we may take our conception of such a whole for granted, but we should bear in mind that this is a limited approach, speaking more of our limitations and need for easy concepts than of reality. Taking a global, more 'synthetic' perspective can be more difficult but may also be more insightful. A more holistic (in the sense of non-reduction or post-reductionist, but nevertheless empirical) approach may be required at both the large and small scale. Such considerations are further justification for the approach attempted here to unify gravity with electromagnetism.

9 Appendix I: Appendix To The Overview

9.1 Introducing The Geometry Of Torsion

5D Cartan torsion is here admitted. This provides extra and required degrees of freedom. It is noted that Einstein-Cartan theory, that adds torsion to the dynamics of relativity theory is most probably a minimal ω-consistent extension of general relativity [13][14] and therefore the use of torsion is not only natural, but arguably a necessity on philosophical and physical grounds. That argument can also be applied here. What we have defined by this addition can be called Kaluza-Cartan theory as it takes Kaluza's theory and adds torsion. We assume that the torsion connection is metric.

For both 5D and 4D manifolds (i.e. dropping the hats and indices notation for a moment), torsion will be introduced into the connection coefficients as follows, using the notation of Hehl [11]. Metricity of the torsion tensor will be assumed [19], the reasonableness of which (in the context of general relativity with torsion) is argued for in [20] and [21]:

$$\frac{1}{2}(\Gamma^k_{ij} - \Gamma^k_{ji}) = S_{ij}{}^k \tag{9.1.1}$$

This relates to the notation of Kobayashi and Nomizu [12] and Wald [7] as follows:

$$T^i{}_{jk} = 2S_{jk}{}^i \equiv \Gamma^i_{jk} - \Gamma^i_{kj} \tag{9.1.2}$$

We have the contorsion tensor $K_{ij}{}^k$ [11] as follows, and a number of relations [11]:

$$\Gamma^k_{ij} = \frac{1}{2} g^{kd}(\partial_i g_{dj} + \partial_j g_{di} - \partial_d g_{ij}) - K_{ij}{}^k = \mathit{\Gamma}^k_{ij} - K_{ij}{}^k \tag{9.1.3}$$

$$K_{ij}{}^k = -S_{ij}{}^k + S_j{}^k{}_i - S^k{}_{ij} = -K_i{}^k{}_j \tag{9.1.4}$$

Notice how the contorsion is antisymmetric in the last two indices.

With torsion included, the auto-parallel equation becomes [11]:

$$\frac{d^2x^k}{ds^2} + \Gamma^k_{(ij)} \frac{dx^i}{ds}\frac{dx^j}{ds} = 0 \tag{9.1.5}$$

$$\Gamma^k_{(ij)} = \mathit{\Gamma}^k_{ij} + S^k{}_{(ij)} - S_{(j}{}^k{}_{i)} = \mathit{\Gamma}^k_{ij} + 2S^k{}_{(ij)} \tag{9.1.6}$$

Only when torsion is completely antisymmetric is this the same as the extremals [11] which give the path of spinless particles and photons in Einstein-Cartan theory: extremals are none other than geodesics with respect to the Levi-Civita connection.

$$\frac{d^2x^k}{ds^2} + \mathit{\Gamma}^k_{ij} \frac{dx^i}{ds}\frac{dx^j}{ds} = 0 \tag{9.1.7}$$

With complete antisymmetry we have many simplifications such as:

$$K_{ij}{}^k = -S_{ij}{}^k \tag{9.1.8}$$

Stress-Energy And Conservation Laws

Inspired by the Belinfante-Rosenfeld procedure [12][15], by defining the torsionless Einstein tensor in terms of torsion bearing components, yields what can be interpreted as extra spin-spin coupling term $\hat{X}_{AB}$:

$$\hat{\mathcal{G}}_{AB} = \hat{G}_{AB} + \hat{V}_{AB} + \hat{X}_{AB} \tag{9.1.9}$$

$$\hat{V}_{AB} = -\frac{1}{2}\hat{\nabla}^C(\hat{\sigma}_{ABC} + \hat{\sigma}_{BAC} + \hat{\sigma}_{CBA}) \tag{9.1.10}$$

Where σ is defined as the spin tensor in Einstein-Cartan theory. However, here we do not start with spin (and some particle Lagrangians), but with the torsion tensor. So instead the spin tensor is defined in terms of the torsion tensor using the Einstein-Cartan equations. Here spin is explicitly defined in terms of torsion:

$$\hat{\sigma}_{ABC} = 2\hat{S}_{ABC} + 2\hat{g}_{AC}\hat{S}_{BD}^{\ \ D} - 2\hat{g}_{BC}\hat{S}_{AD}^{\ \ D} \tag{9.1.11}$$

