In Search Of A Variant Kaluza Theory

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Abstract

Kaluza's 1921 theory of gravity and electromagnetism using a fifth wrapped-up spatial dimension is at the root of many modern attempts to develop new physical theories. Lacking non-null electromagnetic fields however the theory is incomplete. Variants of the theory are explored to find ways to introduce non-null solutions by making the fifth dimension more physical, using alternative, weaker cylinder conditions. The Lorentz force law is investigated starting with a non-Maxwellian definition of charge, this is assumed to be related to Maxwellian charge by ansatz. Order of magnitude methods are used. Provided here is an exhaustive a search as possible. Kaluza theory remains inadequate to support electromagnetism in full, non-null solutions are not readily shown to be admitted. An argument is made in favour of torsion resolving this issue. Postulates are derived from the argument for a variant theory. The charge ansatz is shown to follow from the postulates. It is concluded that Kaluza's 5D space and torsion need to go together.

1 Conventions

The following conventions are adopted unless otherwise specified:

Five dimensional metrics, tensors and pseudo-tensors are given the hat symbol. Five dimensional indices, subscripts and superscripts are given capital Roman letters. So for example the five dimensional Ricci flat 5-dimensional superspace-time of Kaluza theory is given as: $\hat{g}_{AB}$, all other tensors and indices are assumed to be 4 dimensional. Index raising is referred to a metric $\hat{g}_{AB}$ if 5-dimensional, and to g_{ab} if 4-dimensional. The domain of partial derivatives carries to the end of a term without need for brackets, so for example we have $\partial_a g_{db} A_c + g_{db} g_{ac} = (\partial_a (g_{db} A_c)) + (g_{db} g_{ac})$. Terms that might repeat dummy variables or are otherwise in need of clarification use additional brackets. Square brackets can be used to make dummy variables local in scope.

Space-time is given signature $(+, -, -, -)$, Kaluza space $(+, -, -, -, -)$ in keeping with [1]. Under the Wheeler et al [6] nomenclature, the sign conventions used here to correspond with [1] are $[-, ?, -]$. The first dimension (index 0) is always time and the 5^{th} dimension (index 4) is always the topologically closed Kaluza dimension. Universal constants defining physical units: c = 1, and G as a constant. The scalar field component is labelled ϕ^2 (in keeping with the literature) only as a reminder that it is associated with a spatial dimension, and to be taken as positive. The matrix of g_{cd} can be written as $|g_{cd}|$ when considered in a particular coordinate system to emphasize a component view. The Einstein summation convention may be used without special mention.

Some familiar defining equations consistent with [1] (using Roman lower-case for the general case only for ease of reference):

$$\Gamma^c_{ab} = \frac{1}{2} g^{cd}(\partial_a g_{db} + \partial_b g_{da} - \partial_d g_{ab}) \tag{1.0.1}$$

$$R_{ab} = \partial_c \Gamma^c_{ab} - \partial_b \Gamma^c_{ac} + \Gamma^c_{ab}\Gamma^d_{cd} - \Gamma^c_{ad}\Gamma^d_{bc} \tag{1.0.2}$$

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab} = -\frac{8\pi G}{c^4} T_{ab} \tag{1.0.3}$$

$$F_{ab} = \nabla_a A_b - \nabla_b A_a = \partial_a A_b - \partial_b A_a \text{ equally } F = dA \tag{1.0.4}$$

Any 5D exterior derivatives and differential forms could also be given a hat, thus: $\hat{d}\hat{B}$. However, the primary interest here will be 4D forms. $\triangle$ represents the 4D D'Alembertian.

2 Introduction

Kaluza's 1921 theory of gravity and electromagnetism [2][3][4] using a fifth wrapped-up spatial dimension is at the heart of many modern attempts to develop new physical theories [1][5]. From supersymmetry to string theories topologically closed small extra dimensions are used to characterize the various forces of nature. It is therefore at the root of many modern attempts and developments in theoretical physics. However it has a number of foundational problems. It seems sensible to look at these from a classical perspective before looking at more complicated situations such as quantum gravity theories as envisaged by Klein.

The theory assumes a (1,4)-Lorentzian Ricci flat manifold to be the underlying metric, split (analogously to the much later ADM formalism) as follows:

$$\hat{g}_{AB} = \begin{bmatrix} g_{ab} + \phi^2 A_a A_b & \phi^2 A_a \\ \phi^2 A_b & \phi^2 \end{bmatrix} \tag{2.0.1}$$

By inverting this metric as a matrix (readily checked by multiplication) we get:

$$\hat{g}^{AB} = |\hat{g}_{AB}|^{-1} = \begin{bmatrix} g^{ab} & -A^a \\ -A^b & \frac{1}{\phi^2} + A_i A^i \end{bmatrix} \tag{2.0.2}$$

Maxwell's law are automatically satisfied: dF=0 follows from dd = 0. d*F= 4π*J can be set by construction. d*J=0, conservation of charge follows also by dd=0 in most parts of the manifold. However:

In order to write the metric in this form there is a subtle assumption, that g_{ab}, which will be interpreted as the usual four dimensional space-time metric, is itself non-singular. However, this will always be the case for moderate or small values of A_x which will here be identified with the electromagnetic 4-vector potential. The raising and lowering of this 4-vector are defined in the obvious way in terms of g_{ab}. We have also assumed that topology is such as to allow the Hodge star to be defined. This means that near a point charge source the argument that leads to charge conservation potentially breaks down as the potential may cease to be well-defined. Whereas the Toth charge that will be defined in the sequel does not have this problem.

Assume values of ϕ^2 around 1 and relatively low 5-dimensional metric curvatures. We need not therefore concern ourselves with this assumption beyond stating it on the basis that physically these parameters encompass tested theory. Given this proviso A_x is a vector and ϕ^2 is a scalar - with respect to the tensor system defined on any 4-dimensional submanifold that can take the induced metric g.

Kaluza's cylinder condition (KCC) is that all partial derivatives in the 5th dimension i.e. ∂_4 and $\partial_4\partial_4$ etc... of all metric components are 0. A perfect 'cylinder'. This leads to constraints on g_{ab} given in [1] by three equations, the field equations of Kaluza theory, where the Einstein-Maxwell stress-energy tensor can be recognised embedded in the first equation:

$$G_{ab} = \frac{k^2\phi^2}{2}\left\{\frac{1}{4}g_{ab}F_{cd}F^{cd} - F^c_aF_{bc}\right\} - \frac{1}{\phi}\{\nabla_a(\partial_b\phi) - g_{ab}\triangle\phi\} \tag{2.1.1}$$

$$\nabla^a F_{ab} = -3\frac{\partial^a\phi}{\phi}F_{ab} \tag{2.1.2}$$

$$\triangle\phi = \frac{k^2\phi^3}{4}F_{ab}F^{ab} \tag{2.1.3}$$

These will be referred to as the first, second and third field equations respectively. Here there is also a k term, since the formulation in [1] is more general than that used here. In this work k=1 is used unless specified otherwise. Sign conventions should also be checked before using such equations out of context. It is important to note that in the variants of Kaluza theory defined here, these field equations may not apply. They apply fully only to Kaluza's original theory.

By looking at field equation 3 it can be seen that if the scalar field does not vary then only null electromagnetic solutions result. The second field equation then also imposes no charge sources. Here the scalar term could be allowed to vary in order to allow for non-zero $F_{ab}F^{ab}$. This falls within Kaluza's original theory. This potentially allows for non-null electromagnetic solutions, but there are problems to overcome: the field equations cease being necessarily electrovacuum. This remains a problem even when the scalar field terms are set large, as is sometimes done to ensure that field equation 2 is identically zero despite scalar fluctuations.

Another foundational issue of Kaluza theory is that even with a scalar field it does not have convincing sources of mass or charge. Field equation 2 then has some charge sources, but it is far from clear that realistic sources are represented by this equation. Matter (and charge) models in this work will be assumed to be regions of the Kaluza space that are *not* Ricci flat in the otherwise Ricci flat Kaluza space, just as matter/energy is analogously assumed to be in general relativity. That is, where the 5D Einstein tensor of the Kaluza space itself is non-zero.

Charge will be given a possible alternative definition as 5-dimensional momentum, following a known line of reasoning [8] within Kaluza theory. This version of charge will be called Toth charge to make it distinct from Maxwellian charge, their identity is assumed by ansatz in this work, otherwise the Lorentz force law has no obvious explanation in the event that the electrovacuum is no longer valid due to other fields. As momentum the Toth charge is of necessity conserved, provided there are no irregularities in the topology of the Kaluza 5th dimension, similarly the conservation of Maxwellian charge is normally guaranteed by the potential, except that this may not be valid in extreme curvatures where the 4-vector associated with the 4-potential may cease to be a vector.

