In Search Of A Variant Kaluza Theory

Robert Watson

March 24, 2013

Abstract

Kaluza's 1921 theory of gravity and electromagnetism using a fifth wrapped-up spatial dimension is at the root of many modern attempts to develop new physical theories. Lacking non-null electromagnetic fields however the theory is incomplete. Variants of the theory are explored to find ways to introduce non-null solutions by making the fifth dimension more physical, using alternative, weaker cylinder conditions. The Lorentz force law is investigated starting with a non-Maxwellian definition of charge, this is assumed to be related to Maxwellian charge by ansatz. Order of magnitude methods are used. Kaluza theory remains inadequate to support electromagnetism in full, non-null solutions are not readily shown to be admitted. An argument is made in favour of torsion resolving this issue. Postulates are derived from the argument for a variant theory. The charge ansatz is shown to follow from the postulates. It is concluded that Kaluza's 5D space and torsion need to go together in a Kaluza-Cartan theory. Tentatively, generalized Bel super-energy is hypothesized to be a conserved quantity.

1 Conventions

The following conventions are adopted unless otherwise specified:

Five dimensional metrics, tensors and pseudo-tensors are given the hat symbol. Five dimensional indices, subscripts and superscripts are given capital Roman letters. So for example the five dimensional Ricci flat 5-dimensional superspace-time of Kaluza theory is given as: $\hat{g}_{AB}$, all other tensors and indices are assumed to be 4 dimensional. Index raising is referred to a metric $\hat{g}_{AB}$ if 5-dimensional, and to g_{ab} if 4-dimensional. The domain of partial derivatives carries to the end of a term without need for brackets, so for example we have $\partial_a g_{db} A_c + g_{db} g_{ac} = (\partial_a (g_{db} A_c)) + (g_{db} g_{ac})$. Terms that might repeat dummy variables or are otherwise in need of clarification use additional brackets. Square brackets can be used to make dummy variables local in scope.

Space-time is given signature $(+,-,-,-)$, Kaluza space $(+,-,-,-,-)$ in keeping with [1]. Under the Wheeler et al [6] nomenclature, the sign conventions used here to correspond with [1] are [−, ?, −]. The first dimension (index 0) is always time and the 5^{th} dimension (index 4) is always the topologically closed Kaluza dimension. Universal constants defining physical units: c = 1, and G as a constant. The scalar field component is labelled ϕ^2 (in keeping with the literature) only as a reminder that it is associated with a spatial dimension, and to be taken as positive. The matrix of g_{cd} can be written as $|g_{cd}|$ when considered in a particular coordinate system to emphasize a component view. The Einstein summation convention may be used without special mention.

Some familiar defining equations consistent with [1] (using Roman lower-case for the general case only for ease of reference):

$$\Gamma^c_{ab} = \frac{1}{2} g^{cd}(\partial_a g_{db} + \partial_b g_{da} - \partial_d g_{ab}) \tag{1.0.1}$$

$$R_{ab} = \partial_c \Gamma^c_{ab} - \partial_b \Gamma^c_{ac} + \Gamma^c_{ab}\Gamma^d_{cd} - \Gamma^c_{ad}\Gamma^d_{bc} \tag{1.0.2}$$

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab} = -\frac{8\pi G}{c^4} T_{ab} \tag{1.0.3}$$

$$F_{ab} = \nabla_a A_b - \nabla_b A_a = \partial_a A_b - \partial_b A_a \text{ equally } F = dA \tag{1.0.4}$$

Any 5D exterior derivatives and differential forms could also be given a hat, thus: $\hat{d}\hat{B}$. However, the primary interest here will be 4D forms. $\triangle$ represents the 4D D'Alembertian.

2 Introduction

Kaluza's 1921 theory of gravity and electromagnetism [2][3][4] using a fifth wrapped-up spatial dimension is at the heart of many modern attempts to develop new physical theories [1][5]. From supersymmetry to string theories topologically closed small extra dimensions are used to characterize the various forces of nature. It is therefore at the root of many modern attempts and developments in theoretical physics. However it has a number of foundational problems. It seems sensible to look at these from a classical perspective before looking at more complicated situations such as quantum gravity theories as envisaged by Klein.

The theory assumes a (1,4)-Lorentzian Ricci flat manifold to be the underlying metric, split (analogously to the much later ADM formalism) as follows:

$$\hat{g}_{AB} = \begin{bmatrix} g_{ab} + \phi^2 A_a A_b & \phi^2 A_a \\ \phi^2 A_b & \phi^2 \end{bmatrix} \tag{2.0.1}$$

By inverting this metric as a matrix (readily checked by multiplication) we get:

$$\hat{g}^{AB} = |\hat{g}_{AB}|^{-1} = \begin{bmatrix} g^{ab} & -A^a \\ -A^b & \frac{1}{\phi^2} + A_i A^i \end{bmatrix} \tag{2.0.2}$$

Maxwell's law are automatically satisfied: dF=0 follows from dd = 0. d*F= 4π*J can be set by construction. d*J=0, conservation of charge follows also by dd=0 in most parts of the manifold. However:

In order to write the metric in this form there is a subtle assumption, that g_{ab}, which will be interpreted as the usual four dimensional space-time metric, is itself non-singular. However, this will always be the case for moderate or small values of A_x which will here be identified with the electromagnetic 4-vector potential. The raising and lowering of this 4-vector are defined in the obvious way in terms of g_{ab}. We have also assumed that topology is such as to allow the Hodge star to be defined. This means that near a point charge source the argument that leads to charge conservation potentially breaks down as the potential may cease to be well-defined. Whereas the Toth charge that will be defined in the sequel does not have this problem.

Assume values of ϕ^2 around 1 and relatively low 5-dimensional metric curvatures. We need not therefore concern ourselves with this assumption beyond stating it on the basis that physically these parameters encompass tested theory. Given this proviso A_x is a vector and ϕ^2 is a scalar - with respect to the tensor system defined on any 4-dimensional submanifold that can take the induced metric g.

Kaluza's cylinder condition (KCC) is that all partial derivatives in the 5th dimension i.e. ∂_4 and $\partial_4\partial_4$ etc... of all metric components are 0. A perfect 'cylinder'. This leads to constraints on g_{ab} given in [1] by three equations, the field equations of Kaluza theory, where the Einstein-Maxwell stress-energy tensor can be recognised embedded in the first equation:

$$G_{ab} = \frac{k^2\phi^2}{2}\left\{\frac{1}{4}g_{ab}F_{cd}F^{cd} - F^c_aF_{bc}\right\} - \frac{1}{\phi}\{\nabla_a(\partial_b\phi) - g_{ab}\triangle\phi\} \tag{2.1.1}$$

$$\nabla^a F_{ab} = -3\frac{\partial^a\phi}{\phi}F_{ab} \tag{2.1.2}$$

$$\triangle\phi = \frac{k^2\phi^3}{4}F_{ab}F^{ab} \tag{2.1.3}$$

These will be referred to as the first, second and third field equations respectively. Here there is also a k term, since the formulation in [1] is more general than that used here. In this work k=1 is used unless specified otherwise. Sign conventions should also be checked before using such equations out of context. It is important to note that in the variants of Kaluza theory defined here, these field equations may not apply. They apply fully only to Kaluza's original theory.

By looking at field equation 3 it can be seen that if the scalar field does not vary then only null electromagnetic solutions result. The second field equation then also imposes no charge sources. Here the scalar term could be allowed to vary in order to allow for non-zero $F_{ab}F^{ab}$. This falls within Kaluza's original theory. This potentially allows for non-null electromagnetic solutions, but there are problems to overcome: the field equations cease being necessarily electrovacuum. This remains a problem even when the scalar field terms are set large, as is sometimes done to ensure that field equation 2 is identically zero despite scalar fluctuations.

Another foundational issue of Kaluza theory is that even with a scalar field it does not have convincing sources of mass or charge. Field equation 2 then has some charge sources, but it is far from clear that realistic sources are represented by this equation. Matter (and charge) models in this work will be assumed to be regions of the Kaluza space that are *not* Ricci flat in the otherwise Ricci flat Kaluza space, just as matter/energy is analogously assumed to be in general relativity. That is, where the 5D Einstein tensor of the Kaluza space itself is non-zero.

Charge will be given a possible alternative definition as 5-dimensional momentum, following a known line of reasoning [8] within Kaluza theory. This version of charge will be called Toth charge to make it distinct from Maxwellian charge, their identity is initially assumed by ansatz in this work, otherwise the Lorentz force law has no obvious explanation in the event that the electrovacuum is no longer valid due to other fields. As momentum the Toth charge is of necessity conserved, provided there are no irregularities in the topology of the Kaluza 5th dimension, similarly the conservation of Maxwellian charge is normally guaranteed by the potential, except that this may not be valid in extreme curvatures where the 4-vector associated with the 4-potential may cease to be a vector.

We will also assume of necessity a closed 4D spatial hypersurface as is often done in general relativity to ensure 4 dimensional causality. Although 5D causality issues will not be explored.

The leading issue is that Kaluza theory appears to offer only null electromagnetic solutions, non-null electrovacuums more generally are not so easily supported as changes in the scalar field may force divergence of the field equations from those of the electrovacuum (see field equation 1). Null electrovacuums occur under KCC when the scalar is constant as can be seen in the field equations above. That is, non-null solutions, non-radiative electromagnetic fields, seem to have no reserved place within the theory unless one allows the scalar field to oscillate. A standard fix is to set the scalar field to be very large so that its oscillations make little difference to the resulting field equations. However this is limiting, arbitrary, and requires its own explanation. It also does not guarantee that the field equations are electrovacuums unless further arbitrary conditions are added. Without taking such arbitrary measures that cease to be in the spirit of Kaluza's original theory, the stress-energy tensor under scalar field fluctuations is different from the Einstein-Maxwell tensor [6][7] and the accepted derivation of the Lorentz force law (for electrovacuums [6]) can not be assumed. The Lorentz force law is central to the argument.

Note that in addition the derivation of the Lorentz force law within general relativity (from an assumed Einstein-Maxwell stress-energy tensor) is not without problems of principle [6]. *Thus, attempting to build a theory around the ansatz that the Toth charge and Maxwellian charge are equivalent under the variant cylinder conditions is persuasive* - as this leads independently of the electrovacuum to an approximate and provisional Lorentz force law.

The scalar field will be allowed to vary. But order of magnitude limits will be placed on it so as to allow for a variant Lorentz force law. A Lorentz force law, rather than the electrovacuum solutions per se will thus be sought for Toth charges, and an ansatz linking the two definitions of charge assumed here.

3 Preliminary Notes

3.1 Geometrized Units Of Mass

The full metric definition used in [1] was:

$$\hat{g}_{AB} = \begin{bmatrix} g_{ab} + k^2\phi^2 A_a A_b & k\phi^2 A_a \\ k\phi^2 A_b & \phi^2 \end{bmatrix} \tag{3.1.1}$$

This gives null solutions [1] such that $G_{ab} = -\frac{k^2}{2}\ F_{ac}F^c_b$. Comparing this with [7] (where we have $G_{ab} = 2F_{ac}F^c_b$ in geometrized units where G=1) the sign difference is due to the historic use of $[-, ?, -]$ notation here, rather than the more modern [+,+,+] notation as defined by Wheeler et al. [6] (and as also used by Wald [7]).

Although k is set to 1 elsewhere in this work, yielding the metric in the introduction, we need $k = 2$ to get the field equations in the geometrized units of [7] - if the electromagnetic field tensor is to be the same.