This simplifies definition (9.1.10):

$$\hat{V}_{AB} = -\frac{1}{2}\hat{\nabla}^C(\hat{\sigma}_{CBA}) = -\hat{\nabla}^C(\hat{S}_{CBA} + \hat{g}_{CA}\hat{S}_{BD}^{\ \ D} - \hat{g}_{BA}\hat{S}_{CD}^{\ \ D}) \tag{9.1.12}$$

By considering symmetries and antisymmetries we get a divergence law:

$$\hat{\nabla}_B\hat{V}^{AB} = 0 \tag{9.1.13}$$

The Case Of Complete Antisymmetry

Note that the mass-energy-charge divergence law for the torsionless Einstein tensor is in terms of the torsionless connection, but the spin source divergence law here is in terms of the torsion-bearing connection. However, for completely antisymmetric torsion we have:

$$\hat{\nabla}_C\hat{\mathcal{G}}_{AB} = \hat{\triangle}_C\hat{\mathcal{G}}_{AB} + \hat{K}_{CA}^{\ \ \ D}\hat{\mathcal{G}}_{DB} + \hat{K}_{CB}^{\ \ \ D}\hat{\mathcal{G}}_{AD}$$

So,

$$\begin{aligned}\hat{\nabla}^A\hat{\mathcal{G}}_{AB} &= 0 + 0 + \hat{K}^{A\ \ D}_{\ \ B}\hat{\mathcal{G}}_{AD} = -\hat{K}_{B}^{\ \ AD}\hat{\mathcal{G}}_{AD} \\ &= -\hat{K}_{B}^{\ \ AD}\hat{\mathcal{G}}_{DA} = +\hat{K}_{B}^{\ \ DA}\hat{\mathcal{G}}_{DA} = +\hat{K}_{B}^{\ \ AD}\hat{\mathcal{G}}_{AD} = 0\end{aligned} \tag{9.1.14}$$

$$\hat{\nabla}^A(\hat{G}_{AB} + \hat{X}_{AB}) = 0 \tag{9.1.15}$$

And so there is a stress-energy divergence law with respect to the torsion connection also, at least in the completely antisymmetric case.

Further, still assuming complete antisymmetry of torsion, by definition of the Ricci tensor:

$$\begin{aligned}\hat{R}_{AB} &= \hat{\mathcal{R}}_{AB} + \hat{K}_{DA}^{\ \ \ C}\hat{K}_{BC}^{\ \ \ D} - \partial_C\hat{K}_{BA}^{\ \ \ C} - \hat{K}_{BA}^{\ \ \ C}\hat{\Gamma}^{D}_{\ DC} + \hat{K}_{DA}^{\ \ \ C}\hat{\Gamma}^{D}_{\ DC} - \hat{K}_{DB}^{\ \ \ C}\hat{\Gamma}^{D}_{\ AC} \\ &= \hat{\mathcal{R}}_{AB} - \hat{K}_{AD}^{\ \ \ C}\hat{K}_{BC}^{\ \ \ D} - \hat{\nabla}^C\hat{S}_{ABC}\end{aligned} \tag{9.1.16}$$

$$\hat{G}_{[AB]} = \hat{R}_{[AB]} = -\hat{\nabla}^C\hat{S}_{ABC} = -\hat{V}_{AB} \tag{9.1.17}$$

$-\hat{V}_{AB}$ is the antisymmetric part of $\hat{G}_{AB}$ at this limit. And $\hat{X}_{AB}$ is a symmetric spin-torsion coupling adjustment - again only in the case of completely antisymmetric torsion.

These divergence laws can function as 5D Kaluza-Cartan 'conservation' laws, given postulate K2, in the presence of positivity conditions.

Torsion-Normal Coordinates

By using the same argument, verbatim, as in Wald [7] p. 41-42 normal coordinates can be defined about any point also in the presence of torsion using the auto-parallel equation instead of the geodesic equation. Completely anti-symmetric torsion yields the same normal coordinates as without torsion, the paths varying only due to non-completely anti-symmetric terms.

Further, postulate L3 means that local normal and local torsion-normal co-ordinates will be comparable. This gives an interpretation of L3 that is more intuitive in terms of a limit of low significance of the difference between torsionless geodesics and auto-parallels.