We will also assume of necessity a closed 4D spatial hypersurface as is often done in general relativity to ensure 4 dimensional causality. Although 5D causality issues will not be explored.

The leading issue is that Kaluza theory appears to offer only null electromagnetic solutions, non-null electrovacuums more generally are not so easily supported as changes in the scalar field may force divergence of the field equations from those of the electrovacuum (see field equation 1). Null electrovacuums occur under KCC when the scalar is constant as can be seen in the field equations above. That is, non-null solutions, non-radiative electromagnetic fields, seem to have no reserved place within the theory unless one allows the scalar field to oscillate. A standard fix is to set the scalar field to be very large so that its oscillations make little difference to the resulting field equations. However this is limiting, arbitrary, and requires its own explanation. It also does not guarantee that the field equations are electrovacuums unless further arbitrary conditions are added. Without taking such arbitrary measures that cease to be in the spirit of Kaluza's original theory, the stress-energy tensor under scalar field fluctuations is different from the Einstein-Maxwell tensor [6][7] and the accepted derivation of the Lorentz force law (for electrovacuums [6]) can not be assumed.

Note that in addition the derivation of the Lorentz force law within general relativity is not without problems of principle [6]. *Thus, attempting to build a theory around the ansatz that the Toth charge and Maxwellian charge are equivalent under the variant cylinder conditions is persuasive* - as this leads independently of the electrovacuum to an approximate and provisional Lorentz force law.

The scalar field will be allowed to vary. But order of magnitude limits will be placed on it so as to allow for a variant Lorentz force law. A Lorentz force law, rather than the electrovacuum solutions per se will thus be sought for Toth charges, and an ansatz linking the two definitions of charge assumed here.

3 Preliminary Notes

3.1 Geometrized Units Of Mass

The full metric definition used in [1] was:

$$\hat{g}_{AB} = \begin{bmatrix} g_{ab} + k^2\phi^2 A_a A_b & k\phi^2 A_a \\ k\phi^2 A_b & \phi^2 \end{bmatrix} \tag{3.1.1}$$

This gives null solutions [1] such that $G_{ab} = -\frac{k^2}{2}\ F_{ac}F_b^c$. Comparing this with [7] (where we have $G_{ab} = 2F_{ac}F_b^c$ in geometrized units where G=1) the sign difference is due to the historic use of [−, ?, −] notation here, rather than the more modern [+,+,+] notation as defined by Wheeler et al. [6] (and as also used by Wald [7]).

Although k is set to 1 elsewhere in this work, yielding the metric in the introduction, we need $k = 2$ to get the field equations in the geometrized units of [7] - if the electromagnetic field tensor is to be the same.

$G/c^4 = 1 \qquad \text{for k=2}$
$= 6.674 \times 10^{-8} cm^3 g^{-1} s^{-2}/c^4 = 6.674 \times 10^{-8} cm^3 g^{-1} \times (3 \times 10^{10} cm)^{-2}/c^4$
$= 6.674 \times 10^{-8} cm g^{-1} \times (3 \times 10^{10})^{-2} \qquad \text{for c = 1}$

$$1g = 7.42 \times 10^{-29} cm \text{ for k=2, c=1} \tag{3.1.2}$$

Secondly, if we *do* use k = 1, we can adjust G's dimensionless value to accomodate:

$G_{ab} = -2F_{ac}F_b^c = -\frac{8\pi G}{c^4}T_{ab} \quad \text{for k=2, c=1}$
$G_{ab} = -F'_{ac}F'^c_b = -\frac{8\pi G'}{c^4}T'_{ab} \quad \text{for k=1, c=1}$

So if G_{ab} is the same in both cases, then $F'_{ab} = \sqrt{2}F_{ab}$

If we define T'_{ab} to be formally the same as T_{ab}, but with a substitution of F' terms for F terms, it will be twice as big. Then we must adjust $G' = \frac{1}{2}G = \frac{1}{2}$.

With this adjustment to G, we have:

$$1g = 1.48 \times 10^{-28} cm \text{ for k=1, c=1} \tag{3.1.3}$$

So in effect we can change the scale relationship of mass to distance in order to change the electromagnetic tensors. The tensors can represent the same underlying reality but in different units. We can refer to these two schemes of units as k=2 and k=1 respectively.

3.2 Kaluza Theory Is Consistent With Special Relativity Even When 5D Momentum Is Present

In the sequel, one definition of charge (Toth charge) will be identified with 5D momentum. This is already known in the original Kaluza theory to obey a Lorentz force-like law, but will be extended here in scope, noting that the coincidence of Toth charge and Maxwellian charge is not guaranteed prior to the ansatz.

That this is consistent with Special Relativity will be something that anybody seeking confidence in Kaluza theories will want to check. The additions of velocities in Special Relativity, for example, is not obvious. Taking two perpendicular velocities, u and v, and adding them yields:

$$s^2 = u^2 + v^2 - u^2v^2$$

The particle moving in the Kaluza dimension, but stationary with respect to space-time, will have a special relativistic rest mass greater than its Kaluza rest mass. A later result needed here is:

$Q_{toth}/M_0 = -dx_4/d\tau$ relating charge, rest mass and proper Kaluza-velocity

This makes sense only because mass can be written in fundamental units (i.e. in distance or time) and Toth charge will be defined as 5th dimensional momentum.

Using natural conversions between units we get the Kaluza rest mass of any presumed particle with the mass and charge comparable in magnitude to an electron or positron to be about $1.5 \times 10^{-54}g$, which is a lot smaller than the relativistic rest mass used when considering only space-time physics. And its proper Kaluza velocity in natural units is then about $K = 6 \times 10^{26}$, making it highly Kaluza-relativistic.

Much as we may be unfamilar with a Kaluza rest mass (M_{k0}) we can see that it is consistent with the addition of velocities as follows:

$$M_0 = \frac{M_{k0}}{\sqrt{(1-u^2)}} \text{ where } u = Q_{toth}/M_0 \tag{3.2.1}$$

$$M_{rel} = \frac{M_0}{\sqrt{(1-v^2)}} = \frac{M_{k0}}{\sqrt{(1-u^2)}} \times \frac{1}{\sqrt{(1-v^2)}} = \frac{M_{k0}}{\sqrt{(1-u^2-v^2+u^2v^2)}} \tag{3.2.2}$$

where v is the relativistic velocity in space-time

By putting $u = Q_{toth}/M_0$ into the definition of rest mass and solving, we get that charge, whether positive or negative, is a contributor to the relativistic rest mass according to the following formula:

$$ {M_0}^2 = {M_{k0}}^2 + {Q_{toth}}^2 \tag{3.2.3} $$

Here the majority of mass-enery in the rest mass of such an elementary charge is seen as being tied up in its charge. In this work elementary Toth charges will take on these properties. Whether or not these are able to model Maxwellian charges depends on the correspondence between Maxwellian and Toth charge. Whether this further corresponds to real electrons and positrons or whether similar models can describe other fundamental charge sources is left as unascertained experimentally and analytically, but merely a tempting suggestion of this model not to be investigated further here.

Observed electrons have static charge, angular momentum, a magnetic moment, and a flavor. The only thing distinguishing the electron from the muon is apparently the flavor. The mass difference between the muon and the electron is about 105 MeV, perhaps solely due to this difference in flavor. Yet here we have a model where the unit Toth charge's relativistic rest mass is dominated by its Kaluza velocity. Thus at this stage the idealized charge models used here and real particles must be considered not yet correlated.

3.3 Matter And Charge Models, A Disclaimer

The model unit Toth charge presented here, therefore remains a separate entity from any real Maxwellian charges, merely a mathematical device to investigate whether such models are possible. Having said that for the purposes here the ansatz is made that in principle Maxwellian and Toth charges can be identified.

The above analysis has assumed that some sort of particle model of matter and charge is possible, that it can be added to the original theory perhaps without changing the space-time solution, which is impossible no less than in general relativity. Secondly we might imagine that what has been described is a particle whizzing around the fifth dimension like a roller coaster on its spiralled tracks. The cylinder conditions could in fact also be maintained if, instead of a particle, the matter-charge source was rather a 'solid' ring, locked into place around the 5th dimension, rotating at some predetermined Kaluza velocity. An exact solution could even involve changes in the size of the 5th dimension. None of that is investigated here, the aim is to see whether non-null solutions can be found in a Kaluza variant theory without extreme alteration.

It is an essential proviso that a physically realistic matter-charge model has not been detailed, much less formally identified with a real charge source such as an electron. The assumption then that such a hypothetical model would necessarily follow (albeit approximately) geodesics is also therefore just that: an assumption - though not without analogs in other, experimentally valid, classical theories.