$G/c^4 = 1$ for k=2
$= 6.674 \times 10^{-8} cm^3 g^{-1} s^{-2}/c^4 = 6.674 \times 10^{-8} cm^3 g^{-1} \times (3 \times 10^{10} cm)^{-2}/c^4$
$= 6.674 \times 10^{-8} cm g^{-1} \times (3 \times 10^{10})^{-2}$ for c = 1

$$1g = 7.42 \times 10^{-29} cm \text{ for k=2, c=1} \tag{3.1.2}$$

Secondly, if we *do* use k = 1, we can adjust G's dimensionless value to accomodate:

$G_{ab} = -2F_{ac}F^c_b = -\frac{8\pi G}{c^4} T_{ab}$ for k=2, c=1
$G_{ab} = -F'_{ac}F'^c_b = -\frac{8\pi G'}{c^4} T'_{ab}$ for k=1, c=1

So if G_{ab} is the same in both cases, then $F'_{ab} = \sqrt{2} F_{ab}$

If we define T'_{ab} to be formally the same as T_{ab}, but with a substitution of F' terms for F terms, it will be twice as big. Then we must adjust $G' = \frac{1}{2} G = \frac{1}{2}$.

With this adjustment to G, we have:

$$1g = 1.48 \times 10^{-28} cm \text{ for k=1, c=1} \tag{3.1.3}$$

So in effect we can change the scale relationship of mass to distance in order to change the electromagnetic tensors. The tensors can represent the same underlying reality but in different units. We can refer to these two schemes of units as k=2 and k=1 respectively.

3.2 Kaluza Theory Is Consistent With Special Relativity Even When 5D Momentum Is Present

In the sequel, one definition of charge (Toth charge) will be identified with 5D momentum. This is already known in the original Kaluza theory to obey a Lorentz force-like law, but will be extended here in scope, noting that the coincidence of Toth charge and Maxwellian charge is not guaranteed prior to the ansatz.

That this is consistent with Special Relativity will be something that anybody seeking confidence in Kaluza theories will want to check. The additions of velocities in Special Relativity, for example, is not obvious. Taking two perpendicular velocities, u and v, and adding them yields:

$$s^2 = u^2 + v^2 - u^2 v^2$$

The particle moving in the Kaluza dimension, but stationary with respect to space-time, will have a special relativistic rest mass greater than its Kaluza rest mass. A later result needed here is:

$$Q_{toth}/M_0 = -dx_4/d\tau \quad \text{relating charge, rest mass and proper Kaluza-velocity}$$

This makes sense only because mass can be written in fundamental units (i.e. in distance or time) and Toth charge will be defined as 5th dimensional momentum.

Using natural conversions between units we get the Kaluza rest mass of any presumed particle with the mass and charge comparable in magnitude to an electron or positron to be about $1.5 \times 10^{-54} g$, which is a lot smaller than the relativistic rest mass used when considering only space-time physics. And its proper Kaluza velocity in natural units is then about $K = 6 \times 10^{26}$, making it highly Kaluza-relativistic.

Much as we may be unfamilar with a Kaluza rest mass (M_{k0}) we can see that it is consistent with the addition of velocities as follows:

$$M_0 = \frac{M_{k0}}{\sqrt{(1-u^2)}} \text{ where } u = Q_{toth}/M_0 \tag{3.2.1}$$

$$M_{rel} = \frac{M_0}{\sqrt{(1-v^2)}} = \frac{M_{k0}}{\sqrt{(1-u^2)}} \times \frac{1}{\sqrt{(1-v^2)}} = \frac{M_{k0}}{\sqrt{(1-u^2-v^2+u^2v^2)}} \tag{3.2.2}$$

where v is the relativistic velocity in space-time

By putting $u = Q_{toth}/M_0$ into the definition of rest mass and solving, we get that charge, whether positive or negative, is a contributor to the relativistic rest mass according to the following formula:

$$M_0{}^2 = M_{k0}{}^2 + Q_{toth}{}^2 \tag{3.2.3}$$

Here the majority of mass-enery in the rest mass of such an elementary charge is seen as being tied up in its charge. In this work elementary Toth charges will take on these properties. Whether or not these are able to model Maxwellian charges depends on the correspondence between Maxwellian and Toth charge. Whether this further corresponds to real electrons and positrons or whether similar models can describe other fundamental charge sources is left as unascertained experimentally and analytically, but merely a tempting suggestion of this model not to be investigated further here.

Observed electrons have static charge, angular momentum, a magnetic moment, and a flavor. The only thing distinguishing the electron from the muon is apparently the flavor. The mass difference between the muon and the electron is about 105 MeV, perhaps solely due to this difference in flavor. Yet here we have a model where the unit Toth charge's relativistic rest mass is dominated by its Kaluza velocity. Thus at this stage the idealized charge models used here and real particles must be considered not yet correlated.

3.3 Matter And Charge Models, A Disclaimer

The model unit Toth charge presented here, therefore remains a separate entity from any real Maxwellian charges, merely a mathematical device to investigate whether such models are possible. Having said that for the purposes here the ansatz is made that in principle Maxwellian and Toth charges can be identified.

The above analysis has assumed that some sort of particle model of matter and charge is possible, that it can be added to the original theory perhaps without changing the space-time solution, which is impossible no less than in general relativity. Secondly we might imagine that what has been described is a particle whizzing around the fifth dimension like a roller coaster on its spiralled tracks. The cylinder conditions could in fact also be maintained if, instead of a particle, the matter-charge source was rather a 'solid' ring, locked into place around the 5th dimension, rotating at some predetermined Kaluza velocity. An exact solution could even involve changes in the size of the 5th dimension. None of that is investigated here, the aim is to see whether non-null solutions can be found in a Kaluza variant theory without extreme alteration.

It is an essential proviso that a physically realistic matter-charge model has not been detailed, much less formally identified with a real charge source such as an electron. The assumption then that such a hypothetical model would necessarily follow (albeit approximately) geodesics is also therefore just that: an assumption - though not without analogs in other, experimentally valid, classical theories.

The original Kaluza theory almost certainly does not have sufficient degrees of freedom to allow for such a matter model to be embedded within it. More general matter models could be assumed however to have significant enough degrees of freedom to allow for such a model or approximation of such a model in principle. But an actual differential geometrical model of such a matter-charge source is presumed too difficult to produce here, even if possible. In addition, the fact that real charge sources are quantum mechanical may also discourage us, though a classical limit interpretation should be possible regardless.

This work assumes a limited concept of such a charge model and attempts to investigate whether non-null solutions are possible in conjunction with a Lorentz force law. That is, it attempts to replicate all the important features of classical physics, without predicting or imposing its particular model of charge as the correct one.

Geodesic Assumption: That any particle-like matter-charge models derived from the geometry are approximately geodesic. This would also need to follow from any applicable matter-charge model that was ultimately found to describe unit charges. Later when torsion is introduced this assumption should be generalized to auto-parallels, though as it turns out this detail makes little difference.

Charge Ansatz: That the Maxwellian and Toth definitions of charge coincide for the purposes here.

One of the confusing aspects of this work is that the field equations of Kaluza cease to apply when such matter-charge models become part of the solution. Therefore the Kaluza field equations are only used when they can be justified, and going back to the Ricci tensor from scratch is undertaken here as necessary. It is also important to note that in the following the Ricci flat condition of the original Kaluza theory's Kaluza space will not be generally valid due to the presence of matter models, but as with vacuum solutions in general relativity will be usable outside of matter models.

3.4 Duality Invariance Of Kaluza's Original Theory

The dual metrics of $\hat{g}^{AB}$ and g^{ab} will be discussed in this section.

$\hat{g}^{AB}$ will be identified with an alternative dual metric $\hat{h}_{AB}$ for some coordinate system in such a way that their representations as matrices are equal. That is, such that: $|\hat{g}^{AB}| = |\hat{h}_{AB}|$, such an identification will be written $\hat{g}^{AB} \leftrightarrow \hat{h}_{AB}$. It follows that $\hat{g}_{AB} \leftrightarrow \hat{h}^{AB}$ where the two alternative systems $\hat{g}_{AB}$ and $\hat{h}_{AB}$ define their own notions of raising and lowering indices.

$\hat{h}_{AB}$ and $\hat{h}^{AB}$ can be written analogously to the original metrics as follows:

$$\hat{h}^{AB} = \begin{bmatrix} h^{ab} + \phi^2 B^a B^b & \phi^2 B^a \\ \phi^2 B^b & \phi^2 \end{bmatrix} \tag{3.4.1}$$

$$\hat{h}_{AB} = |\hat{h}^{AB}|^{-1} = \begin{bmatrix} h_{ab} & -B_a \\ -B_b & \frac{1}{\phi^2} + B^i B_i \end{bmatrix} \tag{3.4.2}$$

Where analogously to the original system the raising and lowering of indices of 4-vectors is implemented by h_{ab}.

We have the following relations by construction: $g^{ab} \leftrightarrow h_{ab}$ and $A^a \leftrightarrow B_a$.

In other words they are the same system with index conventions swapped around. Or equivalently they are dual systems.

Kaluza's original theory is dual invariant in that if $\hat{g}_{AB}$ is Ricci flat then this is equivalent to $\hat{h}_{AB}$ being Ricci flat. This follows as raising $R_{AB} = 0$ within $\hat{g}_{AB}$ remains 0.

That is the laws defining the system itself are dual invariant, even if the contents of the theory, such as matter models and the various fields themselves are not.

Most matter models and fields have in effect an alternative formulation that would suffice as a physical description by taking the dual system. There is something arbitrary about matter models and fields (also in general relativity) in this respect, from the outset. This duality invariance, or failure thereof, will be explored with respect to the variant theory developed here. Intuitively speaking the idea is that that which is invariant is more likely to form the laws, whilst that which is not invariant we might take to define, in some sense, the contents.

4 The Order of Magnitude of Potentials

4.1 The Electromagnetic Potentials

The contribution to the metric of a typical cgs unit of electromagnetic potential can be calculated: It is actually dimensionless in genuinely natural units, as must be the case for it to be related to metric components in Kaluza theory. Note that human scale units invariably leave the Christoffel symbols small under normal tested conditions.

In cgs units the Coulomb's force law is given by: $F = Q_1 Q_2 / r^2$

Similarly the potential is given by Q/r, that is charge/distance, or esu/cm. Using [7]:

$$([L]^{3/2}[M]^{1/2}/[T])/[L] = ([L]^{1/2}[M]^{1/2})/[T]) \tag{4.1.1}$$

Using $1g = 7.42 \times 10^{-29} cm$ (eq. 3.1.2) gives 1cgs unit of potential (esu or Statvolt) as:

$1cm^{1/2}8.61 \times 10^{-15}\ cm^{1/2}/s$

$= 2.86 \times 10^{-25}(3 \times 10^{10}cm/s)$

$$1esu = 2.86 \times 10^{-25}c \tag{4.1.2}$$

$\approx 10^{-25}$ in natural units where $c = 1$ ($k = 2$)

This is clearly a very small figure and is a comparable order of magnitude for $k = 1$ for the purposes here.

Take a single unit charge (ie of an electron) $Q \approx 5 \times 10^{-10}esu$ and the (Bohr) radius of the hydrogen atom: $r \approx 5 \times 10^{-9}cm$.

The potential $R_a = Q/r = 10^{-1}esu/cm = 10^{-26}$ in natural units. Now this potential corresponds to the strong electrical forces within the atom, but due to the short distances may not represent strong potentials. We might need a more realistic reference for classical potentials (R_c), in order to provide a sort of experimental upper limit for potentials that are also experimentally sure to have well-behaved classical properties such as validating the Lorentz force law. This figure could be given as a ratio $R_c = h.R_a$ to give a sense of proportion, where a typical bound for h will now be estimated.

In any hypothetical experiment we will have an 'earth' that represents (approximately) a zero potential. This can be seen as either the surface experimentally or approximately also the centre of the Earth.

As a case in point imagine we are testing the effects of the Lorentz force law due to forces emanating from the ionosphere. If we set the Earth's surface to be zero potential, i.e. the 'earth', we will end up with high potentials in the ionosphere. However the ionosphere commences about 100km from the Earth, its 'earth', we can take these high potentials as a reasonable description of the bounds of normal conditions for the purposes here.