9.2 Geodesic Motion, An Assumption

This theory assumes some sort of particle model of matter and charge is possible, that it can be added to the original theory without significantly changing the ambient space-time solution and thus its own path, which is approximate here as it is also in general relativity. Here however there are more complications such as the lack of an explicit matter-charge model, and the presence of torsion. We might imagine that what has been described is a particle whizzing around the fifth dimension like a roller coaster on its spiralled tracks. The cylinder conditions can also be maintained if, instead of a 5D particle, the matter and charge sources were rather a 'solid' ring, locked into place around the 5th dimension, rotating at some predetermined proper Kaluza velocity. Of course this is misleading as, as has already been shown, both velocity and rest mass would have to be complex numbers. An exact solution could perhaps involve changes in the size of the 5th dimension. None of that is investigated here, the objective is to see whether non-nullish solutions can be found in a variant Kaluza theory.

In Einstein-Cartan theory geodesics, or extremals, are followed by spinless particles in 4D Einstein-Cartan theory [11]. Other particles follow different paths when interaction with torsion is present. Auto-parallels and extremals are two analogs of geodesics used when torsion is present, but neither of which in the most general case determine the paths followed by all particles in Einstein-Cartan theory. Note that spinless particles according to [11] follow extremals. Extremals coincide with auto-parallels when torsion is completely antisymmetric. Particles with spin may interact in other ways. So the assumption is that torsion-spin coupling does not significantly effect the path of the particle, at least to some approximation. Here we choose to use auto-parallels. Exactly how sensitive this assumption is would require further research. Here however it is packaged into a single clean assumption.

POSTULATE (K4): **Geodesic Assumption**. That any particle-like model, that is to be identified with a charge, approximately follows 5D auto-parallels.

9.3 The Conserved Super-Energy Hypothesis

The Generalized Bel tensor for a Lorentz manifold (or simply Bel tensor) is the super-energy tensor associated with the (torsionless) Riemannian curvature [17]. The discussion here requires metricity from the torsion connection.

The definition of super-energy tensor does not require that torsion be vanishing in either the connection or any of the defining tensors [17], and the important dominant super-energy property [17] follows in all cases. However here the 5D torsionless definition will be primarily used. This leads to the causality of the Rieman tensor [16] under specific conditions without deference to energy conditions. The super-energy tensor definition depends on the antisymmetries of the Riemannian tensor definition, that is [17] that it is a double symmetric (2,2)-form. The definition of the super-energy tensor with respect to basic properties such as it being a 4-tensor are dependent on the admissibility of the interpretation of the Riemann tensor as a (2,2)-form.

Now the Riemann tensor can be written as [12] (not using hat, index or cursive notation, indicating the most general case, but using indices that show compatibility of conventions with eqn 2.0.3):

$$R^{i}_{akb} = \partial_k \Gamma^{i}_{ba} - \partial_b \Gamma^{i}_{ka} + \Gamma^{c}_{ba}\Gamma^{i}_{kc} - \Gamma^{c}_{ka}\Gamma^{i}_{bc} \qquad (9.3.1)$$

It is a (2,2)-form if its antisymmetries are as follows: $R_{[ia][kb]}$. This is clearly the case for [k,b]. For [i,a] it is a known result provided that the torsion-bearing connection is metric. The argument requires the torsion analog of Wald's equation (3.2.12) [7] and then follows for the same reasons as given there for the torsionless case. Thus generalized Bel super-energy is a (2,2)-form whether defined in terms of the torsion or not.

In [16] the derivation of the causality of the fields underlying any particular super-energy tensor is given in terms of the divergence of the field's super-energy tensor. A divergence condition is given that ensures causality of the underlying field associated with any such super-energy tensor. The divergence of the generalized Bel tensor would therefore need to be bounded by this condition if the Riemannian curvature were to remain causal. This condition is theorem 4.2 in [17].

A sufficient case would be if the divergence of the superenergy tensor were zero (and assuming global hyperbolicity -i.e. postulate K2). The important details are on page 4 of [16]. The argument does not require that the definition be torsion free. Thus the vanishing divergence of the generalized Bel tensor would yield causal Riemannian curvature assuming the Riemann tensor remained a (2,2)-form (as indicated above), with no deferment to energy conditions in both the case when torsion is used to define the Bel tensor and when it isn't.

On p24 of [17] we have a calculation of this divergence under vanishing torsion, and it can be seen that when the Ricci curvature is zero that the divergence of super-energy is also zero. This however references symmetry properties (in addition to antisymmetry properties) and thus further consideration of the case with torsion would be required to extend or generalize this theorem. Theorem 6.1 on p25 of [17] may well not apply in the case that the tensors and connection are defined in terms of torsion. Nevertheless it nicely characterizes an important property of the Kaluza vacuum, that it can not be a source of Bel super-energy.

The Conserved Super-Energy Hypothesis: is that the divergence of the Generalized Bel superenergy tensor be vanishing (when defined with respect to the torsionless connection and torsion free tensors) over a Kaluza-Cartan space that *does* have torsion.