The original Kaluza theory almost certainly does not have sufficient degrees of freedom to allow for such a matter model to be embedded within it. More general matter models could be assumed however to have significant enough degrees of freedom to allow for such a model or approximation of such a model in principle. But an actual differential geometrical model of such a matter-charge source is presumed too difficult to produce here, even if possible. In addition, the fact that real charge sources are quantum mechanical may also discourage us, though a classical scale interpretation should be possible regardless.

This work assumes a limited concept of such a charge model and attempts to investigate whether non-null solutions are possible in conjunction with a Lorentz force law. That is, it attempts to replicate all the important features of classical physics, without predicting or imposing its particular model of charge as the correct one.

Geodesic Assumption: That any particle-like matter-charge models derived from the geometry are approximately geodesic. This would need also to follow from any applicable matter-charge model that was ultimately found to describe unit charges.

Charge Ansatz: That the Maxwellian and Toth definitions of charge coincide for the purposes here.

One of the confusing aspects of this work is that the field equations of Kaluza cease to apply when such matter-charge models become part of the solution. Therefore the Kaluza field equations are only used when they can be justified, and going back to the Ricci tensor, from scratch, is undertaken here as necessary. It is also important to note that in the following the Ricci flat condition of the original Kaluza theory's Kaluza space will not be generally valid due to the presence of matter models, but as with vacuum solutions in general relativity will be usable outside of matter models.

3.4 Duality Invariance Of Kaluza's Original Theory

The dual metrics of $\hat{g}^{AB}$ and g^{ab} will be discussed in this section.

$\hat{g}^{AB}$ will be identified with an alternative dual metric $\hat{h}_{AB}$ for some coordinate system in such a way that their representations as matrices are equal. That is, such that: $|\hat{g}^{AB}| = |\hat{h}_{AB}|$, such an identification will be written $\hat{g}^{AB} \leftrightarrow \hat{h}_{AB}$. It follows that $\hat{g}_{AB} \leftrightarrow \hat{h}^{AB}$ where the two alternative systems $\hat{g}_{AB}$ and $\hat{h}_{AB}$ define their own notions of raising and lowering indices.

$\hat{h}_{AB}$ and $\hat{h}^{AB}$ can be written analogously to the original metrics as follows:

$$\hat{h}^{AB} = \begin{bmatrix} h^{ab} + \phi^2 B^a B^b & \phi^2 B^a \\ \phi^2 B^b & \phi^2 \end{bmatrix} \tag{3.4.1}$$

$$\hat{h}_{AB} = |\hat{h}^{AB}|^{-1} = \begin{bmatrix} h_{ab} & -B_a \\ -B_b & \frac{1}{\phi^2} + B^i B_i \end{bmatrix} \tag{3.4.2}$$

Where analogously to the original system the raising and lowering of indices of 4-vectors is implemented by h_{ab}.

We have the following relations by construction: $g^{ab} \leftrightarrow h_{ab}$ and $A^a \leftrightarrow B_a$.

In other words they are the same system with index conventions swapped around. Or equivalently they are dual systems.

Kaluza's original theory is dual invariant in that if $\hat{g}_{AB}$ is Ricci flat then this is equivalent to $\hat{h}_{AB}$ being Ricci flat. This follows as raising $R_{AB} = 0$ within $\hat{g}_{AB}$ remains 0.

The introduction of matter models for Kaluza theory, as with general relativity, disrupts this nice property of Kaluza theory. Every matter model has in effect an alternative formulation that would suffice as a physical description by taking the dual system. There is something arbitrary about matter models (also in general relativity) in this respect from the outset.

4 The Order of Magnitude of Potentials

4.1 The Electromagnetic Potentials

The contribution to the metric of a typical cgs unit of electromagnetic potential can be calculated: It is actually dimensionless in genuinely natural units, as must be the case for it to be related to metric components in Kaluza theory.

In cgs units the Coulomb's force law is given by: $F = Q_1Q_2/r^2$

Similarly the potential is given by Q/r, that is charge/distance, or esu/cm. Using [7]:

$$([L]^{3/2}[M]^{1/2}/[T])/[L] = ([L]^{1/2}[M]^{1/2})/[T]) \tag{4.1.1}$$

Using $1g = 7.42 \times 10^{-29}cm$ (eq. 8.0.3) gives 1cgs unit of potential (esu or Statvolt) as:

$1cm^{1/2}8.61 \times 10^{-15}\ cm^{1/2}/s$

$= 2.86 \times 10^{-25}(3 \times 10^{10}cm/s)$

$$1esu = 2.86 \times 10^{-25}c \tag{4.1.2}$$

$\approx 10^{-25}$ in natural units where $c = 1$ ($k = 2$)

This is clearly a very small figure and is a comparable order of magnitude for $k = 1$ for the purposes here.

Take a single unit charge (ie of an electron) $Q \approx 5 \times 10^{-10}esu$ and the (Bohr) radius of the hydrogen atom: $r \approx 5 \times 10^{-9}cm$.

The potential $R_a = Q/r = 10^{-1} esu/cm = 10^{-26}$ in natural units. Now this potential corresponds to the strong electrical forces within the atom, but due to the short distances may not represent strong potentials. We might need a more realistic reference for classical potentials (R_c), in order to provide a sort of experimental upper limit for potentials with well-behaved classical properties. This figure could be given as a ratio $R_c = h.R_a$ to give a sense of proportion, where a typical bound for h will now be estimated.

We can select the units associated with the Christoffel symbols, or equally the coordinate system if we so choose to label it with units, to be small enough for a usual range of fields so that the various orders of magnitude are set small.

Here we exploit the fact that locally we can choose coordinates, even if only infinitesimally, that are Minkowskian. In talking about potentials in classical physics we are referring to a vector that has gauge freedom to the extent it can be set to 0 at any point and then the potentials in the coordinate system are relative to that origin. Setting coordinates to be locally flat at the origin is the same procedure in Kaluza theory.

This is important since we want to estimate R_c, a sensible bound for typical potentials which have been experimentally tested to satisfy the laws of classical (relativistic) electromagnetism. And so if we can select our origin arbitrarily to be, say, the centre of an experiment, we always have a region with arbitraily low potentials against which the potentials in the experiment are to be defined or measured. This avoids the problem of having an experiment where all the potentials are set large - they can be gauged away with a judicious choice of coordinate system.

As a case in point, imagine we are testing the effects of the Lorentz force law due to the ionosphere. If we set the Earth's surface to be zero potential, we will end up with high potentials in the ionosphere, and vice versa. But since when has an experiment tested the Lorentz force law throughout the entire trajectory of a *slow* charged particle from ionosphere to Earth? We may test it repeatedly over a range of a few metres if the particle is slow and over a long distance if the particle is very fast. Generally a slow particle will not stay slow if it is accelerated by a field over a great distance, and if it does the experiment will not be clean as the slowness will be caused by impacts with other particles or fields.

The reason for this distinction is that the Lorentz force law will later be derived such that it is conditional on the velocity of the particle. If it is fast it will satisfy the Lorentz force law more accurately, if it is slow it may not and other terms might be involved - depending on R_c. In the ionosphere for example we have approximately 150V/m wherever we may be. It is whether this figure is large or small relative to the spatial extent of the experiment that becomes the issue under this consideration, not the potentials per se. And therefore it relates to the velocities of the particles. Similarly we may get very high fields indeed under experiments in the laboratory that are of necessity far less extensive than the distance from Earth to the ionophere.

Fast particles therefore present one problem over longer distances, and the scale of the experiment (and the corresponding potentials) can be set accordingly shorter for slower particles.

In order to estimate our h in $R_c = h.R_a$ we need to define the experiment we are looking at. For these purposes we can look at two extremes that have comparable energies: Firstly 150V/m over a vacuum of 1km (comparable to the potential differences created by the ionosphere over long distances), secondly 150KV over a vacuum of 1m (a high voltage experiment if it is to be sustained for any duration of time). They are both high in terms of the potentials involved and will act as a guide to the estimate of h. The second is perhaps easier to deal with experimentally. A clean experiment over larger scales may be unfeasible except perhaps in space.

We can then look at the maximum 4-potential that the second experiment defines and define this to be R_c, from which we can estimate h relative to this level of experimental testing. As it pertains to the orders of magnitude that will be used here an approximate idea of this figure is useful to make the scales meaningful. If the accepted tested level of classical electromagnetism is higher or lower than the 150KV used here it is a simple matter to scale h accordingly.

Both the above set-ups have a 150KV potential difference, which is about 500 Statvolts. This gives a h of about 5000, and,

$$O(f) \approx R_c \approx 10^{-22}, k = 2 \tag{4.1.3}$$

4.2 The Metric Components of $O(v)$

Using k=2 we can estimate one possible $O(v)$ by looking at the the Schwarzchild Solution for the Earth. The differences from unity (or negative unity) of the terms depends on $2GM/r$, in this case $2M/r$ in natural units.

$$2 \times [6 \times 10^{24} kg \times 1000 g/kg]/[6000 km \times 10^5 cm/km] = 2 \times 10^{19} g/cm$$

Using $1g = 7.42 \times 10^{-29} cm$, and when k=2, with a comparable order of magnitude for k=1:

$$O(v) \approx 10^{-9} \tag{4.2.1}$$

$O(v)$ is considerably larger in significance than $O(f)$.