In order to estimate our h in $R_c = h.R_a$ we need to define the experiment we are looking at. For these purposes we can look at two extremes that have comparable energies: Firstly 150V/m over a vacuum of 100km (comparable to the potential differences created between the ionosphere and the Earth over long distances), secondly 15MV over, say, a vacuum of 1m (a high voltage experiment if it is to be sustained for any duration of time). It is the potential difference relative to the 'earth' that sets the potential. They are both high in terms of the potentials involved and will act as a guide to the estimate of h. The second is perhaps easier to deal with experimentally. A clean experiment over larger scales may be unfeasible except perhaps in space.

We can then look at the maximum 4-potential that the second experiment defines and define this to be R_c, from which we can estimate h relative to this level of experimental testing. As it pertains to the orders of magnitude that will be used here an approximate idea of this figure is useful to make the scales meaningful. If the accepted tested level of classical electromagnetism is higher or lower than the 15MV used here it is a simple matter to scale h accordingly. If we are dealing with specific situations where the order of magnitudes are much lower, same again.

Both the above set-ups have a 15MV potential difference, which is about 50000 Statvolts. This gives an h of about 500000, and,

$$O(f) \approx R_c \approx 10^{-20}, k = 2 \tag{4.1.3}$$

We can do a similar calculation with a very low voltage for comparative purposes, say 1.5KV, over which the Lorentz force is guaranteed. This leads to a different reference figure:

$$O(f) \approx R_w \approx 10^{-24}, k = 2 \tag{4.1.4}$$

These different figures correspond to different ensembles of experiments. In the first case we have a level where we would expect the Lorentz force law to generally be experimentally valid, but above which we may not be quite as sure, as it is outside normal experimental experience. In the second case we have guaranteed compliance with the Lorentz force law (barring perhaps contrived scenarios), and the strong liklihood in any case of compliance above that level. Crudely, the first is the level above which deviation may well occur for all we know since it represents exceptional experimental scenarios, the second a level below which deviation does not occur to the extent that it represents normal experimental experience.

4.2 The Metric Components of $O(v)$

Using k=2 we can estimate one possible $O(v)$ by looking at the the Schwarzchild Solution for the Earth. The differences from unity (or negative unity) of the terms depends on $2GM/r$, in this case $2M/r$ in natural units.

$$2 \times [6 \times 10^{24} kg \times 1000 g/kg]/[6000 km \times 10^5 cm/km] = 2 \times 10^{19} g/cm$$

Using $1g = 7.42 \times 10^{-29} cm$, and when k=2, with a comparable order of magnitude for k=1:

$$O(v) \approx 10^{-9} \tag{4.2.1}$$

$O(v)$ is considerably larger in significance than $O(f)$. Notice that also in this calculation, in effect, the gravitational 'earth' has been taken to be the centre of the earth.

5 The Cylinder Conditions And Electromagnetic Limits

5.1 Introducing Orders of Magnitude

We will start with a weak field limit that can be assumed at the usual classical scale. Terms such as $A_a A_b$ will be bounded $O(v^2)$ as opposed to terms not thus multiplied such as $\hat{g}_{ab}$ or simply A_a. Metric terms either being bounded $O(v)$ or the difference from 1 being bounded $O(v)$. Our Kaluza space-time solutions, at the usual classical limit, are to be approximately 5-Lorentzian. $O(v)$ will be taken to be a small term at the usual classical limit. In addition, the units will be assumed to be such that derivatives of O(v) i.e. O(v+) and O(v++) will also all be small, where O(v+) and O(v++) are the order of magnitude of first and second derivatives (with corresponding units) respectively.

An electromagnetic limit will be assumed as required, where the scalar field is set to being approximately the identity: $\phi^2 \approx 1$

Now, this simple declaration turns out to be quite complicated. For one thing at the usual classical scale it will be automatically approximately 1 by the weak field limit as a minimum constraint, at least to $O(v)$. But this in itself won't make it any closer to unity than the electric potentials are to 0 relative to other weak fields. For the electromagnetic limit we want more than that.

We will define it in terms of three orders of magnitude:

$\phi^2 = 1$ to O(s), and,

$\partial_A \phi^2$ is O(s+). We can also have:

$\partial_A \partial_B \phi^2$ is $O(s++)$

(5.1.1)

This distinction will be of fundamental importance later. O(s) will be no more significant than O(v), with the possibility of turning out to be a lot smaller in significance. O(s+) and O(s++) can be assigned units s^{-1} and s^{-2} respectively and will thus not be comparable to O(s) in a simple way. Similarly we can have the additional levels of bounding for the derivatives of electromagnetic tensors defined as follows:

A^a is $O(f)$, and,

$\partial_A A^a$ as $O(f+)$. We can also have:

$\partial_A \partial_B A^a$ as $O(f++)$

(5.1.2)

Noting that in any situation the tightest of any two applied bounds dominates. Noting also the same unit considerations as previously.

5.2 Duality And Orders Of magnitude

In order to investigate where these orders of magnitude are duality invariant, the following approximate conditions will be used as a comparative case. That is, duality invariance would require:

$$O(f^2) \leq O(s) \tag{5.2.1}$$

$$O(f^2) \leq O(v) \tag{5.2.2}$$

The latter follows from previous considerations in any case. The first will in general *not* be true in this work. Where this is the case we will make a note of it.

5.3 The Cylinder Conditions

The other fields, the 4D metric and the electromagnetic potential vector will be given a similar order of magnitude constraint so that their derivatives in the direction of the Kaluza dimension are bounded as follows:

$$\partial_4 \hat{g}_{AB} \text{ is } O(\delta+) \tag{5.3.1}$$

KCC is the limiting case where it is identically 0. This implies also that $\partial_4 \hat{g}^{AB}$ is $O(\delta+)$ via a consideration of divergence of the metric being 0. Now of course this bound as defined above is not yet a constraint until $O(\delta+)$ is defined. So this device allows us to weaken the cylinder condition as required.

A basic reasonable constraint might be to ensure that at least such oscillations are not greater than the general order of magnitude of the electromagnetic fields. This will in any case be justified later. Thus:

$$O(\delta+) \leq O(f+) \tag{5.3.2}$$

The use of the symbol + to identify when a derivative has been taken is easy to misuse. A symbol such as $O(X+)$ will be related to $O(X)$ via proportion to a constant that will depend not only on the terms and functions in question, such as frequencies, but also on the units being used. Later such orders of magnitude with the same units will be multiplied by each other. This can be made more concrete by giving $O(f)$ a numerical value to set the scale. By default we can set this value to some classical reference potential, a figure that gives a reasonable bound to the level for which the well-behaved properties of a classical system have actually been tested. We might similarly have another classical reference to define O(v). Such quantities were investigated previously, but depend on the ensemble of solutions being investigated or experimented with.

Similarly, relative to this (by some constant and some unit) $O(f+)$ will be taken to be defined, though any constants of proportionality and any units will not be specified. Typical estimates for such an order of magnitude could also be taken from an ensemble of solutions.

Two other complementary limits can be defined: the strong electromagnetic and the strong scalar limit respectively. These will both be when oscillations in the other are absolute zero.

A weak cylinder condition will later be defined where KCC does not hold, but it will remain within the electromagnetic limit.

5.4 Some Rules For Orders Of Magnitude

The use of the $\leq$ and $<<$ symbols in comparing orders of magnitudes will be used to express the idea that one order of magnitude can be or is smaller than the other when compared numerically, units allowing.

We also need a few rules for dealing with more complex situations. One could be that if $O(X) = O(Y)$ and $O(W) \leq O(Z)$ then generally $O(X+) = O(Y+)$ and $O(W+) \leq O(Z+)$ respectively, and so on. This will be called *proportionality*. It should of course be invoked with care and may not in the most general case be valid, for example if frequencies vary to a sufficiently extreme extent. It will be used to the extent that it can be reasonably justified on grounds of physicality.

Similarly, *distributivity*: terms such as $O(f+)O(f+)$ and $O(f)O(f++)$, where the terms are all based on the same underlying order O(f) and where it seems as if all that has happened is a displacement of some + terms, will be considered the same order of magnitude without further consideration. We might observe the reasonableness of this by considering the chain rule. Heuristically it corresponds to the idea that $O(f++)$ should be as far from $O(f+)$ as $O(f+)$ is from $O(f)$ in order of magnitude. Consideration of a sine wave oscillating about zero shows the good sense of this in terms of physicality.

Similarly we could also extend distributivity over different underlying terms. This however will be used more cautiously in the same way as proportionality, and indeed it would follow from proportionality. Here's the proof:

Let's consider whether $O(P+)O(F) = O(F+)O(P)$. First, invent a dummy order of magnitude such that:

$O(P) = O(X)O(F)$ and $O(F) = O(P)/O(X)$

We have by the chain rule,

$O(P+)O(F) = O(X+)O(F)O(F) + O(X)O(F+)O(F) = O(X+)O(F)O(F) + O(F+)O(P)$

Similarly by the quotient rule (with sign made plus as we are dealing with orders of magnitude):

$O(F+)O(P) = O(P)x[O(P+)O(X) + O(P)O(X+)]/[O(X)O(X)]$
$= O(P)O(P+)/O(X) + O(P)O(P)O(X+)/O(X)O(X) = O(F)O(P+) + O(F)O(F)O(X+)$

So, these two equations can only be true if the $O(F)O(F)O(X+)$ term is bounded by the same order of magnitude as the largest of $O(P+)O(F)$ and $O(F+)O(P)$ and thus $O(P+)O(F) = O(P)O(F+)$ as required. □

5.5 A Resonant Cylinder Condition

Further a Resonant Cylinder Condition is defined. This needs some discussion.

The objective of the Resonant Cylinder Condition is to weaken KCC but not in the same way as having simply an order of magnitude limit for derivative terms in the direction of the Kaluza dimension. The Resonant Cylinder Condition (RCC) takes a loop around the Kaluza 5th dimension (one that is locally normal to the supposed space-time embeddings). The idea is then that various components, derivatives and tensors oscillate around this loop, in a way reminiscent of resonance, whilst maintaining an average value that can be represented (approximately) as a tensor in a representative space-time. This will be applied to any tensors, pseudo-tensors or related terms that might be meaningful on a sample 4D manifold chosen to represent space-time. In particular any terms which consist of (possibly multiple) differentiation of another term, and where one or more of those derivatives is a derivative in the direction of the Kaluza dimension, must average 0. This is because the tensor thus differentiated must start at a certain value, and in passing round the loop return to it. Terms constructing by compounding such zeroed terms need not however necessarily average to 0.

So we need to:

(i) Impose that the average of the oscillations of such a simply resonant term to be 0.

(ii) Consider the oscillations of simply resonant terms non-zero at any particular point round the loop, and thus with respect to compound terms.

In this work only point (i) will be used.

The original Kaluza theory assumed KCC. So weakening KCC (whether by RCC or otherwise) requires particular care in that the original field equations and conclusions derived from them can nolonger be assumed, in particular the Kaluza field equations cease to be valid.

6 Charge, 5D Momentum And The Lorentz Force

Toth [8] derives a Lorentz-like force law where there is a static scalar field and Kaluza's cylinder condition applies. The resulting 'charge' is the momentum term in the fifth dimension and it is not apparent how this relates to the Maxwell current, except as Toth states via 'formal equivalence'. While this result is not new, Toth's calculation is used here as the starting point for a more detailed calculation.