This then ensures the causality sought for Kaluza-Cartan vacuums, as well as over any Kaluza-Cartan matter.

It can be noted that in 4D and 5D in particular, see p29 of [17], the torsionless generalized Bel tensor has the nice property of being completely symmetric. It is curious that it should be completely symmetric precisely in the 4D and 5D cases.

Theorem 6.1 of [17] (not proven for its torsion analog) links divergence of torsionless generalized Bel super-energy with what Senovilla et al [17] call the matter current.

The divergence of the Bel super-energy is given as follows [17] p.25 (still not using the hat, cursive or index notation but understanding that torsion is omitted from the definitions):

$$\nabla_a B^{ablm} = R^{b\ l}_{\ r\ s} J^{msr} + R^{b\ m}_{\ r\ s} J^{lsr} - \frac{1}{2} g^{lm} R^{b}_{\ rsy} J^{syr} \tag{9.3.2}$$

$$J_{lmb} = -J_{mlb} \equiv \nabla_l R_{mb} - \nabla_m R_{lb} \tag{9.3.3}$$

$J_{lmb} = 0$ then implies conservation of (torsionless) generalized Bel super-energy. We can put this in the 5D hat notation used elsewhere in this paper as follows:

$$\hat{\triangle}_A \hat{\mathcal{B}}^{ABLM} = \hat{\mathcal{R}}^{B\ L}_{\ R\ S} \hat{\mathcal{J}}^{MSR} + \hat{\mathcal{R}}^{B\ M}_{\ R\ S} \hat{\mathcal{J}}^{LSR} - \frac{1}{2} \hat{g}^{LM} \hat{\mathcal{R}}^{B}_{\ RSY} \hat{\mathcal{J}}^{SYR} \tag{9.3.4}$$

$$\hat{\mathcal{J}}_{LMB} = -\hat{\mathcal{J}}_{MLB} \equiv \hat{\triangle}_L \hat{\mathcal{R}}_{MB} - \hat{\triangle}_M \hat{\mathcal{R}}_{LB} \tag{9.3.5}$$

By the conserved super-energy hypothesis the following would have to be satisfied even by matter and charge models to ensure causality:

$$\hat{\triangle}_A \hat{\mathcal{B}}^{ABLM} = 0 \tag{9.3.6}$$

Which holds for example when [17]:

$$\hat{\mathcal{J}}_{LMB} = 0 \tag{9.3.7}$$

9.4 Consistency With Special Relativity

Kinetic charge is identified with 5D momentum in a space-time rest frame. This is already known in the original Kaluza theory to obey a Lorentz-like force law, but will be extended to the current setting.

That this is consistent with special relativity can be investigated: the relativistic mass created by momentum in the 5th dimension is kinematically identical to the relativistic rest mass.

The additions of velocities in special relativity is not obvious. Assume a flat 5D Kaluza space (i.e without geometric curvature or torsion, thus analogously to special relativity at a flat space-time limit, a 5D Minkowski limit). Space-time can be viewed as a 4D slice (or series of parallel slices) perpendicular to the 5th Kaluza dimension that minimizes the length of any loops that are perpendicular to it. Taking a particle and an inertial frame, the relativistic rest frame where the particle is stationary with respect to space-time but moving with velocity u in the 5th dimension, and a second frame where the charge is now moving in space-time at velocity v, but still with velocity u in the 5th dimension, then the total speed squared of the particle in the second frame is according to relativistic addition of orthogonal velocities:

$$s^2 = u^2 + v^2 - u^2v^2 \tag{9.4.1}$$

The particle moving in the Kaluza dimension with velocity u, but stationary with respect to 4D space-time, will have a special relativistic 4D rest mass (m_0) normally greater than its 5D Kaluza rest mass (m_{k0}). We can see that the Kaluza rest mass definition (m_{k0}) is consistent with the orthogonal addition of velocities as follows:

$$m_0 = \frac{m_{k0}}{\sqrt{(1-u^2)}} \text{ where } u = \tanh[\sinh^{-1}(Q^*/(m_{k0}))] \tag{9.4.2}$$

$$m_{rel} = \frac{m_0}{\sqrt{(1-v^2)}} = \frac{m_{k0}}{\sqrt{(1-u^2)}} \times \frac{1}{\sqrt{(1-v^2)}} = \frac{m_{k0}}{\sqrt{(1-u^2-v^2+u^2v^2)}} \tag{9.4.3}$$