5 The Cylinder Conditions And Electromagnetic Limits

We will start with a weak field limit that can be assumed at the usual classical scale. Terms such as $A_a A_b$ will be bounded $O(v^2)$ as opposed to terms not thus multiplied such as $\hat{g}_{ab}$ or simply A_a. The latter either being bounded $O(v)$ or the difference from 1 being bounded $O(v)$. Our Kaluza space-time solutions, at the usual classical limit, are to be approximately 5-Lorentzian. $O(v)$ will be taken to be a small term at the usual classical limit. In addition, the units will be assumed to be such that derivatives of O(v) i.e. O(v+) and O(v++) will also all be small, where O(v+) and O(v++) are the order of magnitude of first and second derivatives (with corresponding units) respectively.

An electromagnetic limit will be assumed as required, where the scalar field is set to being approximately the identity: $\phi^2 \approx 1$

Now, this simple declaration turns out to be quite complicated. For one thing at the usual classical scale it will be automatically approximately 1 by the weak field limit as a minimum constraint, at least to $O(v)$. But this in itself won't make it any closer to unity than the electric potentials are to 0 relative to other weak fields. For the electromagnetic limit we want more than that.

We will define it in terms of three orders of magnitude:

$\phi^2 = 1$ to O(s), and,

$\partial_A \phi^2$ is O(s+). We can also have:

$\partial_A \partial_B \phi^2$ is $O(s++)$

(5.0.1)

This distinction will be of fundamental importance later. O(s) will be no more significant than O(v), with the possibility of turning out to be a lot smaller in significance. O(s+) and O(s++) could have units s^{-1} and s^{-2} respectively and will thus not be comparable to O(s) in a simple way. Similarly we can have the additional levels of bounding for the derivatives of electromagnetic tensors defined as follows:

A^a is $O(f)$, and,

$\partial_A A^a$ as $O(f+)$. We can also have:

$\partial_A \partial_B A^a$ as $O(f++)$

(5.0.2)

Noting that in any situation the tightest of any two applied bounds dominates. Noting also the same unit considerations as previously.

The other fields, the 4D metric and the electromagnetic potential vector will be given a similar order of magnitude constraint so that their derivatives in the direction of the Kaluza dimension are bounded as follows:

$\partial_4 \hat{g}_{AB}$ is $O(\delta+)$

(5.0.3)

KCC is the limiting case where it is identically 0. This implies also that $\partial_4 \hat{g}^{AB}$ is $O(\delta+)$ via a consideration of divergence of the metric being 0. Now of course this bound as defined above is not yet a constraint until $O(\delta+)$ is defined. So this device allows us to weaken the cylinder condition as required.

A basic reasonable constraint might be to ensure that at least such oscillations are not greater than the general order of magnitude of the electromagnetic fields, otherwise we would have another limit. Thus:

$$O(\delta+) \leq O(f+) \tag{5.0.4}$$

The use of the symbol $+$ to identify when a derivative has been taken is easy to misuse. A symbol such as $O(X+)$ will be related to $O(X)$ via proportionality with some constant that will depend not only on the terms and functions in question, such as frequencies, but also on the units being used. Later such orders of magnitude with the same units will be multiplied by each other. This can be made more concrete by giving $O(f)$ a numerical value to set the scale. By default we can set this value to some classical reference potential, a figure that gives a reasonable bound to the level for which the well-behaved properties of a classical system have actually been tested. We might similarly have another classical reference to define O(v).

Similarly, relative to this (by some constant and some unit) $O(f+)$ will be taken to be defined, though any constants of proportionality and any units will not be specified. Typical estimates for such an order of magnitude could be taken from an ensemble of experiments of interest. The use of the $\leq$ and $<<$ symbols in comparing orders of magnitudes will be used to express the idea that one order of magnitude can be or is smaller than the other when compared numerically.

We also need a few rules for dealing with orders of magnitude. One could be that if $O(X) = O(v)$ and $O(X) \leq O(v)$ then generally $O(X+) = O(v+)$ and $O(X+) \leq O(v+)$ respectively. This will be called *proportionality* . It should of course be invoked with care and may not in the most general case be valid.

Similarly, *distributivity*: terms such as $O(f+)O(f+)$ and $O(f)O(f++)$ - where the terms are all based on the same underlying order O(f) - will be considered the same order of magnitude without further consideration. We might observe the reasonableness of this by considering the chain rule. Similarly for consistency we also need to extend it over different underlying terms. This however will be used more cautiously.

Two other complementary limits can be defined: the strong electromagnetic and the strong scalar limit respectively. These will both be when oscillations in the other are absolute zero. The scalar limit would not usually be applied at the same time as either of the electromagnetic limits.

Further a resonant cylinder condition is defined. This needs some discussion before stating the final definition.

The objective of the Resonant Cylinder Condition is to weaken KCC but not in the same way as having simply an order of magnitude limit for derivative terms in the direction of the Kaluza dimension. The Resonant Cylinder Condition (RCC) takes a loop around the Kaluza 5th dimension (one that is locally normal to the supposed space-time embeddings). The idea is then that various components, derivatives and tensors oscillate around this loop, in a way reminiscent of resonance, whilst maintaining an average value that can be represented (approximately) as a tensor in a representative space-time. This will be applied to any tensors, pseudo-tensors or related terms that might be meaningful on a sample 4D manifold chosen to represent space-time, and any raisings or lowering of those terms in terms of the 4D metric. In particular any terms which consist of repeated differentiation of another term, and where one or more of those derivatives is a covariant derivative in the direction of the Kaluza dimension, must average to less significant than would otherwise be expected. This is because the tensor thus differentiated must start at a certain value, and in passing round the loop return to it, give or take small errors.

In this work we shall be primarily interested in derivatives or double derivatives of tensors whose elements are constructed from the metric components that contain a ∂_4 operator or can be written in such a way. Such derivatives will average a small order of magnitude round a loop, with a small error that will be O(v+) times the significance of the original tensor (of which the derivatives are taken) due to the Christoffel symbol used in converting between partial and covariant derivatives. We have $O(Y+) = O(v+)O(X)$ where the ostensible order of magnitude of term $O(Y+)$ was $O(X+)$. So for example if it is the derivative in the Kaluza dimension of a normal metric component $O(v)$, then the derivative is ostensibly $O(v+)$, but the formula here gives $O(v+)O(v)$ due to the cylindricity.

We will model this as follows, also embracing the idea that the size of the Kaluza dimension has an upper bound: Take an $O(X+)$ resonant term (i.e. a derivative in the fifth dimension of another term of order $O(X)$). Consider the situation where this term itself has been multiplied by another term of magnitude $O(v)$, and whose derivatives are therefore $O(v+)$, and thus the compound term of the $O(X+)$ term and the $O(v)$ term may not be as insignificant due to the skewed nature of multiplication when taking averages. Let this multiple then be controlled or delimited by a constant B that is inversely proportional to the size of the Kaluza dimension.

We could then give a compound term that would otherwise locally be about $O(X+)O(v)$, by looking at its constituents, a reduced effective order of magnitude $O(Y+)O(v) \approx O(X+)O(v)/B$ on account of the small Kaluza dimension constraint and averaging around the loop. We might generalize this without loss: a resonant term that should otherwise be $O(X+)$ is instead given $O(Y+) \approx O(X+)/B$, for some B approximately inversely related to the size of the Kaluza dimension, and we might further rescale this so that $B = A \times O(v+)/O(f+)$ for future convenience. This is now a fairly simple definition for RCC.

This might be particularly important, and different from a simple bound, in the case that a compound term were given averaged order $O(X+)O(X+)/B$, but having two constituents both of averaged order $O(X+)/B$. However such a distinction is not consistent with the distributivity property if required. So the term must in fact be $O(X+)O(X+)/B^2$ to be consistent with distributivity.

For double derivatives we would obtain $O(X++)/B^2$ with respect to RCC.

The original Kaluza theory assumed KCC. So weakening KCC (whether by RCC or otherwise) requires particular care in that the original Kaluza's field equations and conclusions derived from them can nolonger be assumed, in particular the Kaluza field equations.

6 Charge, 5D Momentum And The Lorentz Force

Toth [8] derives a Lorentz-like force law where there is a static scalar field and Kaluza's cylinder condition applies. The resulting 'charge' is the momentum term in the fifth dimension and it is not apparent how this relates to the Maxwell current, except as Toth states via 'formal equivalence'. While this result is not new, Toth's calculation is used here as the starting point for a more detailed calculation.