A derivation is given of the Lorentz force law applicable to the Toth current. Toth makes several assumptions in his calculation. First that the scalar field is constant near the charge, and secondly the Kaluza cylinder condition (KCC). Toth also assumes a single point particle, not necessarily the case here, and constant mass-charge. These issues relate in this context to finding such a matter model as a solution. Here the KCC is relaxed, and both the Resonant Cylinder Condition is applied and an alternative Weak Cylinder Condition defined. Results are compared. The Geodesic Assumption must also be made for matter-charge models in this context. Oscillations of the scalar field are included (indices have been omitted from order of magnitude terms for clarity of presentation). Details of Chrisfoffel symbol terms can be found in a later section for reference.

$$\begin{aligned}
&\hat{\Gamma}^c_{4b} = \tfrac{1}{2}g^{cd}(\delta_4\hat{g}_{bd} + \delta_b\hat{g}_{4d} - \delta_d\hat{g}_{4b}) + \tfrac{1}{2}\hat{g}^{c4}(\delta_4\hat{g}_{b4} + \delta_b\hat{g}_{44} - \delta_4\hat{g}_{4b}) = \\
&\tfrac{1}{2}g^{cd}[\delta_b(\phi^2 A_d) - \delta_d(\phi^2 A_b)] + \tfrac{1}{2}g^{cd}\delta_4\hat{g}_{bd} + \tfrac{1}{2}\hat{g}^{c4}\delta_b\hat{g}_{44} = \\
&\tfrac{1}{2}\phi^2 g^{cd}[\delta_b A_d - \delta_d A_b] + \tfrac{1}{2}g^{cd}A_d\delta_b\phi^2 - \tfrac{1}{2}g^{cd}A_b\delta_d\phi^2 + \tfrac{1}{2}g^{cd}\delta_4\hat{g}_{bd} + \tfrac{1}{2}\hat{g}^{c4}\delta_b\phi^2 = \\
&\tfrac{1}{2}\phi^2 F^c_b + \tfrac{1}{2}g^{cd}A_d\delta_b\phi^2 - \tfrac{1}{2}g^{cd}A_b\delta_d\phi^2 + \tfrac{1}{2}g^{cd}\delta_4\hat{g}_{bd} + \tfrac{1}{2}\hat{g}^{c4}\delta_b\phi^2 = \\
&\tfrac{1}{2}\phi^2 F^c_b - \tfrac{1}{2}g^{cd}A_b\delta_d\phi^2 + \tfrac{1}{2}g^{cd}\delta_4\hat{g}_{bd} = \\
&\tfrac{1}{2}\phi^2 F^c_b + O(s+)O(f) + O(\delta+)
\end{aligned} \tag{6.0.1}$$

$$\hat{\Gamma}^c_{44} = \tfrac{1}{2}\hat{g}^{\mathrm{cD}}(\delta_4\hat{g}_{4D} + \delta_4\hat{g}_{4D} - \delta_D\hat{g}_{44}) = O(\delta+) \text{ - } \tfrac{1}{2}g^{\mathrm{cd}}\delta_d\phi^2 = O(\delta+) + O(s+) \tag{6.0.2}$$

We have:

$$\begin{aligned}
&\hat{\Gamma}^c_{ab} = \tfrac{1}{2}g^{cd}(\delta_a g_{db} + \delta_b g_{da} - \delta_d g_{ab}) \\
&+\tfrac{1}{2}g^{cd}(\delta_a(\phi^2 A_a A_b) + \delta_b(\phi^2 A_a A_b) - \delta_d(\phi^2 A_a A_b)) + \tfrac{1}{2}\hat{g}^{c4}(\delta_a\hat{g}_{4b} + \delta_b\hat{g}_{4a} - \delta_4\hat{g}_{ab}) \\
&= \Gamma^c_{ab} + O(f^2)O(\delta+) + (O(f+) + O(\delta+))O(f)
\end{aligned} \tag{6.0.3}$$

So:

$$\begin{aligned}
&0 = \frac{d^2x^a}{d\tau^2} + \hat{\Gamma}^a_{BC}\frac{dx^B}{d\tau}\frac{dx^C}{d\tau} \\
&= \frac{d^2x^a}{d\tau^2} + \hat{\Gamma}^a_{bc}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} + \hat{\Gamma}^a_{4c}\frac{dx^4}{d\tau}\frac{dx^c}{d\tau} + \hat{\Gamma}^a_{b4}\frac{dx^b}{d\tau}\frac{dx^4}{d\tau} + \hat{\Gamma}^a_{44}\frac{dx^4}{d\tau}\frac{dx^4}{d\tau}
\end{aligned}$$

$= \frac{d^2x^a}{d\tau^2} + \hat{\Gamma}^a_{bc}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} + \phi^2 F^a_b \frac{dx^b}{d\tau}\frac{dx^4}{d\tau} + (O(\delta+) + O(s+)O(f))\frac{dx^b}{d\tau}\frac{dx^4}{d\tau} + (O(\delta+) + O(s+))\frac{dx^4}{d\tau}\frac{dx^4}{d\tau}$

(6.0.4)

Setting the Toth charge-to-mass ratio to:

$$Q_t/m = -\phi^2 \frac{dx^4}{d\tau} \tag{6.0.5}$$

Or equally setting the Toth charge to $-\phi^2 m \frac{dx^4}{d\tau}$ where m is the rest mass of the charge carrier, we derive a Lorentz-like law:

$\frac{d^2x^a}{d\tau^2} + (\Gamma^a_{bc} + ((O(f+) + O(\delta+))O(f) + O(f^2)O(\delta+)))\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = (Q_t/m)F^a_b\frac{dx^b}{d\tau} + (O(\delta+) + O(f)O(s+))\frac{dx^b}{d\tau}\frac{dx^4}{d\tau} + (O(\delta+) + O(s+))\frac{dx^4}{d\tau}\frac{dx^4}{d\tau}$

(6.0.6)

We would need to zero most of the orders of magnitude terms here to get the Lorentz force law itself. This however would be the same constraints as the null solutions of Kaluza's original theory. That is, Kaluza's original theory is suggestive of a link between Toth and Maxwellian charge by making the Lorentz force law for Toth charge apparent. This is assumed as required via the Charge Ansatz. The order of magnitude terms on the left can all be removed as less significant than the $O(v+)$ Christoffel symbol elements in general.

$\frac{d^2x^a}{d\tau^2} + \Gamma^a_{bc}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = (Q_t/m)F^a_b\frac{dx^b}{d\tau} + (O(\delta+) + O(f)O(s+))\frac{dx^b}{d\tau}\frac{dx^4}{d\tau} + (O(\delta+) + O(s+))\frac{dx^4}{d\tau}\frac{dx^4}{d\tau}$

(6.0.7)

One other immediate observation is that throwing away the first $O(\delta+)$ term would simplify (6.0.7). This could be done by KCC automatically, or RCC with provisos.

We might also simplify it without RCC, guessing explicitly rather than implicitly the applicability of the same constraint to get rid of the first $O(\delta+)$ term, calling the result the Weak Cylinder Condition (WCC):

$$O(\delta+) << O(f+) \qquad \text{[WCC] (6.0.8)}$$

The result in any case is:

$\frac{d^2x^a}{d\tau^2} + \Gamma^a_{bc}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = (Q_t/m)F^a_b\frac{dx^b}{d\tau} + O(f)O(s+)\frac{dx^b}{d\tau}\frac{dx^4}{d\tau}$
$+ (O(\delta+) + O(s+))\frac{dx^4}{d\tau}\frac{dx^4}{d\tau}$

(6.0.9)

We can then apply the following reasonable constraint for both RCC and WCC, in fact we have to apply it. It is reasonable and necessary in that without it we can not obtain the Lorentz force law, and in any case it is useful to have an equation that defines better the required electromagnetic limit:

$$O(f)O(s+) << O(f+) \tag{6.0.10}$$

To get:

$$\frac{d^2x^a}{d\tau^2} + \Gamma^a_{bc}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = (Q_t/m)F^a_b\frac{dx^b}{d\tau} + (O(\delta+) + O(s+))\frac{dx^4}{d\tau}\frac{dx^4}{d\tau} \tag{6.0.11}$$

So for the two variant cylinder conditions investigated here, there is only one term which strays from a formal equivalence to the experimentally valid Lorentz force law. We must however remember that actual equivalence depends on the important difference between Toth and Maxwell charges, so we make use of the Charge Ansatz and set B_v appropriately.

This is the $(O(\delta+) + O(s+))$ term:

$-\hat{\Gamma}^c_{44} = -\frac{1}{2}\hat{g}^{\mathrm{cD}}(\delta_4\hat{g}_{4D} + \delta_4\hat{g}_{4D} - \delta_D\hat{g}_{44}) = O(\delta+) + \frac{1}{2}g^{\mathrm{cd}}\delta_d\phi^2 = O(\delta+) + O(s+)$
$= -g^{\mathrm{cD}}\delta_4 g_{4D} + \frac{1}{2}g^{\mathrm{cD}}\delta_D\phi^2$
$= -g^{\mathrm{cd}}\delta_4 g_{4d} - \frac{1}{2}g^{\mathrm{c4}}\delta_4\phi^2 + \frac{1}{2}g^{\mathrm{cd}}\delta_d\phi^2$

(6.0.12)

Whether using RCC or WCC:

$$\frac{d^2x^a}{d\tau^2} + \Gamma^a_{bc}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = (Q_t/m)F^a_b\frac{dx^b}{d\tau} - (g^{\mathrm{cD}}\delta_4 g_{4D} - \frac{1}{2}g^{\mathrm{cD}}\delta_D\phi^2)\frac{dx^4}{d\tau}\frac{dx^4}{d\tau}$$

[LFL1] (6.0.13)

Note that however we constrain the size of the Kaluza dimension with RCC we can not eliminate the O(s+) term $\frac{1}{2}g^{\mathrm{cd}}\delta_d\phi^2$ unless we discard the scalar field fluctuations also. We therefore keep all the relevant terms in hope that we may later somehow cancel them out. In this way an exhaustive a search as possible is undertaken. It follows however from the preceding that we most likely have:

$$O(\delta+) <<<< O(f+) \tag{6.0.14}$$

$$O(s+) <<<< O(f+) \tag{6.0.15}$$

Where the $<<<<$ symbol expresses the extra extreme constraint imposed by the uncancellable and inescapably large $\frac{dx^4}{d\tau}$ term. Or an equivalent formulation with averages using the RCC. It might be noted that this extra strong condition starts to make the RCC condition look moot, as we have here a condition that seems to suggest WCC is needed in anycase. However there is an alternative. The alternative, whether RCC is used or not, is a more specialized contraint:

$$O(\hat{\Gamma}^c_{44}) <<<< O(f+) \tag{6.0.16}$$

To be precise about the $<<<<$ symbol: $O(X) <<<< O(Y)$ is such that $O(X) << O(Y)\frac{dx^a}{d\tau}/\frac{dx^4}{d\tau}$ for some smallest electron velocity $\frac{dx^a}{d\tau}$ at which the Lorentz force law works, ie has been tested to be accurate. For $\frac{dx^a}{d\tau}/\frac{dx^4}{d\tau}$ to be equal to $O(f)$, where $O(f) = R_c$ as estimated previously, requires $\frac{dx^a}{d\tau} = 10^6$. But the Lorentz force law certainly works for lower proper velocities than this! Using the very low $O(f) = R_w$ we are not much better off with $\frac{dx^a}{d\tau} = 10^2$. We have $O(X) <<<< O(f+)$ as stronger than $O(X) << O(f+)O(f)$ by some margin. It can't quite be made to fit into $<< O(f+)O(f)$, it remains a tighter constraint even than that.

Under such constraints however we have the Lorentz force law proper:

$$\frac{d^2x^a}{d\tau^2} + \Gamma^a_{bc}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} \approx (Q_t/m)F^a_b \qquad \text{[LFL2] (6.0.17)}$$

We may note that 6.0.15 appears to be highly non-duality invariant, as for example the original Kaluza theory is when the scalar field is set constant.

7 Introducing Torsion

7.1 The Basic Equations

Later Cartan torsion will be admitted leading to a Kaluza-Cartan space-time. But only after showing the failure of the other routes considered. In this way any introduction of the additional complexity of torsion will be empirically necessary. Empirically necessary, that is, provided that its addition solves the problems not otherwise solved. This is done in a later section.