By putting $u = \tanh[\sinh^{-1}(Q^*/(m_{k0}))]$ (keeping the hyperbolics to recall the conversion between unidirectional proper and coordinate velocities) into the definition of relativistic rest mass in terms of Kaluza rest mass and solving, we get that charge, whether positive or negative, is related to the relativistic rest mass according to the following formula:

$$\begin{aligned}\cosh[\sinh^{-1}(Q^*/(m_{k0}))] = m_0/m_{k0} &= \frac{dt_0}{d\tau^*} \\ &= \sqrt{(Q^*/(m_{k0}))^2+1}\end{aligned} \tag{9.4.4}$$

Using $k = 2$ we also have, for a typical unit charge:

$$m_e = 9.1094 \times 10^{-28} g \tag{9.4.5}$$

$$Q^* = 4.8032 \times 10^{-10} statcoulomb = 4.8032 \times 3.87 \times 10^{-10+3} g = 1.859 \times 10^{-6} g \tag{9.4.6}$$

If we take these figures and equate $m_e = m_0$ then we end up with imaginary m_{k0} and imaginary proper Kaluza velocity. Obviously to detail this the Kaluza-Cartan space would have to be adapted further in some way. However, what is important physically is that the figures we know to be physical in 4D remain so. And this is so.

Further issues are pertinent.

Observed electrons have static charge, angular momentum, a magnetic moment, and a flavor. The only thing distinguishing the electron from the muon is the flavor. The mass difference between the muon and the electron is about 105 MeV, perhaps solely due to this difference in flavor. The issue of modeling particles within a classical theory is, not surprisingly, a difficult one! Thus at this stage the idealized hypothetical charges used here, and real particles, can only be tentatively correlated.

It is possible to proceed without concern for the foundational issues of such charge models or attempting to interpret this quandary, instead simply developing the mathematics 'as is' and seeing where it leads without prejudging it.

Proper Kaluza Velocity As A Scalar

This section shows that the proper velocity $\mathbf{W}$ (written as a vector) with only one component in the Kaluza dimension is invariant under 4D space-time boosts orthogonal to it. The proper Kaluza velocity therefore is a constant with respect to *local* coordinate changes within a Kaluza atlas. It could be claimed that this result should follow in any case from the definition of proper velocity if the local coordinate transformation is only in the 4 dimensions of space-time, however this is not true for rotations - a rigorous proof is always better. The result here simply says that with respect to the Kaluza atlas the value is a scalar.

$W_4 = dx_4/d\tau$ proper velocity in a stationary space-time frame, but following the particle

$$U_4 = \frac{W_4}{\sqrt{1 + {W_4}^2}} \text{ coordinate velocity using proper velocity formula}$$

Using orthogonal addition of coordinate velocities formula to boost space-time frame by orthogonal coordinate velocity $\mathbf{V}$:

$$\mathbf{V} = (V, 0, 0, 0)$$
$$\mathbf{U} = (0, 0, 0, U_4)$$

Coordinate velocity vector in new frame, using the orthogonal velocity addition formula:

$$\overline{\mathbf{U}} = \mathbf{V} + \sqrt{1 - V^2}\ \mathbf{U}$$

So,

$$\overline{U}_4 = \sqrt{1 - V^2}\frac{W_4}{\sqrt{1 + W_4^2}}$$

Define proper velocity in new frame: $\overline{\mathbf{W}}$, using proper velocity definition:

$$\begin{aligned}
\overline{W}_4 &= \frac{\overline{U}_4}{\sqrt{1 - V^2 - \overline{U}_4{}^2}} \\
&= \frac{\sqrt{1 - V^2}\frac{W_4}{\sqrt{1+W_4^2}}}{\sqrt{1 - V^2 - (\sqrt{1 - V^2}\frac{W_4}{\sqrt{1+W_4^2}})^2}} \\
&= \frac{W_4}{\sqrt{1 + W_4^2}\sqrt{1 - (\frac{W_4}{\sqrt{1+W_4^2}})^2}} \\
&= \frac{W_4}{\sqrt{1 + W_4^2 - W_4^2}} = W_4
\end{aligned} \tag{9.4.7}$$

$\overline{W_4} = W_4$ is the result required

9.5 Geometrized Charge

The geometrized units, Wald [7], give a relation for mass in terms of fundamental units. This leads to an expression for kinetic charge in terms of Kaluza momentum when $k = 2$ and $\mathbf{G} = 1$.