A derivation is given of the Lorentz force law applicable to the Toth current. Toth makes several assumptions in his calculation. First that the scalar field is constant near the charge, and secondly the Kaluza cylinder condition (KCC). Toth also assumes a single point particle, not necessarily the case here, and constant mass-charge. These issues relate in this context to finding such a matter model as a solution. Here the KCC is relaxed, and both the Resonant Cylinder Condition is applied and an alternative Weak Cylinder Condition defined. Results are compared. The Geodesic Assumption must also be made for matter-charge models in this context. Oscillations of the scalar field are included (indices have been omitted from order of magnitude terms for clarity of presentation). Details of Chrisfoffel symbol terms can be found in a later section for reference.

$$\begin{aligned}
&\hat{\Gamma}^c_{4b} = \tfrac{1}{2}g^{cd}(\delta_4\hat{g}_{bd} + \delta_b\hat{g}_{4d} - \delta_d\hat{g}_{4b}) + \tfrac{1}{2}\hat{g}^{c4}(\delta_4\hat{g}_{b4} + \delta_b\hat{g}_{44} - \delta_4\hat{g}_{4b}) = \\
&\tfrac{1}{2}g^{cd}[\delta_b(\phi^2 A_d) - \delta_d(\phi^2 A_b)] + \tfrac{1}{2}g^{cd}\delta_4\hat{g}_{bd} + \tfrac{1}{2}\hat{g}^{c4}\delta_b\hat{g}_{44} = \\
&\tfrac{1}{2}\phi^2 g^{cd}[\delta_b A_d - \delta_d A_b] + \tfrac{1}{2}g^{cd}A_d\delta_b\phi^2 - \tfrac{1}{2}g^{cd}A_b\delta_d\phi^2 + \tfrac{1}{2}g^{cd}\delta_4\hat{g}_{bd} + \tfrac{1}{2}\hat{g}^{c4}\delta_b\phi^2 = \\
&\tfrac{1}{2}\phi^2 F^c_b + \tfrac{1}{2}g^{cd}A_d\delta_b\phi^2 - \tfrac{1}{2}g^{cd}A_b\delta_d\phi^2 + \tfrac{1}{2}g^{cd}\delta_4\hat{g}_{bd} + \tfrac{1}{2}\hat{g}^{c4}\delta_b\phi^2 = \\
&\tfrac{1}{2}\phi^2 F^c_b - \tfrac{1}{2}g^{cd}A_b\delta_d\phi^2 + \tfrac{1}{2}g^{cd}\delta_4\hat{g}_{bd} = \\
&\tfrac{1}{2}\phi^2 F^c_b + O(s+)O(f) + O(\delta+)
\end{aligned} \tag{6.0.1}$$

$$\hat{\Gamma}^c_{44} = \tfrac{1}{2}\hat{g}^{\mathrm{cD}}(\delta_4\hat{g}_{4D} + \delta_4\hat{g}_{4D} - \delta_D\hat{g}_{44}) = O(\delta+) - \tfrac{1}{2}g^{\mathrm{cd}}\delta_d\phi^2 = O(\delta+) + O(s+) \tag{6.0.2}$$

We have:

$$\begin{aligned}
&\hat{\Gamma}^c_{ab} = \tfrac{1}{2}g^{cd}(\delta_a g_{db} + \delta_b g_{da} - \delta_d g_{ab}) \\
&+\tfrac{1}{2}g^{cd}(\delta_a(\phi^2 A_a A_b) + \delta_b(\phi^2 A_a A_b) - \delta_d(\phi^2 A_a A_b)) + \tfrac{1}{2}\hat{g}^{c4}(\delta_a\hat{g}_{4b} + \delta_b\hat{g}_{4a} - \delta_4\hat{g}_{ab})
\end{aligned}$$

$$= \Gamma^c_{ab} + O(f^2)O(\delta+) + (O(f+) + O(\delta+))O(f) \tag{6.0.3}$$

So:

$$0 = \frac{d^2x^a}{d\tau^2} + \hat{\Gamma}^a_{BC}\frac{dx^B}{d\tau}\frac{dx^C}{d\tau}$$
$$= \frac{d^2x^a}{d\tau^2} + \hat{\Gamma}^a_{bc}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} + \hat{\Gamma}^a_{4c}\frac{dx^4}{d\tau}\frac{dx^c}{d\tau} + \hat{\Gamma}^a_{b4}\frac{dx^b}{d\tau}\frac{dx^4}{d\tau} + \hat{\Gamma}^a_{44}\frac{dx^4}{d\tau}\frac{dx^4}{d\tau}$$
$$= \frac{d^2x^a}{d\tau^2} + \hat{\Gamma}^a_{bc}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} + \phi^2 F^a_b\frac{dx^b}{d\tau}\frac{dx^4}{d\tau} + (O(\delta+) + O(s+)O(f))\frac{dx^b}{d\tau}\frac{dx^4}{d\tau} + (O(\delta+) + O(s+))\frac{dx^4}{d\tau}\frac{dx^4}{d\tau} \tag{6.0.4}$$

Setting the Toth charge-to-mass ratio to:

$$Q_t/m = -\phi^2\frac{dx^4}{d\tau} \tag{6.0.5}$$

Or equally setting the Toth charge to $-\phi^2 m\frac{dx^4}{d\tau}$ where m is the rest mass of the charge carrier, we derive a Lorentz-like law:

$$\frac{d^2x^a}{d\tau^2} + (\Gamma^a_{bc} + ((O(f+) + O(\delta+))O(f) + O(f^2)O(\delta+)))\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = (Q_t/m)F^a_b\frac{dx^b}{d\tau} + (O(\delta+) + O(f)O(s+))\frac{dx^b}{d\tau}\frac{dx^4}{d\tau} + (O(\delta+) + O(s+))\frac{dx^4}{d\tau}\frac{dx^4}{d\tau} \tag{6.0.6}$$

We would need to zero most of the orders of magnitude terms here to get the Lorentz force law itself. This however would be the same constraints as the null solutions of Kaluza's original theory. That is, Kaluza's original theory is suggestive of a link between Toth and Maxwellian charge by making the Lorentz force law for Toth charge apparent. This is assumed throughout this work via the Charge Ansatz. The order of magnitude terms on the left can all be removed as less significant than the $O(v+)$ Christoffel symbol elements in general.

$$\frac{d^2x^a}{d\tau^2} + \Gamma^a_{bc}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = (Q_t/m)F^a_b\frac{dx^b}{d\tau} + (O(\delta+) + O(f)O(s+))\frac{dx^b}{d\tau}\frac{dx^4}{d\tau} + (O(\delta+) + O(s+))\frac{dx^4}{d\tau}\frac{dx^4}{d\tau} \tag{6.0.7}$$

One other immediate observation is that throwing away the first $O(\delta+)$ term would simplify (6.0.7). This could be done by KCC automatically, or RCC with provisos.

In the case of RCC we would seem to need to have $O(\delta+) << O(f+)$. $O(v+) \approx O(f+)$ would ensure this as $O(\delta+) << O(v+)$ by RCC. The trouble is we do not want to be constrained to $O(v+) \approx O(f+)$.

We therefore instead need B to be sufficiently large, $B >> O(v+)/O(f+)$. So $O(v+_4)/B \equiv$ averaged$[O(\delta+)] << O(f+)$. That is, $A >> 1$. Where the subscript 4 is to be interpreted as a specialization to derivatives in the Kaluza dimension.

We might also simplify it without RCC, guessing the applicability of the following constraint to get rid of the first $O(\delta+)$, and calling the result the Weak Cylinder Condition (WCC):

$$O(\delta+) << O(f+) \qquad \text{[WCC] (6.0.8)}$$

The result in any case is:

$$\frac{d^2x^a}{d\tau^2}+\Gamma^a_{bc}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = (Q_t/m)F^a_b\frac{dx^b}{d\tau}+O(f)O(s+)\frac{dx^b}{d\tau}\frac{dx^4}{d\tau}+(O(\delta+)+O(s+))\frac{dx^4}{d\tau}\frac{dx^4}{d\tau} \qquad (6.0.9)$$

We can then apply the following reasonable constraint (reasonable in that it follows from $O(s+) \leq O(f+)$ which simply means we are at an electromagnetic rather than a scalar limit):

$$O(f)O(s+) << O(f+) \qquad (6.0.10)$$

To get:

$$\frac{d^2x^a}{d\tau^2} + \Gamma^a_{bc}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = (Q_t/m)F^a_b\frac{dx^b}{d\tau} + (O(\delta+) + O(s+))\frac{dx^4}{d\tau}\frac{dx^4}{d\tau} \qquad (6.0.11)$$

So for the two variant cylinder conditions investigated here, there is only one term which strays from a formal equivalence to the experimentally valid Lorentz force law, remembering the important difference between Toth and Maxwell charges, provided we make use of the Charge Ansatz.