For both 5D and 4D manifolds (i.e. dropping the hats and indices notation for a moment), torsion will be introduced into the Christoffel symbols as follows, using the notation of Hehl [11]. Metricity of the torsion tensor will be assumed [19], the reasonableness of which (in the context of general relativity with torsion) is argued for in [20] and [21]:

$$\frac{1}{2}(\Gamma^k_{ij} - \Gamma^k_{ji}) = S_{ij}{}^k \tag{7.1.1}$$

This relates to the notation of Kobayashi and Nomizu [12] and Wald [7] as follows:

$$T^i{}_{jk} = 2S_{jk}{}^i \equiv \Gamma^i_{jk} - \Gamma^i_{kj} \tag{7.1.2}$$

We have the contorsion tensor $K_{ij}{}^k$ [11] as follows, and a number of relations [11]:

$$\Gamma^k_{ij} = \frac{1}{2} g^{kd}(\partial_i g_{dj} + \partial_j g_{di} - \partial_d g_{ij}) - K_{ij}{}^k = \{^{\,k}_{ij}\} - K_{ij}{}^k \tag{7.1.3}$$

$$K_{ij}{}^k = -S_{ij}{}^k + S_j{}^k{}_i - S^k{}_{ij} = -K_i{}^k{}_j \tag{7.1.4}$$

With torsion included, the geodesic/auto-parallel equation becomes [11]:

$$\frac{d^2x^k}{ds^2} + \Gamma^k_{(ij)} \frac{dx^i}{ds} \frac{dx^j}{ds} = 0 \tag{7.1.5}$$

$$\Gamma^k_{(ij)} = \{^{\,k}_{ij}\} + S^k{}_{(ij)} - S_{(j}{}^k{}_{i)} \tag{7.1.6}$$

This leaves the Lorentz force law unchanged when torsion is *completely* antisymmetric, because the geodesics equation is unchanged. Also auto-parallels and extremals remain identifiable [11].

Antisymmetry of Torsion Ansatz: total antisymmetry of the torsion tensor is to be assumed in the sense that $S_{(ij)}{}^k = S_{(i}{}^k{}_{j)} = S^k{}_{(ij)} = 0$

It is noted that Einstein-Cartan theory, that adds torsion to the dynamics of relativity theory is most probably a minimal ω-consistent extension of general relativity [13][14] and therefore the use of (completely antisymmetric [19]) torsion is not only natural, but arguably a necessity on philosophical and physical grounds. What we have here is better termed Kaluza-Cartan theory as it takes Kaluza's theory and adds torsion. Further justification for using completely antisymmetric torsion is given in [19] where complete anti-symmetry is shown to be a corollary of metricity. This argument can also be inverted so that we also have an argument, albeit a loose suggestive one, for metricity in terms of the need for complete anti-symmetry.

Torsion will be bounded in order of magnitude by the Christoffel symbols, that is, torsion will be given a $O(v+)$ bound on magnitude. We might consider the further constraint that ensures the non-dominance of torsion in the Christoffel symbols (and similarly extended derivatives, with the same justification):

$$O(\hat{S}_{AB}{}^C) \leq O(\hat{\Gamma}^C_{AB} + \hat{K}_{AB}{}^C) \quad \forall A, B \text{ and } C \in \{\text{a,b,c,4}\} \tag{7.1.7}$$

7.2 A Brief Consistency Check

Having secured the Lorentz force law it is worth looking also at the assumption that the velocity of the charge does not change. Whilst under the charge ansatz momentum is indeed conserved, it is still necessary to show that lack of proper acceleration of the charged particle in the Kaluza dimension is feasible: that such a model has a chance of being consistent. We therefore look at this acceleration as with the Lorentz force law:

$$0 = \frac{d^2x^4}{d\tau^2} + \hat{\Gamma}^4_{(BC)} \frac{dx^B}{d\tau} \frac{dx^C}{d\tau} \tag{7.2.1}$$

Once again by complete antisymmetry we have no change in the geodesic equation due to torsion. Ignoring torsion, we have:

$$\begin{aligned} 0 &= \frac{d^2x^4}{d\tau^2} + \hat{\Gamma}^4_{BC} \frac{dx^B}{d\tau} \frac{dx^C}{d\tau} \\ &= \frac{d^2x^4}{d\tau^2} + \hat{\Gamma}^4_{bc} \frac{dx^b}{d\tau} \frac{dx^c}{d\tau} + \hat{\Gamma}^4_{4c} \frac{dx^4}{d\tau} \frac{dx^c}{d\tau} + \hat{\Gamma}^4_{b4} \frac{dx^b}{d\tau} \frac{dx^4}{d\tau} + \hat{\Gamma}^4_{44} \frac{dx^4}{d\tau} \frac{dx^4}{d\tau} \end{aligned} \tag{7.2.2}$$

The last term, as with the Lorentz force law, vanishes only for sufficiently small $O(s+)$ and $O(\delta+)$ or if we make it a special constraint.

$2\hat{\Gamma}^4_{4c} \frac{dx^4}{d\tau} \frac{dx^c}{d\tau}$ - the two middle terms added together, give:

$$-(A^d F_{cd} + O(f)O(\delta+) + O(f)O(s+) + \hat{g}^{44}\partial_c\phi^2)\ \frac{dx^4}{d\tau} \frac{dx^c}{d\tau}$$

That is significantly smaller than the comparable term in the Lorentz force law. That is, given small enough $O(s+) << O(f+)$ for the last term and sufficiently small $O(\delta+)$. Thus, if $\frac{dx^c}{d\tau}$ is small relative to $\frac{dx^4}{d\tau}$ (so that we can also discount the first Christoffel symbol terms) we have that the 5th dimensional acceleration of the charge is small relative to any Lorentz forces.

Here we have additional reason for asserting complete anti-symmetry of torsion.

7.3 Belinfante-Rosenfeld Stress-Energy Tensor

The Einstein tensor defined using a torsion bearing connection will be labelled $\kappa\hat{P}$, it need not be symmetric. The constant is included here only because of the Gravitational constant, to be consistent with the literature. $\hat{P}$ is the Einstein-Cartan stress-energy or canonical energy-momentum tensor.

The Belinfante-Rosenfeld [12] stress-energy tensor $\hat{B}$ is a symmetric adjustment of $\hat{P}$ that adjusts for spin currents as sources. It can be defined equally for the 5D case. It is the torsion equivalent according to Belinfante and Rosenfeld of the original Einstein tensor $\hat{G}$ [12] but is formed explicitly from the torsion bearing connection using $\hat{P}$ and adjustments.

7.4 The Generalized Bel Super-Energy Tensor

Taking torsion as part of the model, that is, given a Kaluza-Cartan space-time, we can here investigate tentatively a way to look at the energy conditions of general relativity. Noting that the use of torsion has not yet been justified, but some dependent arguments can in any case be made prior.

The Generalized Bel tensor for a Lorentz manifold is the super-energy tensor associated with the Riemannian curvature [17]. The definition of super-energy tensor does not require that torsion be null in either the connection or any of the defining tensors [17], and the important dominant super-energy property [17] follows in all cases. This leads to the causality of the Rieman tensor [16] under specific conditions. The torsion is involved as an input in defining the Riemannian curvature, but the properties of the connection with torsion are not invoked to obtain these results. However, the super-energy tensor definition does depend on the anti-symmetries of the Riemannian tensor definition, that is, [17], that it is a double symmetric (2,2)-form. The double symmetric property playing a secondary role is not involved in this section. Whereas the definition of the super-energy tensor in terms of very elementary properties such as it being a 4-tensor are dependent on admissibility of the interpretation of the Riemann tensor as a (2,2)-form.

In [16] the derivation of the causality of the fields underlying any particular super-energy tensor is given in terms of the divergence of the field's super-energy tensor. A divergence condition is given that ensures causality of the underlying field associated with any such super-energy tensor. The divergence of the generalized Bel tensor would therefore need to be bounded by this condition if the Riemannian curvature were to remain causal. This condition is theorem 4.2 in [17].

A sufficient case would be if the divergence of the superenergy tensor were 0 (and assuming global hyperbolicity). The important details are on page 4 of [16]. The argument does not require that the connection be torsion free. Thus the null divergence of the generalized Bel tensor would yield causal Riemannian curvature, assuming the Riemann tensor remained a (2,2)-form. On p24 of [17] we have a calculation of this divergence in the torsion-free case, and it can be seen that when the Ricci curvature is 0 that the divergence of super-energy is also always 0. This however references symmetry properties (in addition to antisymmetry properties) and thus further consideration of the case with torsion would be required to extend or generalize this theorem. Theorem 6.1 on p25 of [17] may well not apply in the presence of torsion. Nevertheless it nicely characterizes an important property of the Kaluza vacuum (or let us say the Kaluza-Cartan vacuum in the absence of torsion), that it can not be a source of Bel super-energy (allowing 'generalized' to be dropped from here on).

This shows the reasonableness of Bel super-energy as a controlling and limiting function of any possible separation of negative and positive energies within matter models: thus hypothetically bounding negative energy within a causal construction, and limiting its presence, leading to the approximate validity of the energy conditions. The constraint could therefore be generalized to matter/charge models and also to any areas where torsion is present to provide exactly the limiting constraints characteristic of energy conditions:

The Conserved Bel Hypothesis will be that the divergence of the Generalized Bel superenergy tensor be null (when defined with respect to the torsion connection and torsion containing tensors) over all of the Kaluza-Cartan space-time. Thus including matter/charge models and torsion.

We might also have *the nearly Conserved Bel Hypothesis* such that the divergence of the Generalized Bel superenergy tensor be bounded (when defined with respect to the torsion connection and torsion containing tensors) over all of the Kaluza-Cartan space-time. Thus including matter/charge models and torsion. The bound here being defined by theorem 4.2 of [17]. A nearly conserved Bel tensor, if shown to be necessary theoretically or empirically, could then be indicative of further refinements or extensions of the space-time model.

These 2 tentative suggestions would have explanatory power if shown to be correct as a way of rationalizing the various energy conditions used in general relativity, and in explaining some aspects of classical scale causality. This will now be left to one side until the concluding sections. Considerable work would need to be done on these hypotheses to bring them actively into play.

It can be noted that in 4D and 5D in particular p29 of [17] the generalized Bel tensor (torsion not mentioned, so presumably it may not be true in the general case) has the nice property of being completely symmetric. It is curious that it should be completely symmetric precisely in the 4D and 5D cases.

Finally, as stated, the above tentative ideas require that the Riemann tensor be a (2,2)-form in the sense of Senovilla [17] also with torsion. Now the Riemann tensor can be written as [12]:

$$R^i_{jkl} = \partial_k \Gamma^i_{lj} - \partial_l \Gamma^i_{kj} + \Gamma^m_{lj}\Gamma^i_{km} - \Gamma^m_{kj}\Gamma^i_{lm} \tag{7.4.1}$$

It is a (2,2)-form if its antisymmetries are as follows: $R_{[ij][kl]}$. This is clearly the case for [k,l]. For [i,j] it is a known result when the connection is metric, even with torsion.

This section is dependent on a presumed Riemannian-Cartan geometry, a Kaluza-Cartan space-time in particular, admitting (completely antisymmetric) torsion. Why this is preferable to varying the cylinder conditions will now be investigated in detail.

8 Analysing Alternative Field Equations

Alternative field equations, or constraints on them at least, under varying cylinder conditions will now be investigated in order to identify where we might find non-null electromagnetic fields within Kaluza theory. Noting that we are not considering matter models here, but the electrovacuum, or electroscalar 'Kaluza' vacuum.

The field equations in the introduction are related to the results here, but here we do not assume KCC and investigate the effect of the other possible cylinder conditions, thus the Kaluza field equations can not be assumed. This investigation is done by setting the Ricci tensor to zero and looking at its components via the Christoffel symbols. Initially torsion is not invoked, as it is the effect of varying the cylinder conditions alone that we are initially interested in.