$\mathbf{G} = 1 = 6.674 \times 10^{-8} cm^3 g^{-1} s^{-2} = 6.674 \times 10^{-8} cm^3 g^{-1} \times (3 \times 10^{10} cm)^{-2}$
$= 6.674 \times 10^{-8} cm g^{-1} \times (3 \times 10^{10})^{-2}$

$$1g \approx 7.42 \times 10^{-29} cm \text{ for c=1, } \mathbf{G}\text{=1} \tag{9.5.1}$$

$$1g \approx G/c^2 cm \text{ for c=1, } \mathbf{G}\text{=1} \tag{9.5.2}$$

For $k = 2$, c=1, $\mathbf{G}$=1 we have:

$$1 statcoulomb = 1 cm^{3/2} s^{-1} g^{1/2} = cm^{1/2} \times (7.42 \times 10^{-29} cm)^{1/2} / (3.00 \times 10^{10})$$

$$= 8.61 \times 10^{-15} cm/(3 \times 10^{10}) \approx 2.87 \times 10^{-25} cm \approx 3.87 \times 10^3 g \tag{9.5.3}$$

Using cgs (Gaussian) units and the cgs versions of $\mathbf{G}$ and c, ie $\mathbf{G} = 6.67 \times 10^{-7} cm^3 g^{-1} s^{-1}$ and $c = 3 \times 10^{10} cms^{-1}$, the charge can be written in terms of 5D proper momentum P_4 as follows:

$$1 statcoulomb = 1cm^{3/2}s^{-1}g^{1/2} = 1(cm/s)cm^{1/2}g^{1/2} = \frac{c}{\sqrt{\mathbf{G}}} g.cm/s$$

$$Q^* = \frac{c}{\sqrt{\mathbf{G}}} P_4 \tag{9.5.4}$$

9.6 A Complete Set of Postulates

By way of summary a complete set of postulates can be given for this version of Kaluza-Cartan theory:

{K1, K2, K3, K4, B1, L1, L2, L3, SE1}

Of these only {K1, K3} need be fundamental to the theory. {K2, K4, B1} relate to well-behaved assumptions. {L1, L2, L3} to a weak field limit. And {SE1} takes the place of the energy conditions of general relativity and may not be unique or universal, pending experimental testing or deeper analysis.

It would be nice however to give a clearer physical interpretation of some of these postulates. K2, K4, B1 need no special justification. With the exception of B1 there are classical analogs. The sensitivity of the theory to K4 is bounded by L3 in that when the non-completely antisymmetric terms are zero there is no difference between extremals and auto-parallels. B1 is shown to be a consequence of non-zero potential field components in the Appendix, and so a near consequence of a variable electromagnetic field, making it a surprisingly reasonable physical postulate. K1 and K3 are foundational to the theory, core postulates with no immediate need for interpretation. SE1, if true, stands out as requiring specific empirical testing, and other similar energy conditions may turn out to be better either theoretically or empirically.

Heuristically L1 gives a scale and a proportionality to tensors, in the sense that there is a balance of contributions from different types of tensors as a result. L2 continues this sense of proportionality so that, all other things being equal, such tensors as for example the Einstein tensors (with and without torsion) can be expected to be comparable in order of magnitude, in significance, to each other. This has an impression of physical reasonableness about it.

This leaves only L3 as in any way enigmatic. A possible interpretation is suggested in the Appendix under Torsion-Normal Coordinates, but exactly why it should be so important in this formulation is not clear, but it has many consequences here.

10 Appendix II: Appendix To The Field Equations And Further Working

10.1 The Christoffel Symbols And Connection Coefficients

Here we assume only the definitions of the Christoffel symbols and the cylinder condition. (Without torsion terms shown, k set to 1)

$$2\hat{\Gamma}^A_{BC} = \sum_d \hat{g}^{AD}(\partial_B \hat{g}_{CD} + \partial_C \hat{g}_{DB} - \partial_D \hat{g}_{BC})$$

$$\begin{aligned} &= \sum_d \hat{g}^{Ad}(\partial_B \hat{g}_{Cd} + \partial_C \hat{g}_{dB} - \partial_d \hat{g}_{BC}) \\ &\quad + \hat{g}^{A4}(\partial_B \hat{g}_{C4} + \partial_C \hat{g}_{4B} - \partial_4 \hat{g}_{BC}) \end{aligned}$$

$$\begin{aligned} 2\hat{\Gamma}^A_{bc} &= \sum_d \hat{g}^{Ad}(\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc}) \\ &+ \sum_d \hat{g}^{Ad}(\partial_b \phi^2 A_c A_d + \partial_c \phi^2 A_d A_b - \partial_d \phi^2 A_b A_c) \\ &+ \hat{g}^{A4}(\partial_b \phi^2 A_c + \partial_c \phi^2 A_b - \partial_4 g_{bc} - \partial_4 \phi^2 A_b A_c) \end{aligned}$$

$$2\hat{\Gamma}^A_{4c} = \sum_d \hat{g}^{Ad}(\partial_4 g_{cd} + \partial_4 \phi^2 A_c A_d + \partial_c \phi^2 A_d - \partial_d \phi^2 A_c) + \hat{g}^{A4}\partial_c \phi^2$$

$$2\hat{\Gamma}^A_{44} = 2\sum_d \hat{g}^{Ad}\partial_4 \phi^2 A_d - \sum_d \hat{g}^{Ad}\partial_d \phi^2 + \hat{g}^{A4}\partial_4 \phi^2 \tag{10.1.1}$$