This is the $(O(\delta+) + O(s+))$ term:

$-\hat{\Gamma}^c_{44} = -\frac{1}{2}\hat{g}^{\mathrm{cD}}(\delta_4\hat{g}_{4D} + \delta_4\hat{g}_{4D} - \delta_D\hat{g}_{44}) = O(\delta+) + \frac{1}{2}g^{\mathrm{cd}}\delta_d\phi^2 = O(\delta+) + O(s+)$
$= -g^{\mathrm{cD}}\delta_4 g_{4D} + \frac{1}{2}g^{\mathrm{cD}}\delta_D\phi^2$
$= -g^{\mathrm{cd}}\delta_4 g_{4d} - \frac{1}{2}g^{\mathrm{c4}}\delta_4\phi^2 + \frac{1}{2}g^{\mathrm{cd}}\delta_d\phi^2$

(6.0.12)

Whether using RCC or WCC:

$$\frac{d^2x^a}{d\tau^2} + \Gamma^a_{bc}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = (Q_t/m)F^a_b\frac{dx^b}{d\tau} - (g^{\mathrm{cD}}\delta_4 g_{4D} - \frac{1}{2}g^{\mathrm{cD}}\delta_D\phi^2)\frac{dx^4}{d\tau}\frac{dx^4}{d\tau} \qquad \text{[LFL1] (6.0.13)}$$

Note that however we constrain the size of the Kaluza dimension with RCC we can not eliminate the O(s+) term $\frac{1}{2}g^{\mathrm{cd}}\delta_d\phi^2$ unless we discard the scalar field fluctuations also, we therefore keep all the relevant terms in hope that we may later somehow cancel them out. In this way an exhaustive a search as possible is undertaken. It follows, however, from the preceding that we most likely have:

$$O(\delta+) <<<< O(f+) \tag{6.0.14}$$

$$O(s+) <<<< O(f+) \tag{6.0.15}$$

Where the $<<<<$ symbol expresses the extra extreme constraint imposed by the uncancellable and inescapably large $\frac{dx^4}{d\tau}$ term. Or an equivalent formulation with averages using the RCC. It might be noted that this extra strong condition starts to make the RCC condition look moot, as we have here a condition that seems to suggest WCC is needed in anycase. The only alternative, whether RCC is averaged or not, is a more specialized contraint:

$$O(\hat{\Gamma}^c_{44}) <<<< O(f+) \tag{6.0.16}$$

To be precise $O(A) <<<< O(B)$ is such that $O(A) << O(B)\frac{dx^a}{d\tau}/\frac{dx^4}{d\tau}$ for some smallest electron velocity $\frac{dx^a}{d\tau}$ at which the Lorentz force law works, ie has been tested to be accurate. We might need to set this arbitrarily small on the one hand, or settle for a lower bound on the other. For $\frac{dx^a}{d\tau}/\frac{dx^4}{d\tau}$ to be equal to $O(f)$, where $O(f) = R_c$ as estimated previously, requires $\frac{dx^a}{d\tau} = 10^4$. But the Lorentz force law certainly works for lower proper velocities than this! - despite the tantalising relative proximity to a more acceptable figure, relative that is to the very large numbers being used here. We have $O(A) <<<< O(f+)$ as stronger than $O(A) << O(f+)O(f)$ by some margin. It frustratingly can't quite be made to be absorbed into $<< O(f+)O(f)$, it remains a tighter constraint even than that.

Under such constraints we have the Lorentz force law proper:

$$\frac{d^2x^a}{d\tau^2} + \Gamma^a_{bc}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} \approx (Q_t/m)F^a_b \qquad \text{[LFL2] (6.0.17)}$$

7 Introducing Torsion

7.1 The Basic Equations

Where necessary Cartan torsion will be admitted. But only where all other routes have failed. In this way any introduction of the additional complexity of torsion will be empirically necessary under the ansatz. Empirically necessary, that is, only if it is shown to work.

For both 5D and 4D manifolds (i.e. dropping the hats and indices notation for a moment), torsion will be introduced into the Christoffel symbols as follows, using the notation of Hehl [11]:

$$\frac{1}{2}(\Gamma^k_{ij} - \Gamma^k_{ji}) = S_{ij}{}^k \tag{7.1.1}$$

This relates to the notation of Kobayashi and Nomizu [12] and Wald [7] as follows:

$$T^i{}_{jk} = 2S_{jk}{}^i \equiv \Gamma^i_{jk} - \Gamma^i_{kj} \tag{7.1.2}$$

We have the contorsion tensor $K_{ij}{}^k$ [11] as follows, and a number of relations [11]:

$$\Gamma^k_{ij} = \frac{1}{2}g^{kd}(\partial_i g_{dj} + \partial_j g_{di} - \partial_d g_{ij}) - K_{ij}{}^k = \{{}^{\,k}_{ij}\} - K_{ij}{}^k \tag{7.1.3}$$

$$K_{ij}{}^k = -S_{ij}{}^k + S_j{}^k{}_i - S^k{}_{ij} = -K_i{}^k{}_j \tag{7.1.4}$$

With torsion included, the geodesic/auto-parallel equation becomes [11]:

$$\frac{d^2x^k}{ds^2} + \Gamma^k_{(ij)}\frac{dx^i}{ds}\frac{dx^j}{ds} = 0 \tag{7.1.5}$$

Reintroducing the hat and index dimension, the following simple constraint prevents the torsion tensor dominating in the Lorentz force law:

$$O(\hat{S}_{AB}{}^C) \leq O(\hat{\Gamma}^C_{AB} + \hat{K}_{AB}{}^C) \quad \forall A, B \text{ and } C \in \{\text{a,b,c,4}\} \tag{7.1.6}$$

On the left hand side of $[LFL1]$(6.0.13) the torsion term is part of the 4D Christoffel symbol (noting its dependence here on torsion), whereas the right hand term contribution is:

$$-\hat{\Gamma}^a_{(4c)}\frac{dx^4}{d\tau}\frac{dx^c}{d\tau} - \hat{\Gamma}^a_{(b4)}\frac{dx^b}{d\tau}\frac{dx^4}{d\tau} = (2\hat{K}^a{}_{(4b)} + 2\widehat{\{{}^{\,a}_{4b}\}})\frac{dx^b}{d\tau}\frac{dx^4}{d\tau} \tag{7.1.7}$$

This leaves the Lorentz force law formally unchanged when torsion is *totally* antisymmetric. Empirically the torsion component of the Christoffel symbol in $[LFL2]$(6.0.17), at the classical limit, must be considered small relative to the Christoffel symbol. And thus the geodesics, and hence the classical scale geometries, both with and without torsion, must be for some reason approximately equivalent. By (7.1.3),(7.1.4),(7.1.5),(7.1.7) and total antisymmetry the geodesic equations do indeed become the same.

Antisymmetry of Torsion Ansatz: total antisymmetry of the torsion tensor is to be assumed at the usual classical limit in the sense that $S_{(ij)}{}^k = S_{(i}{}^k{}_{j)} = S^k{}_{(ij)}$.

It is noted that Einstein-Cartan theory, that adds torsion to the dynamics of relativity theory is most probably a minimal ω-consistent extension of general relativity [13][14] and therefore the use of (fully antisymmetric) torsion is not only natural, but arguably a necessity on philosophical and physical grounds.

7.2 A Brief Consistency Check

Having secured the Lorentz force law it is worth looking also at the assumption that the velocity of the charge does not change. Whilst under the charge ansatz momentum is indeed conserved, it is still necessary to show that lack of proper acceleration of the charged particle in the Kaluza dimension is feasible: that such a model has a chance of being consistent. We therefore look at this acceleration in exactly the same way as the previous investigation of the Lorentz force law:

$$\begin{aligned} 0 &= \frac{d^2x^4}{d\tau^2} + \hat{\Gamma}^4_{BC}\frac{dx^B}{d\tau}\frac{dx^C}{d\tau} \\ &= \frac{d^2x^4}{d\tau^2} + \hat{\Gamma}^4_{bc}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} + \hat{\Gamma}^4_{4c}\frac{dx^4}{d\tau}\frac{dx^c}{d\tau} + \hat{\Gamma}^4_{b4}\frac{dx^b}{d\tau}\frac{dx^4}{d\tau} + \hat{\Gamma}^4_{44}\frac{dx^4}{d\tau}\frac{dx^4}{d\tau} \end{aligned} \tag{7.2.1}$$

The last term once again vanishes only for sufficiently small $O(s+)$ and $O(\delta+)$ or if we make it a special constraint. The torsion, if admitted, of the two middle terms, cancels as before, when antisymmetry is invoked.