8.1 The Christoffel Symbols

(Without torsion terms)

$$2\hat{\Gamma}^A_{BC} = \sum_d \hat{g}^{AD}(\partial_B \hat{g}_{CD} + \partial_C \hat{g}_{DB} - \partial_D \hat{g}_{BC})$$

$$= \sum_d \hat{g}^{Ad}(\partial_B \hat{g}_{Cd} + \partial_C \hat{g}_{dB} - \partial_d \hat{g}_{BC}) + \hat{g}^{A4}(\partial_B \hat{g}_{C4} + \partial_C \hat{g}_{4B} - \partial_4 \hat{g}_{BC})$$

$$\begin{aligned}2\hat{\Gamma}^A_{bc} = &\sum_d \hat{g}^{Ad}(\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc}) \\ &+ \sum_d \hat{g}^{Ad}(\partial_b \phi^2 A_c A_d + \partial_c \phi^2 A_d A_b - \partial_d \phi^2 A_b A_c) \\ &+ \hat{g}^{A4}(\partial_b \phi^2 A_c + \partial_c \phi^2 A_b - \partial_4 g_{bc} - \partial_4 \phi^2 A_b A_c)\end{aligned}$$

$$2\hat{\Gamma}^A_{4c} = \sum_d \hat{g}^{Ad}(\partial_4 g_{cd} + \partial_4 \phi^2 A_c A_d + \partial_c \phi^2 A_d - \partial_d \phi^2 A_c) + \hat{g}^{A4}\partial_c \phi^2$$

$$2\hat{\Gamma}^A_{44} = 2\sum_d \hat{g}^{Ad}\partial_4 \phi^2 A_d - \sum_d \hat{g}^{Ad}\partial_d \phi^2 + \hat{g}^{A4}\partial_4 \phi^2$$

The Strong Electromagnetic Limit $\boldsymbol{\phi^2 = 1}$

$$\begin{aligned}2\hat{\Gamma}^A_{bc} &= \sum_d \hat{g}^{Ad}(\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc}) + \sum_d \hat{g}^{Ad}(\partial_b A_c A_d + \partial_c A_d A_b - \partial_d A_b A_c) \\ &\quad + \hat{g}^{A4}(\partial_b A_c + \partial_c A_b - \partial_4 g_{bc} - \partial_4 A_b A_c) \\ 2\hat{\Gamma}^A_{4c} &= \sum_d \hat{g}^{Ad}(\partial_4 g_{cd} + \partial_4 A_c A_d + \partial_c A_d - \partial_d A_c) \\ \hat{\Gamma}^A_{44} &= \sum_d \hat{g}^{Ad}\partial_4 A_d\end{aligned}$$

Simplifying...

$$\begin{aligned}2\hat{\Gamma}^a_{bc} &= 2\Gamma^a_{bc} + \sum_d g^{ad}(A_b F_{cd} + A_c F_{bd}) + A^a \partial_4 g_{bc} + A^a \partial_4 A_b A_c \\ 2\hat{\Gamma}^4_{bc} &= -\sum_d A^d(\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc}) - \sum_d A^d(A_b F_{cd} + A_c F_{bd}) \\ &\quad - (1 + \sum_i A_i A^i)(\partial_4 g_{bc} + \partial_4 A_b A_c) + (\partial_b A_c + \partial_c A_b) \\ 2\hat{\Gamma}^a_{4c} &= \sum_d g^{ad}(\partial_4 g_{cd} + \partial_4 A_c A_d) + \sum_d g^{ad} F_{cd} \\ 2\hat{\Gamma}^4_{4c} &= -\sum_d A^d(\partial_4 g_{cd} + \partial_4 A_c A_d) - \sum_d A^d F_{cd} \\ \hat{\Gamma}^a_{44} &= \sum_d g^{ad}\partial_4 A_d \\ \hat{\Gamma}^4_{44} &= -\sum_d A^d \partial_4 A_d\end{aligned}$$

The Strong Scalar Limit $A_i = 0$

$$\begin{aligned}2\hat{\Gamma}^A_{bc} &= \sum_d \hat{g}^{Ad}(\partial_b g_{cd} + \partial_c g_{ab} - \partial_d g_{bc}) - \hat{g}^{A4}\partial_4 g_{bc} \\ 2\hat{\Gamma}^A_{4c} &= \sum_d \hat{g}^{Ad}\partial_4 g_{cd} + \hat{g}^{A4}\partial_c \phi^2 \\ 2\hat{\Gamma}^A_{44} &= -\sum_d \hat{g}^{Ad}\partial_d \phi^2 + \hat{g}^{A4}\partial_4 \phi^2\end{aligned}$$

Simplifying...

$\hat{\Gamma}^a_{bc} = \Gamma^a_{bc}$
$2\hat{\Gamma}^4_{bc} = -\frac{1}{\phi^2}\partial_4 g_{bc}$
$2\hat{\Gamma}^a_{4c} = \sum_d g^{ad}\partial_4 g_{cd}$
$2\hat{\Gamma}^4_{4c} = \frac{1}{\phi^2}\partial_c \phi^2$
$2\hat{\Gamma}^a_{44} = -\sum_d g^{ad}\partial_d \phi^2$
$2\hat{\Gamma}^4_{44} = \frac{1}{\phi^2}\partial_4 \phi^2$

8.2 Constraints On The Ricci tensor

(Including torsion as necessary)

5D Ricci curvature $\hat{R}_{AB} = 0$ (outside of matter models) produces the following:

$$\hat{R}_{ab} = \partial_C \hat{\Gamma}^C_{ab} - \partial_b \hat{\Gamma}^C_{aC} + \hat{\Gamma}^C_{ab}\hat{\Gamma}^D_{CD} - \hat{\Gamma}^C_{aD}\hat{\Gamma}^D_{bC} = 0 \qquad (8.2.1)$$

$$R_{ab} = \partial_c \Gamma^c_{ab} - \partial_b \Gamma^c_{ac} + \Gamma^c_{ab}\Gamma^d_{cd} - \Gamma^c_{ad}\Gamma^d_{bc} \qquad (8.2.2)$$

$$\begin{aligned} R_{ab} = R_{ab} - \hat{R}_{ab} = {} & (\partial_c \Gamma^c_{ab} - \partial_c \hat{\Gamma}^c_{ab}) - \partial_4 \hat{\Gamma}^4_{ab} + (-\partial_b \Gamma^c_{ac} + \partial_b \hat{\Gamma}^c_{ac}) + \partial_b \hat{\Gamma}^4_{a4} \\ & + (\Gamma^c_{ab}\Gamma^d_{cd} - \hat{\Gamma}^c_{ab}\hat{\Gamma}^d_{cd}) - \hat{\Gamma}^c_{ab}\hat{\Gamma}^4_{c4} - \hat{\Gamma}^4_{ab}\hat{\Gamma}^4_{44} - \hat{\Gamma}^4_{ab}\hat{\Gamma}^d_{4d} \\ & + (-\Gamma^c_{ad}\Gamma^d_{bc} + \hat{\Gamma}^c_{ad}\hat{\Gamma}^d_{bc}) + [\hat{\Gamma}^c_{a4}\hat{\Gamma}^4_{bc} + \hat{\Gamma}^4_{ad}\hat{\Gamma}^d_{b4}] + \hat{\Gamma}^4_{a4}\hat{\Gamma}^4_{b4} \end{aligned}$$

$$= -\partial_4 \hat{\Gamma}^4_{ab} + \partial_b \hat{\Gamma}^4_{a4} - \hat{\Gamma}^c_{ab}\hat{\Gamma}^4_{c4} - \hat{\Gamma}^4_{ab}\hat{\Gamma}^4_{44} - \hat{\Gamma}^4_{ab}\hat{\Gamma}^c_{4c} + [\hat{\Gamma}^c_{a4}\hat{\Gamma}^4_{bc} + \hat{\Gamma}^c_{b4}\hat{\Gamma}^4_{ac}] + \hat{\Gamma}^4_{a4}\hat{\Gamma}^4_{b4}$$

([SEM1] 8.2.3)

By inspection of the above (without torsion), we have that $\partial_b \hat{\Gamma}^4_{a4}$ is symmetric. This symmetry however follows from the Christoffel symbol definition: the symmetry presents no constraint. We have simply 10 constraints on 10 unknowns, which corresponds to the 4D field equations, field equation 1 in the original Kaluza theory.

[SEM1] will not be used here to draw conclusions about the validity of WCC or RCC. We simply note that it defines the 4D stress energy tensor.

Further possible constraints on the field equations may be obtained by inspecting the other components of $\hat{R}_{AB}$:

$$\hat{R}_{44} = \partial_C \hat{\Gamma}^C_{44} - \partial_4 \hat{\Gamma}^C_{4C} + \hat{\Gamma}^C_{44}\hat{\Gamma}^D_{CD} - \hat{\Gamma}^C_{4D}\hat{\Gamma}^D_{4C} = 0 \qquad ([SEM2]\ 8.2.4)$$

$$\hat{R}_{a4} = \partial_C \hat{\Gamma}^C_{a4} - \partial_4 \hat{\Gamma}^C_{aC} + \hat{\Gamma}^C_{a4}\hat{\Gamma}^D_{CD} - \hat{\Gamma}^C_{aD}\hat{\Gamma}^D_{4C} = 0 \qquad ([SEM3]\ 8.2.5)$$

The first equation $[SEM2]8.2.4$ means that all electromagnetic fields must be null when there is no torsion, no scalar field, nor a physical Kaluza dimension, ie at the Kaluza limit, as will become clear in the following (Related to Kaluza's third field equation). The second relates toand Kaluza's second field equation. Initially we include the torsion dependent terms, but take advantage of complete antisymmetry, so that the working can be re-used later.

$$\begin{aligned}
&\hat{R}_{44} = 0 = \partial_c \hat{\Gamma}^c_{44} + \partial_4 \hat{\Gamma}^4_{44} - \partial_4 \hat{\Gamma}^c_{4c} - \partial_4 \hat{\Gamma}^4_{44} + \hat{\Gamma}^C_{44} \hat{\Gamma}^D_{CD} - \hat{\Gamma}^C_{4D} \hat{\Gamma}^D_{4C} \\
&= \partial_c \hat{\Gamma}^c_{44} - \partial_4 \hat{\Gamma}^c_{4c} + \hat{\Gamma}^c_{44} \hat{\Gamma}^D_{cD} + \hat{\Gamma}^4_{44} \hat{\Gamma}^D_{4D} - \hat{\Gamma}^c_{4D} \hat{\Gamma}^D_{4c} - \hat{\Gamma}^4_{4D} \hat{\Gamma}^D_{44} \\
&= \partial_c \hat{\Gamma}^c_{44} - \partial_4 \hat{\Gamma}^c_{4c} + \hat{\Gamma}^c_{44} \hat{\Gamma}^d_{cd} + \hat{\Gamma}^4_{44} \hat{\Gamma}^d_{4d} - \hat{\Gamma}^c_{4d} \hat{\Gamma}^d_{4c} - \hat{\Gamma}^4_{4d} \hat{\Gamma}^d_{44} - \hat{\Gamma}^c_{44} K_{c4}{}^4 + \hat{\Gamma}^c_{44} K_{4c}{}^4 \\
&= \partial_c \hat{\Gamma}^c_{44} - \partial_4 \hat{\Gamma}^c_{4c} + \hat{\Gamma}^c_{44} \hat{\Gamma}^d_{cd} + \hat{\Gamma}^4_{44} \hat{\Gamma}^d_{4d} - \hat{\Gamma}^c_{4d} \hat{\Gamma}^d_{4c} - \hat{\Gamma}^4_{4d} \hat{\Gamma}^d_{44} + 2\hat{\Gamma}^c_{44} \hat{S}_{c4}{}^4
\end{aligned} \tag{8.2.6}$$

By complete antisymmetry of the torsion tensor:

$$0 = \partial_c \hat{\Gamma}^c_{44} - \partial_4 \hat{\Gamma}^c_{4c} + \hat{\Gamma}^c_{44} \hat{\Gamma}^d_{cd} + \hat{\Gamma}^4_{44} \hat{\Gamma}^d_{4d} - \hat{\Gamma}^c_{4d} \hat{\Gamma}^d_{4c} - \hat{\Gamma}^4_{4d} \hat{\Gamma}^d_{44} \tag{8.2.7}$$