The Electromagnetic Limit $\phi^2 = 1$

Now putting in $\phi^2 = 1$,

$$\begin{aligned} 2\hat{\Gamma}^A_{bc} &= \sum_d \hat{g}^{Ad}(\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc}) + \sum_d \hat{g}^{Ad}(\partial_b A_c A_d + \partial_c A_d A_b - \partial_d A_b A_c) \\ &\quad + \hat{g}^{A4}(\partial_b A_c + \partial_c A_b - \partial_4 g_{bc} - \partial_4 A_b A_c) \\ 2\hat{\Gamma}^A_{4c} &= \sum_d \hat{g}^{Ad}(\partial_4 g_{cd} + \partial_4 A_c A_d + \partial_c A_d - \partial_d A_c) \\ \hat{\Gamma}^A_{44} &= \sum_d \hat{g}^{Ad}\partial_4 A_d \end{aligned} \tag{10.1.2}$$

Simplifying...

$$\begin{aligned} 2\hat{\Gamma}^a_{bc} &= 2\Gamma^a_{bc} + \sum_d g^{ad}(A_b F_{cd} + A_c F_{bd}) + A^a \partial_4 g_{bc} + A^a \partial_4 A_b A_c \\ 2\hat{\Gamma}^4_{bc} &= -\sum_d A^d(\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc}) - \sum_d A^d(A_b F_{cd} + A_c F_{bd}) \\ &\quad - (1 + \sum_i A_i A^i)(\partial_4 g_{bc} + \partial_4 A_b A_c) + (\partial_b A_c + \partial_c A_b) \\ 2\hat{\Gamma}^a_{4c} &= \sum_d g^{ad}(\partial_4 g_{cd} + \partial_4 A_c A_d) + \sum_d g^{ad} F_{cd} \\ 2\hat{\Gamma}^4_{4c} &= -\sum_d A^d(\partial_4 g_{cd} + \partial_4 A_c A_d) - \sum_d A^d F_{cd} \\ \hat{\Gamma}^a_{44} &= \sum_d g^{ad}\partial_4 A_d \\ \hat{\Gamma}^4_{44} &= -\sum_d A^d \partial_4 A_d \end{aligned} \tag{10.1.3}$$

The Scalar Limit $A_i = 0$

The scalar limit is similarly defined,

$$2\hat{\Gamma}^A_{bc} = \sum_d \hat{g}^{Ad}(\partial_b g_{cd} + \partial_c g_{ab} - \partial_d g_{bc}) - \hat{g}^{A4}\partial_4 g_{bc}$$

$$2\hat{\digamma}^{A}_{4c} = \sum_d \hat{g}^{Ad}\partial_4 g_{cd} \qquad + \hat{g}^{A4}\partial_c\phi^2$$
$$2\hat{\digamma}^{A}_{44} = -\sum_d \hat{g}^{Ad}\partial_d\phi^2 \qquad + \hat{g}^{A4}\partial_4\phi^2$$

$$(10.1.4)$$

Simplifying...

$$\begin{aligned}
\hat{\digamma}^{a}_{bc} &= \digamma^{a}_{bc}\\
2\hat{\digamma}^{4}_{bc} &= -\tfrac{1}{\phi^2}\partial_4 g_{bc}\\
2\hat{\digamma}^{a}_{4c} &= \textstyle\sum_d g^{ad}\partial_4 g_{cd}\\
2\hat{\digamma}^{4}_{4c} &= \tfrac{1}{\phi^2}\partial_c\phi^2\\
2\hat{\digamma}^{a}_{44} &= -\textstyle\sum_d g^{ad}\partial_d\phi^2\\
2\hat{\digamma}^{4}_{44} &= \tfrac{1}{\phi^2}\partial_4\phi^2
\end{aligned} \tag{10.1.5}$$