$2\hat{\Gamma}^4_{4c}\frac{dx^4}{d\tau}\frac{dx^c}{d\tau}$ - the two middle terms added together with a cancelling torsion, give: $-(A^dF_{cd} + O(f)O(\delta+) + O(f)O(s+) + \hat{g}^{44}\partial_c\phi^2)\ \frac{dx^4}{d\tau}\frac{dx^c}{d\tau}$. That is significantly smaller than the comparable term in the Lorentz force law. That is, given small enough $O(s+) << O(f+)$ for the last term and sufficiently small $O(\delta+)$. Thus, if $\frac{dx^c}{d\tau}$ is small relative to $\frac{dx^4}{d\tau}$ (so that we can also discount the first Christoffel symbol terms) we have that the 5th dimensional acceleration of the charge is small relative to any Lorentz forces.

This reasoning here is independent of whether there is a torsion tensor (even when not totally antisymmetric) as the torsion components are constrained to be of the same orders of magnitude as the Christoffel symbols.

7.3 Belinfante-Rosenfeld Stress Energy Tensor

The Einstein tensor defined using a torsion bearing connection will be labelled $\kappa\hat{P}$, it need not be symmetric. The constant is included here only because of the Gravitational constant, to be consistent with the literature. $\hat{P}$ is the Einstein-Cartan stress-energy or canonical energy-momentum tensor.

The Belinfante-Rosenfeld [12] stress-energy tensor $\hat{B}$ is a symmetric adjustment of $\hat{P}$ that adjusts for spin currents as sources. It can be defined equally for the 5D case. It is equivalent to the original Einstein tensor $\hat{G}$ [12] but is formed explicitly from the torsion bearing connection via $\hat{P}$.

8 Analysing The Alternative Field Equations

The next step is to investigate the alternative field equations in a little more detail in order to identify where we might find non-null electromagnetic fields. Or at least the various constraints that may be placed on them that could lead to inconsistencies. Noting that we are not yet considering matter models, but still the electrovacuum, or electroscalar 'Kaluza' vacuum.

The field equations in the introduction are related, but here we do not assume KCC but investigate the effect of the other possible cylinder conditions. This is done by setting the Ricci tensor to zero and looking at its components via the Christoffel symbols. This is done here over a range of limits and constraints. Initially torsion is not invoked.

The Christoffel Symbols Before Taking A Limit

$$2\hat{\Gamma}^A_{BC} = \sum_d \hat{g}^{AD}(\partial_B \hat{g}_{CD} + \partial_C \hat{g}_{DB} - \partial_D \hat{g}_{BC})$$

$$= \sum_d \hat{g}^{Ad}(\partial_B \hat{g}_{Cd} + \partial_C \hat{g}_{dB} - \partial_d \hat{g}_{BC}) + \hat{g}^{A4}(\partial_B \hat{g}_{C4} + \partial_C \hat{g}_{4B} - \partial_4 \hat{g}_{BC})$$

$$2\hat{\Gamma}^A_{bc} = \sum_d \hat{g}^{Ad}(\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc}) + \sum_d \hat{g}^{Ad}(\partial_b \phi^2 A_c A_d + \partial_c \phi^2 A_d A_b - \partial_d \phi^2 A_b A_c) + \hat{g}^{A4}(\partial_b \phi^2 A_c + \partial_c \phi^2 A_b - \partial_4 g_{bc} - \partial_4 \phi^2 A_b A_c)$$

$$2\hat{\Gamma}^A_{4c} = \sum_d \hat{g}^{Ad}(\partial_4 g_{cd} + \partial_4 \phi^2 A_c A_d + \partial_c \phi^2 A_d - \partial_d \phi^2 A_c) + \hat{g}^{A4}\partial_c \phi^2$$

$$2\hat{\Gamma}^A_{44} = 2\sum_d \hat{g}^{Ad}\partial_4 \phi^2 A_d - \sum_d \hat{g}^{Ad}\partial_d \phi^2 + \hat{g}^{A4}\partial_4 \phi^2$$

The Strong Electromagnetic Limit $\boldsymbol{\phi^2 = 1}$

$$2\hat{\Gamma}^A_{bc} = \sum_d \hat{g}^{Ad}(\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc}) + \sum_d \hat{g}^{Ad}(\partial_b A_c A_d + \partial_c A_d A_b - \partial_d A_b A_c) + \hat{g}^{A4}(\partial_b A_c + \partial_c A_b - \partial_4 g_{bc} - \partial_4 A_b A_c)$$
$$2\hat{\Gamma}^A_{4c} = \sum_d \hat{g}^{Ad}(\partial_4 g_{cd} + \partial_4 A_c A_d + \partial_c A_d - \partial_d A_c)$$
$$\hat{\Gamma}^A_{44} = \sum_d \hat{g}^{Ad}\partial_4 A_d$$

Simplifying...

$$2\hat{\Gamma}^a_{bc} = 2\Gamma^a_{bc} + \sum_d g^{ad}(A_b F_{cd} + A_c F_{bd}) + A^a \partial_4 g_{bc} + A^a \partial_4 A_b A_c$$
$$2\hat{\Gamma}^4_{bc} = -\sum_d A^d(\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc}) - \sum_d A^d(A_b F_{cd} + A_c F_{bd}) - (1 + \sum_i A_i A^i)(\partial_4 g_{bc} + \partial_4 A_b A_c) + (\partial_b A_c + \partial_c A_b)$$
$$2\hat{\Gamma}^a_{4c} = \sum_d g^{ad}(\partial_4 g_{cd} + \partial_4 A_c A_d) + \sum_d g^{ad} F_{cd}$$
$$2\hat{\Gamma}^4_{4c} = -\sum_d A^d(\partial_4 g_{cd} + \partial_4 A_c A_d) - \sum_d A^d F_{cd}$$
$$\hat{\Gamma}^a_{44} = \sum_d g^{ad}\partial_4 A_d$$
$$\hat{\Gamma}^4_{44} = -\sum_d A^d \partial_4 A_d$$

$2\hat{\Gamma}^a_{bc} = 2\Gamma^a_{bc} + g^{ad}(A_b F_{cd} + A_c F_{bd}) +$ (terms $<< O(f)O(f+)$), whether using WCC or averaged RCC.

The Strong Scalar Limit: $A_i = 0$

$2\hat{\Gamma}^A_{bc} = \sum_d \hat{g}^{Ad}(\partial_b g_{cd} + \partial_c g_{ab} - \partial_d g_{bc}) - \hat{g}^{A4}\partial_4 g_{bc}$
$2\hat{\Gamma}^A_{4c} = \sum_d \hat{g}^{Ad}\partial_4 g_{cd} \qquad + \hat{g}^{A4}\partial_c\phi^2$
$2\hat{\Gamma}^A_{44} = -\sum_d \hat{g}^{Ad}\partial_d\phi^2 \qquad + \hat{g}^{A4}\partial_4\phi^2$

Simplifying...

$\hat{\Gamma}^a_{bc} = \Gamma^a_{bc}$
$2\hat{\Gamma}^4_{bc} = -\frac{1}{\phi^2}\partial_4 g_{bc}$
$2\hat{\Gamma}^a_{4c} = \sum_d g^{ad}\partial_4 g_{cd}$
$2\hat{\Gamma}^4_{4c} = \frac{1}{\phi^2}\partial_c\phi^2$
$2\hat{\Gamma}^a_{44} = -\sum_d g^{ad}\partial_d\phi^2$
$2\hat{\Gamma}^4_{44} = \frac{1}{\phi^2}\partial_4\phi^2$

Constraints on the Ricci tensor

5D Ricci curvature $\hat{R} = 0$ (outside of matter models) produces the following:

$$\hat{R}_{ab} = \partial_C\hat{\Gamma}^C_{ab} - \partial_b\hat{\Gamma}^C_{aC} + \hat{\Gamma}^C_{ab}\hat{\Gamma}^D_{CD} - \hat{\Gamma}^C_{aD}\hat{\Gamma}^D_{bC} = 0 \tag{8.0.1}$$

$$R_{ab} = \partial_c\Gamma^c_{ab} - \partial_b\Gamma^c_{ac} + \Gamma^c_{ab}\Gamma^d_{cd} - \Gamma^c_{ad}\Gamma^d_{bc} \tag{8.0.2}$$

$$\begin{aligned} R_{ab} = R_{ab} - \hat{R}_{ab} = {} & (\partial_c\Gamma^c_{ab} - \partial_c\hat{\Gamma}^c_{ab}) - \partial_4\hat{\Gamma}^4_{ab} + (-\partial_b\Gamma^c_{ac} + \partial_b\hat{\Gamma}^c_{ac}) + \partial_b\hat{\Gamma}^4_{a4} \\ & (\Gamma^c_{ab}\Gamma^d_{cd} - \hat{\Gamma}^c_{ab}\hat{\Gamma}^d_{cd}) - \hat{\Gamma}^c_{ab}\hat{\Gamma}^4_{c4} - \hat{\Gamma}^4_{ab}\hat{\Gamma}^4_{44} - \hat{\Gamma}^4_{ab}\hat{\Gamma}^d_{cd} \\ & (-\Gamma^c_{ad}\Gamma^d_{bc} + \hat{\Gamma}^c_{ad}\hat{\Gamma}^d_{bc}) + [\hat{\Gamma}^c_{a4}\hat{\Gamma}^4_{bc} + \hat{\Gamma}^4_{ad}\hat{\Gamma}^d_{b4}] + \hat{\Gamma}^4_{a4}\hat{\Gamma}^4_{b4} \end{aligned} \tag{8.0.3}$$