At the Kaluza limit we have nullity as follows, from the fifth term on the right-hand side:

$$0 = -F_{cd} F^{dc}$$

Now we look at the other terms not at the Kaluza limit (but either without torsion, or torsion present but bound by 7.1.7), and using (6.0.14) and (6.0.15), or imposing the additional (somewhat arbitrary) constraint of (6.0.16). (6.0.16) is in any case a corollary of (6.0.14) and (6.0.15) and so can be used liberally here. Whether using WCC, or RCC, we can discount all but the second and the fifth term as follows:

$0 = \partial_c \hat{\Gamma}^c_{44}$ is of order $<< O(f+)O(f+)$ by a reasonable use of distributivity
$-\partial_4 \hat{\Gamma}^c_{4c}$
$+[<< O(f+)O(f)O(v+)]$ by (6.0.16)
$+[O(\delta+) + O(s+)O(f)\ + O(f+)O(\delta+)][O(\delta+) + O(f+) + O(f+)O(s+)]$
$+[F_{cd} F^{cd} + O(f)O(s+)O(f)O(s+)]$
$+[<< O(f+)O(f)O(v+)]$ by (6.0.16)

Now, it is necessary to assume a degree of proportionality to make: $O(f)$ $O(v+) \leq O(f+)$, but not a lot, as full proportionality would allow: $O(f)O(v+)=$ $O(f+)O(v)$. This is reasonable as $O(v+)/O(v) \leq O(f+)/O(f)$ means that percentage gains of the electromagnetic potential are generally greater than or equal to those of the gravitational potential. Similarly we need $O(f)$ $O(s+)$ $<< O(f+)$ for part of the fourth and fifth terms, but this very order of magnitude bound (6.0.10) was necessary to derive the Lorentz force law, and so can be used also here to give:

$$\partial_4 \hat{\Gamma}^c_{4c} = F_{cd} F^{cd} \tag{8.2.8}$$

A possible weak link in this derivation is perhaps the reasonable use of distributivity in the first derivative term. So to clarify we should consider what would happen if the term oscillated such that distributivity/proportionality did not hold. The underlying Christoffel symbol term is $<< O(f+)O(f) = O(f+)O(f)/C$ where $C >> 1$ is a measure of how well the Lorentz force law has been tested or is assumed valid. In order for its derivative to be $O(f+)O(f+)$ we must have a gain at least C times longer than a typical O(f+)O(f) term. Such a typical term, for comparison, being an electromagnetic field times a potential. To the extent that this gain is continued for such a time-distance that it is significant in the above equation, we must have a proportionately higher frequency (or shorter wavelength) of oscillation so as to maintain the Christoffel symbol term within its prescribed bounds. As such the net gain in one direction of the higher frequency term will be limited. Thus we have a crude heuristic argument that leads to a contradiction. In this sense proportionality is reasonable here.

We can now look at the equation above and claim that using WCC we can only have 0, or else we have RCC. If we do not have 0 we would need the Christoffel symbol term to average 0 around the loop in some way: but if we are doing such an averaging process we are really invoking RCC. Thus WCC leads to RCC. And under RCC point (i) can be taken as 0.

We have therefore shown the insufficiency of trying to weaken the Kaluza cylinder condition.

8.3 Non-Null Solutions And Degrees of Freedom

At this point we can put the torsion tensor back in (with respect to the significant orders of magnitude) as follows, assuming the additional constraint of (7.1.7) for simplification:

$$\begin{aligned}
0 &= \partial_4 \hat{\Gamma}^c_{4c} = \partial_4 \widehat{\{{}^{\,c}_{4c}\}} - \partial_4 \hat{K}_{4c}{}^{c} = -[g^{ce}(F_{de}) - \hat{K}_{4d}{}^{c}][g^{df}(F_{cf}) - \hat{K}_{4c}{}^{d}] \\
&= -[(F^c_d) - \hat{K}_{4d}{}^{c}][(F^d_c) - \hat{K}_{4c}{}^{d}] \\
&= -F_{cd}F^{cd} + 2\hat{K}_{4d}{}^{c} F^d_c - \hat{K}_{4d}{}^{c} \hat{K}_{4c}{}^{d} \quad \text{where all terms are O(f+)O(f+)}
\end{aligned}$$

([SEM2b] 8.3.1)

In order to check that we do indeed have enough degrees of freedom to allow for non-null fields, $[SEM3]$8.2.5 will now be investigated using the same assumptions.

$$\hat{R}_{a4} = \partial_C \hat{\Gamma}^C_{a4} - \partial_4 \hat{\Gamma}^C_{aC} + \hat{\Gamma}^C_{a4}\hat{\Gamma}^D_{CD} - \hat{\Gamma}^C_{aD}\hat{\Gamma}^D_{4C} = 0 \qquad \text{([SEM3] 8.3.2)}$$

Torsion terms will be assumed present in the Christoffel symbols of both 4 and 5 dimensions. Terms strictly less than $O(f++)$ will be discarded.

$$\begin{aligned}\hat{R}_{a4} &= \partial_c \hat{\Gamma}^c_{a4} + \partial_4 \hat{\Gamma}^4_{a4} - \partial_4 \hat{\Gamma}^c_{ac} - \partial_4 \hat{\Gamma}^4_{a4} + \hat{\Gamma}^c_{a4}\hat{\Gamma}^D_{cD} + \hat{\Gamma}^4_{a4}\hat{\Gamma}^D_{4D} - \hat{\Gamma}^c_{aD}\hat{\Gamma}^D_{4c} - \hat{\Gamma}^4_{aD}\hat{\Gamma}^D_{44} \\ &= \partial_c \hat{\Gamma}^c_{a4} - \partial_4 \hat{\Gamma}^c_{ac} + \hat{\Gamma}^c_{a4}\hat{\Gamma}^d_{cd} + \hat{\Gamma}^c_{a4}\hat{\Gamma}^4_{c4} + \hat{\Gamma}^4_{a4}\hat{\Gamma}^d_{4d} - \hat{\Gamma}^c_{ad}\hat{\Gamma}^d_{4c} - \hat{\Gamma}^c_{a4}\hat{\Gamma}^4_{4c} - \hat{\Gamma}^4_{ad}\hat{\Gamma}^d_{44} \\ &= \partial_c \hat{\Gamma}^c_{a4} - \partial_4 \hat{\Gamma}^c_{ac} + \hat{\Gamma}^c_{a4}\hat{\Gamma}^d_{cd} + \hat{\Gamma}^4_{a4}\hat{\Gamma}^d_{4d} - \hat{\Gamma}^c_{ad}\hat{\Gamma}^d_{4c} - \hat{\Gamma}^4_{ad}\hat{\Gamma}^d_{44} + \hat{\Gamma}^c_{a4}(\hat{K}_{4c}{}^4 - \hat{K}_{c4}{}^4)\end{aligned}$$

Removing small terms due to Postulates, and assuming 7.1.7:

$$\hat{R}_{a4} = 0 = \partial_c \hat{\Gamma}^c_{a4} + \hat{\Gamma}^c_{a4}\hat{\Gamma}^d_{cd} - \hat{\Gamma}^c_{ad}\hat{\Gamma}^d_{4c}$$

$$\begin{aligned}\hat{\Gamma}^c_{a4} &= \tfrac{1}{2}[\hat{g}^{cd}(\partial_4 g_{ad} + \partial_4 \phi^2 A_a A_d + \partial_a \phi^2 A_d - \partial_d \phi^2 A_a) + \hat{g}^{c4}\partial_a \phi^2] - \hat{K}_{a4}{}^c \\ \hat{\Gamma}^d_{4c} &= \tfrac{1}{2}[\hat{g}^{de}(\partial_4 g_{ce} + \partial_4 \phi^2 A_c A_e + \partial_c \phi^2 A_e - \partial_e \phi^2 A_c) + \hat{g}^{d4}\partial_c \phi^2] - \hat{K}_{4c}{}^d\end{aligned}$$

$$\begin{aligned}&\partial_c \hat{\Gamma}^c_{a4} = \tfrac{1}{2}\partial_c F^c_a - \partial_c \hat{K}_{a4}{}^c \\ &\hat{\Gamma}^c_{a4}\hat{\Gamma}^d_{cd} = [\tfrac{1}{2}F^c_a - \hat{K}_{a4}{}^c][\Gamma^d_{cd}] \\ &-\hat{\Gamma}^c_{ad}\hat{\Gamma}^d_{4c} = -[\Gamma^c_{ad}][\tfrac{1}{2}F^d_c - \hat{K}_{c4}{}^d]\end{aligned}$$

$$0 = \frac{1}{2}\partial_c F^c_a - \partial_c \hat{K}_{a4}{}^c + [\frac{1}{2}F^c_a - \hat{K}_{a4}{}^c][\Gamma^d_{cd}] - [\Gamma^c_{ad}][\frac{1}{2}F^d_c - \hat{K}_{c4}{}^d] \quad \text{([SEM3b] 8.3.3)}$$

Setting $\hat{K}_{a4}{}^c$ to 0 or to $\frac{1}{2}F^c_a$ does nothing but rederive null solutions via $[SEM2b]$8.3.1 and antisymmetry. These are not the solutions we are looking for. Releasing the additional constraint 7.1.7 also makes no difference as complete antisymmetry dispatches most of the torsion terms in any case. What we have here is 4 constraints on the 6 unknowns of the contorsion tensor where one index is 4. We have of course also $[SEM2b]$8.3.1. Thus there are in fact 5 constraints in total on the 6 unknowns. [SEM1] consists of 10 constraints, but at most these are used in constraining the 10 unknowns of the antisymmetric $\hat{K}_{ab}{}^c$.

Thus we have the degrees of freedom required.

9 Postulates

We have arrived at a 5D Kaluza theory with torsion, that is, where the Ricci flat part of Kaluza space includes the torsion tensor in the defining connection, and total antisymmetry of torsion is assumed. The Charge Ansatz has been made throughout so far, but will be shown in a later section to actually be a corollary of this Kaluza-Cartan theory. The following order of magnitude constraints are imposed on various terms to provide an electromagnetic/classical limit:

$O(\delta+) <<<< O(f+)$
$O(s+) <<<< O(f+)$
$O(\hat{S}_{AB}^{\ C}) \leq O(v+)$

That is: (6.0.14),(6.0.15) and a reasonable constraint on torsion so that it does not dominate curvature.

Other broad assumptions were made and are still needed: the Geodesic Assumption, causality (and the existence of a hypersurface), suitable topology and so on. But these, although needed for the limit of general relativity, are not part of the theory per se. It turns out that this theory is not duality invariant, not only because of matter/charge sources as with general relativity but also due to the scalar field.

Despite the length and complexity of the argument we have in fact arrived at straight forward postulates, and require none of the messiness previously entertained. The froth has boiled down to a few simple ideas. Nor indeed, it has been argued, are there other clearly satisfactory ways forward. An exhaustive a search as possible has been attempted. The above, or substantially similar postulates, appear to be the most natural selection of postulates and resolve the foundational problems of Kaluza theory investigated here (notwithstanding the Charge Ansatz to be dealt with in the sequel). The first two postulates can be weakened by (6.0.16), the second can be tightened by (7.1.7). Further the first one could be a corollary of proportionality, and the third one could result from more general considerations also. Finally, the second one could simply be the electromagnetic limit of a more general case. We might consider the non-duality invariance of the constraint to be a sign of the reality of the scalar field, though this would need further investigation.

The important end result is that the core postulates are simply the addition of torsion to Kaluza theory and a vanishing scalar field at the electromagnetic limit. But a number of peripheral postulates that have a more general nature are also needed.

The outstanding issue is the Charge Ansatz. Until this point no attempt to justify this ansatz has been made, apart from the fact that its assumption has been necessary. This will now be investigated.