The Electromagnetic Limit And Cylinder Condition

By applying equations (5.1.13) and the cylinder condition in order to simplify terms of the electromagnetic limit with $k = 1$, and without torsion noting that these Christoffel symbols are symmetric in the lower indices:

$$\begin{aligned}
2\hat{\digamma}^{a}_{bc} &= 2\digamma^{a}_{bc} + (A_b F_c{}^a + A_c F_b{}^a)\\
2\hat{\digamma}^{4}_{bc} &= -\textstyle\sum_d A^d(\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc}) + (\partial_b A_c + \partial_c A_b)\\
&= -2A_d\digamma^{d}_{bc} + (\partial_b A_c + \partial_c A_b)\\
2\hat{\digamma}^{a}_{4c} &= F_c{}^a\\
\hat{\digamma}^{4}_{4c} &= -\tfrac{1}{2}A^d F_{cd} \text{ and } \hat{\digamma}^{a}_{44} = \hat{\digamma}^{4}_{44} = 0
\end{aligned} \tag{10.1.6}$$

Now with torsion used to define the connection coefficients, using equations (5.1.13) and others from that section, and noting that these connection coefficients are not necessarily symmetric in the lower indices:

$$\begin{aligned}
2\hat{\Gamma}^{a}_{bc} &= g^{ad}(\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc}) + (A_b F_c{}^a + A_c F_b{}^a) - 2\hat{K}_{bc}{}^a\\
&= 2\digamma^{a}_{bc} + (A_b F_c{}^a + A_c F_b{}^a) - 2\hat{K}_{bc}{}^a\\
2\hat{\Gamma}^{4}_{bc} &= -\textstyle\sum_d A^d(\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc}) + (\partial_b A_c + \partial_c A_b)\\
&\quad - A^d(A_b F_{cd} + A_c F_{bd}) - 2\hat{K}_{bc}{}^4\\
&= -2A_d\digamma^{d}_{bc} + (\partial_b A_c + \partial_c A_b) - A^d(A_b F_{cd} + A_c F_{bd}) - 2\hat{K}_{bc}{}^4\\
\hat{\Gamma}^{a}_{4c} &= 0 \text{ and } \hat{\Gamma}^{4}_{4c} = \hat{\Gamma}^{a}_{44} = \hat{\Gamma}^{4}_{44} = 0\\
\hat{\Gamma}^{4}_{c4} &= -\tfrac{1}{2}A^d F_{cd}\\
\hat{\Gamma}^{a}_{c4} &= \tfrac{1}{2}F_c{}^a - \hat{K}_{c4}{}^a
\end{aligned} \tag{10.1.7}$$

10.2 Raised Levi-Civita Christoffel Symbols

Following the same procedure as with Christoffel symbols of the first and second kind a raised version of the Christoffel symbols can be derived, a third kind. It takes the value that would be guessed at (a guess because you can not raise across partial derivatives without caution) by inspecting the elements of the Christoffel symbol of the second kind and raising each element individually.

Starting from the covariant derivative of the raised metric tensor being zero:

$$\nabla^i g^{jk} = 0 = \partial^i g^{jk} + \Gamma^{jik} + \Gamma^{kij}$$

Cycling indices we have:

$$0 = \partial^k g^{ij} + \Gamma^{ikj} + \Gamma^{jki}$$
$$0 = \partial^j g^{ki} + \Gamma^{kji} + \Gamma^{ijk}$$

Adding the first two and subtracting the third:

$$\partial^i g^{jk} + \partial^k g^{ij} - \partial^j g^{ki} = 2\Gamma^{jik}$$

So, exactly as would be guessed:

$$\Gamma^{ijk} = \frac{1}{2}(\partial^j g^{ik} + \partial^k g^{ji} - \partial^i g^{jk}) \tag{10.2.1}$$

10.3 The Physicality Of The Constancy Of The Scalar Field

For this section the Einstein summation rule is not applied. Using eqn(5.1.3) and the anti-symmetry of the last two indices of contorsion:

$$0 = 2\hat{K}_{4c}{}^{c} = \hat{g}^{cd}(\partial_c \phi^2 A_d - \partial_d \phi^2 A_c) + \hat{g}^{c4}\partial_c \phi^2 \tag{10.3.1}$$

$$= g^{cd}\phi^2(\partial_c A_d - \partial_d A_c) + g^{cd}(A_d \partial_c \phi^2 - A_c \partial_d \phi^2) - A^c \partial_c \phi^2 \tag{10.3.2}$$

Using properties of derivatives of scalar fields ([7] eqn 3.1.10):

$$= g^{cd}\phi^2(\partial_c A_d - \partial_d A_c) + g^{cd}(A_d \triangle_c \phi^2 - A_c \triangle_d \phi^2) - A^c \partial_c \phi^2 \tag{10.3.3}$$

From this we get:

$$0 = \hat{A}^c \partial_c \phi^2 \tag{10.3.4}$$

This works to the extent ϕ^2 can be decomposed from the metric as a scalar field. Without Einstein summation this shows that the postulate B1 is physically reasonable when the vector potential components are non-zero.

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12 References

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