10 The Nature Of Charge

In general relativity at a weak field limit the conservation of momentum-energy can be given in terms of the stress-energy tensor as follows [9]. Energy:

$$\frac{\partial \hat{T}^{00}}{\partial t} + \frac{\partial \hat{T}^{i0}}{\partial x^i} = 0 \tag{10.0.1}$$

Momentum in the j direction:

$$\frac{\partial \hat{T}^{0j}}{\partial t} + \frac{\partial \hat{T}^{ij}}{\partial x^i} = 0 \tag{10.0.2}$$

This can be applied equally to Kaluza theory (with matter models that are not Ricci flat in the Kaluza space). It needs to be applied to the underlying (not here Ricci flat) Kaluza space. We have a description of conservation of momentum in the 5th dimension as follows:

$$\frac{\partial \hat{T}^{04}}{\partial t} + \frac{\partial \hat{T}^{i4}}{\partial x^i} = 0 \tag{10.0.3}$$

This is accurate at the weak field limit, and so is valid at the usual classical scale. We also have by the postulates the situation where i=4 can be treated as small. Thus the conservation of 'charge' becomes the property of a 4-vector current at the usual classical scale, which we know to be conserved:

$$V = (\hat{T}^{04}, \hat{T}^{14}, \hat{T}^{24}, \hat{T}^{34}) \tag{10.0.4}$$

$$\partial_0 \hat{T}^{04} + \partial_1 \hat{T}^{14} + \partial_2 \hat{T}^{24} + \partial_3 \hat{T}^{34} = 0 \tag{10.0.5}$$

(V is a vector for the same reasons as the vector potential is a vector.)

At this point we can calculate this 'current' in terms of the metric and Ricci tensor, noting that even with torsion present the original Einstein tensor constructed from the torsion free connection and metric can still be used. We have two connections on one manifold. And two possible Einstein tensors. The Einstein-Cartan stress-energy tensor $\hat{P}$ with torsion may be antisymmetric.

For the moment we shall ignore torsion, and revert to the Einstein tensor $\hat{G}$ and torsion free connection.

We can identify the current with the following components of the 5D (torsion free) Einstein tensor by discounting small terms, and thus consistently with (6.0.5) and the previous derivations of the Lorentz force law:

$$V \approx (\hat{R}^{04} - 1/2\hat{g}^{04}\hat{R}, \hat{R}^{14} - 1/2\hat{g}^{14}\hat{R}, \hat{R}^{24} - 1/2\hat{g}^{24}\hat{R}, \hat{R}^{34} - 1/2\hat{g}^{34}\hat{R}) \tag{10.0.6}$$

$$V \approx (\hat{R}^{04}, \hat{R}^{14}, \hat{R}^{24}, \hat{R}^{34}) \text{ due to O(f) terms in the metric.} \tag{10.0.7}$$

The following parts of the Ricci tensor will be looked at due to its significance $O(f++)$:

$$\hat{X}^{a4} = \partial_C \hat{\Gamma}^{Ca4} - \partial^4 \hat{\Gamma}^{Ca}{}_C \tag{10.0.8}$$

The other part of the Ricci tensor will have a significance $<< O(f++)$ since, being always compounded of two Christoffel symbols, it starts off bounded by $O(v+)O(v+)$. We need only one of the Christoffel symbols to be O(f+) and we have already an order of significance less than $O(f++)$ via distributivity (as defined in section 5). That there is always such a term follows from the fact that at least one of each pair of Christoffel symbols in the remaining part of the Ricci tensor will have an index that is 4, providing there is no torsion. This then also makes insignificant any contribution from the corresponding torsion tensors. The terms are simplified and discarded using the Postulates and distributivity, and by comparing relative significances.

$$\begin{aligned}
&2V \approx 2\hat{X}^{a4} = \partial_C[\hat{g}^C{}_D(\partial^a \hat{g}^{D4} + \partial^4 \hat{g}^{Da} - \partial^D \hat{g}^{a4})] - \partial^4[\hat{g}_{CD}(\partial^a \hat{g}^{DC} + \partial^C \hat{g}^{Da} - \partial^D \hat{g}^{aC})]\\
&\approx \partial_C[\hat{g}^C{}_D(\partial^a \hat{g}^{D4} + \partial^4 \hat{g}^{Da} - \partial^D \hat{g}^{a4})]\\
&\approx \partial_C[\hat{g}^C{}_D(\partial^a \hat{g}^{D4} - \partial^D \hat{g}^{a4})]\\
&\approx \partial_c[\hat{g}^c{}_d(\partial^a \hat{g}^{d4} - \partial^d \hat{g}^{a4})]\\
&\approx \partial_c(\partial^a \hat{g}^{c4} - \partial^c \hat{g}^{a4})\\
&\approx -\partial_c(\partial^a A^c - \partial^c A^a)\\
&\approx -\partial_c F^{ac}
\end{aligned} \tag{10.0.9}$$

This, as the explicit equation of Maxwellian charge sources (albeit approximate), provides justification for the approximate, but consistent, association of Toth charge with Maxwell charge. It is no longer an ansatz as such at all, but now follows from the Postulates, at least given the additional proviso of distributivity.

However we have here ignored the torsion tensor. This doesn't necessarily matter as the sought for property of zero divergence in the (torsion free) Einstein tensor will still present a conservation law, just one relative to the torsion free connection rather than the torsion bearing connection. By using the Belinfante-Rosenfeld procedure [12] however we can derive a tensor, the Belinfante-Rosenfeld tensor, that is indeed conserved under the new torsion connection. The Belinfante-Rosenfeld tensor being equivalent to the Einstein tensor according to [12] as it is precisely the symmetric Hilbert energy-momentum tensor in adjusted form. With the Belinfante-Rosenfeld stress-energy tensor we have indeed a symmetric stress-energy tensor which is explicitly conserved relative to the torsion connection, and that has a momentum portion approximately equal to the charge under the reasonable assumptions used here. Whether an analogous link exists when the torsion terms are directly considered has not been investigated. Nevertheless a convincing link between Maxwellian and Toth charge has been established even without this.

The assumption of distributivity/proportionality was assumed in this section, that is, that the orders of magnitude are well-behaved as discussed in section 5.

11 Conclusion

Kaluza's 1921 theory of gravity and electromagnetism using a fifth wrapped-up spatial dimension is at the root of many modern attempts to develop new physical theories. Lacking non-null electromagnetic fields however the theory is incomplete.

This work sought to investigate the issue of non-null electromagnetic solutions in Kaluza variant theories by the method of an order of magnitude analysis under various assumptions, such as variant cylinder conditions, and making the ansatz that charge can be identified with 5D momentum - the initial justification for which was the ease of derivation of the Lorentz force law under limited circumstances.

The non-null solutions were not found by relaxing the cylinder condition or allowing for scalar fields, and further the Lorentz force law was maintained best with respect to Toth charges by maintaining a tight cylinder condition and very limited scalar field oscillations. This was despite a search that tried to be as exhaustive as possible. Attempts at using both the scalar field and various 5th dimensional oscillations proved ultimately ineffective. And the program to find non-null solutions as a result of these two factors failed.

The derivation of the Lorentz force law was not, however, impaired by the admission of completely antisymmetric torsion. Further, enough degrees of freedom for the sought for non-null solutions were found. And this was achieved without the help of weakening the cylinder conditions or a significant fluctuating scalar field. Explicit examples of non-null solutions, however, were not provided.

We can therefore conclude that when 5D momentum is to be identified with Maxwellian charge, and when there is no (or very weak) scalar field, that the cylinder condition must be more or less as given by Kaluza, and that torsion is necessary to get non-null electromagnetic fields, such as static electric fields. It was also noted that in any case Einstein-Cartan theory is a natural and necessary extension of general relativity via ω-consistency, thus the use of torsion is really very natural, and all the more so with the resulting resolution to the foundational issues of Kaluza theory.

The identification of 5D momentum and charge is shown to be a consequence of the new Postulates developed through the paper, and in particular the presence of torsion that allows the Postulates to offer the full range of electromagnetic solutions. Additional well-behaved properties were required for this identification.

Some theorists investigate relativity theory with torsion, and some theorists investigate Kaluza or Kaluza Klein theories. Here it is shown why it makes sense to investigate both together: why Kaluza theory should have torsion added.

This work has resolved foundational issues associated with classical Kaluza theory and provides motivation for further investigation. Further, the coupling such as it is between the torsion tensor and other tensors (primarily the electromagnetic field) means that there are, in principle, testable phenomena, though the effects may be small.

The overall purpose of the theory is the same as that of Kaluza, to provide an explanation for as much classical electromagnetic phenomena in terms of geometry as possible. The variant theory merges two serious attempts at unifying electromagnetism and gravity. The argument justifies this. It is reasonable to refer to this type of theory as a Kaluza-Cartan theory.

Finally, and tentatively, it was suggested that the generalized Bel superenergy tensor or similar, via the Conserved Bel Hypothesis (or a weaker alternative), could be applied as an alternative to the energy conditions of general relativity in the 5D context. This is because a conservation law (or an approximate conservation law) on the Bel tensor results in the causality of the Riemannian curvature, and this is so even in the presence of negative energies.

12 Aknowledgements

With thanks to Viktor Toth.

13 References

A bibliography of references:

[1] Overduin, J.M., Wesson, P.S., Kaluza-Klein Gravity, arXiv:gr-qc/9805018v1, 1998

[2] Kaluza, T. Zum Unitatsproblem der Physik, Stz. Preuss. Akad.Wiss.Phys. Math. K1 (1921) 966 (Eng trans in [3][4][5])

[3] Unified field theories of more than 4 dimensions, proc. international school of cosmology and gravitation (Erice), eds. V. De Sabbata and E. Schmutzer (World Scientific, Singapore, 1983)

[4] An introduction to Kaluza-Klein theories, proc. Chalk River workshop on Kaluza-Klein theories, ed. H.C. Lee (World Scientific, Singapore, 1984)

[5] Modern Kaluza-Klein theories, eds T. Applequist, A Chodos and P.G.O. Freund (Addison-Wesley, MendoPark,1987)

[6] Misner, C.W., Thorne, K.S., Wheeler, J.A., Gravitation, Freeman, New York, 1970

[7] Wald, R.M., General Relativity, The University of Chicago press, Chicago, 1984

[8] Toth, V., Kaluza-Klein theory, http://www.vttoth.com/CMS/physics-notes/118-kaluza, 2004

[9] Baez, J., Muniain, J., Gauge Fields, Knots and Gravity, World Scientific, New Jersey, 1994

[10] Azreg-Ainou, M., Clement, G., Constantinidis, C.P., and Fabris, J.C., Electrostatic Solutions In Kaluza-Klein Theory: Geometry and Stability, Gravitation and Cosmology, Vol. 6 (2000), No. 3 (23), pp 207-218, Russian Gravitational Society

[11] Hehl, F.W., von der Heyde, P., Kerlick, G.D. and Nester, J.M. (1976), General relativity with spin and torsion: Foundations and prospects. Rev. Mod. Phys. 48:393-416

[12] Kobayashi, S., and Nomizu, K., Foundations of Differential Geometry, vol.s I and II, Wiley Classics Library, New York (republished 1996)

[13] Petti, R. J. (1986). "On the Local Geometry of Rotating Matter" Gen. Rel. Grav. 18, 441-460.

[14] Adamowicz, W. (1975). Bull. Acad. Polon. Sci. Sr. Sci. Math. Astronom. Phys. 23 (1975), no. 11, 1203-1205.

[15] Belinfante, F., 1939. On the current and the density of the electric charge, the energy, the linear momentum, and the angular momentum of arbitray fields, Physica 6, 887 and 7, 449-474

[16] Bergqvist, Goran., Senovilla, J.M.M., On the Causal Propagation Of Fields, Class. Quantum Grav. 16, L55-L61, 1999

[17] Senovilla, J.M.M., Super-energy Tensors, Class. Quantum Grav. 17, pp2799-2842, 2000

[18] Senovilla, J.M.M., Remarks on Super-energy Tensors, arXiv: gr-qc/9901019v1, 1999

[19] Fabbri, L., On A Completely Antisymmetric Cartan Torsion Tensor, arXiv: gr-qc/0608090v5, 2012

[20]F.Hehl, E. Kroner, Z. Physik 187, 478 (1965)

[21]F. Hehl, Abhandt. Braunschweiger Wiss. Ges. 18, 98 (1966)