These produce 10 constraints plus a symmetry condition, or equivalently 16 constraints component-by-component. Looking at it as 10 constraints by a symmetry condition is more useful, the 10 constraints corresponding to the 10 unknowns of the metric, the symmetry condition then being looked at separately: by inspection of the above (without torsion), we have that $\partial_b\hat{\Gamma}^4_{a4}$ is symmetric at the strong scalar limit. A similar symmetry condition holds at the strong electromagnetic limit, and more generally. There is, however, no constraint imposed on the strong scalar limit:

$$\begin{aligned} & \partial_b\hat{\Gamma}^4_{a4} = \partial_b(\tfrac{1}{\phi^2}\partial_a\phi^2) = \partial_a(\tfrac{1}{\phi^2}\partial_b\phi^2) \\ & \longleftrightarrow (\partial_b\tfrac{1}{\phi^2})\partial_a\phi^2 + \tfrac{1}{\phi^2}\partial_b\partial_a\phi^2 = (\partial_a\tfrac{1}{\phi^2})\partial_b\phi^2 + \tfrac{1}{\phi^2}\partial_a\partial_b\phi^2 \\ & \longleftrightarrow (\partial_b\tfrac{1}{\phi^2})\partial_a\phi^2 = (\partial_a\tfrac{1}{\phi^2})\partial_b\phi^2 \end{aligned} \tag{8.0.4}$$

Which is already symmetric by the chain rule. Whereas more generally (without torsion) we have the symmetry of:

$$-\partial_b \Gamma^c_{ac} + \partial_b \hat{\Gamma}^c_{ac} + \partial_b \hat{\Gamma}^4_{a4} \qquad \text{([SEM1] 8.0.5)}$$

This leads to gauge-like constraints on the fields which may well be contrary to both WCC and RCC. However this was not proven by the author.

[SEM1] will not be used here to draw conclusions about WCC or RCC

Further possible constraints on the field equations may be obtained by inspecting the other components of $\hat{R}$:

$$\hat{R}_{44} = \partial_C \hat{\Gamma}^C_{44} - \partial_4 \hat{\Gamma}^C_{4C} + \hat{\Gamma}^C_{44} \hat{\Gamma}^D_{CD} - \hat{\Gamma}^C_{4D} \hat{\Gamma}^D_{4C} = 0 \qquad \text{([SEM2] 8.0.6)}$$

$$\hat{R}_{a4} = \partial_C \hat{\Gamma}^C_{a4} - \partial_4 \hat{\Gamma}^C_{aC} + \hat{\Gamma}^C_{a4} \hat{\Gamma}^D_{CD} - \hat{\Gamma}^C_{aD} \hat{\Gamma}^D_{4C} = 0 \qquad \text{([SEM3] 8.0.7)}$$

The first equation $[SEM2]$8.0.6 means that all electromagnetic fields must be null when there is no torsion, no scalar field, nor a physical Kaluza dimension as will become clear in the following. The second relates to the Charge Ansatz. For future references and full generality we include the torsion dependent terms.

$$\begin{aligned}
&\hat{R}_{44} = 0 = \partial_c \hat{\Gamma}^c_{44} + \partial_4 \hat{\Gamma}^4_{44} - \partial_4 \hat{\Gamma}^c_{4c} - \partial_4 \hat{\Gamma}^4_{44} + \hat{\Gamma}^C_{44} \hat{\Gamma}^D_{CD} - \hat{\Gamma}^C_{4D} \hat{\Gamma}^D_{4C} \\
&= \partial_c \hat{\Gamma}^c_{44} - \partial_4 \hat{\Gamma}^c_{4c} + \hat{\Gamma}^c_{44} \hat{\Gamma}^D_{cD} + \hat{\Gamma}^4_{44} \hat{\Gamma}^D_{4D} - \hat{\Gamma}^c_{4D} \hat{\Gamma}^D_{4c} - \hat{\Gamma}^4_{4D} \hat{\Gamma}^D_{44} \\
&= \partial_c \hat{\Gamma}^c_{44} - \partial_4 \hat{\Gamma}^c_{4c} + \hat{\Gamma}^c_{44} \hat{\Gamma}^d_{cd} + \hat{\Gamma}^4_{44} \hat{\Gamma}^d_{4d} - \hat{\Gamma}^c_{4d} \hat{\Gamma}^d_{4c} - \hat{\Gamma}^4_{4d} \hat{\Gamma}^d_{44} - \hat{\Gamma}^c_{44} K_{c4}{}^4 + \hat{\Gamma}^c_{44} K_{4c}{}^4 \\
&= \partial_c \hat{\Gamma}^c_{44} - \partial_4 \hat{\Gamma}^c_{4c} + \hat{\Gamma}^c_{44} \hat{\Gamma}^d_{cd} + \hat{\Gamma}^4_{44} \hat{\Gamma}^d_{4d} - \hat{\Gamma}^c_{4d} \hat{\Gamma}^d_{4c} - \hat{\Gamma}^4_{4d} \hat{\Gamma}^d_{44} + 2\hat{\Gamma}^c_{44} \hat{S}_{c4}{}^4
\end{aligned} \qquad (8.0.8)$$

Whether using (6.0.14), (6.0.15) or imposing the additional (somewhat arbitrary) constraint of (6.0.16), the $\partial_c \hat{\Gamma}^c_{44}$ term makes no significant contribution (a significant contribution has to be at least $O(f+)O(f+)$ as will become apparent). And this is true whether using WCC, RCC or even adding a torsion tensor.

The $-\partial_4 \hat{\Gamma}^c_{4c}$ term, independently of (6.0.14), (6.0.15) and (6.0.16), shows the difficulty of using WCC to adapt to double derivatives: $O(\delta++)$ should be proportionately smaller than simply the $<< O(f++)$ that WCC gives, in order to take into account the small Kaluza dimension. But there is not yet an algorithm associated with the definition of WCC to investigate this, so we extend it such that $O(\delta++) = O(\delta+)O(\delta+)$ to be consistent across the chain rule, i.e. via distributivity. Similar problems exist when taking it over potentials. Hereafter full expansion of WCC across the chain rule using distributivity is assumed. This makes WCC more similar to RCC than originally expected. However in what follows WCC fails to provide the necessary terms and is discontinued. Assuming (6.0.16), RCC is however shown to be borderline with respect to admitting sufficient terms to allow non-null solutions (We can ignore torsion by (7.1.6), its addition will add the same order of magnitude again):

$$2\partial_4\hat{\Gamma}^c_{4c} = \partial_4(g^{cd}(\partial_4 g_{cd})) + \partial_4 g^{cd}(\partial_4 A_c A_d) + \partial_4(g^{cd}F_{cd}) + \partial_4(g^{c4}\partial_c\phi^2) \tag{8.0.9}$$

We have under WCC, with chain rule algorithm (distributivity), some small terms:

$$O(\delta+)O(\delta+) + O(\delta+)O(\delta+)O(f) + O(\delta+)O(f+) + O(\delta+)O(f)O(s+)$$

Where we have been channelled into multiplying by an extra O(f) in the last term due to the preceding discussion on consistency across the chain rule and distributivity. Without such the last term might have been $O(\delta+)O(s+)$. But still not significant under WCC. If however the distributivity property was disallowed, then the first term might become significant. In such cases however we are really dealing with the resonance concept for which RCC was introduced.

We have under RCC with (6.0.16) some terms as follows (assuming distributivity again):

$$O(v++)/B^2 + O(v+)O(f)O(f+)/B^2 + O(v+)O(f+)/B + O(f+)O(v+)/B^2$$

Whilst the others are small, the first term is:

$$O(v++)O(f+)O(f+)/A^2O(v+)O(v+) = O(f+)O(f+)/A^2O(v)$$

Although it seems very close to call using orders of magnitude estimates, we can see that this term could be O(f+)O(f+) if A were not very large. This is at a level where experimental concerns start to be significant. This tricky situation is tantalising and led to many forced attempts by the author to find a way to make it work. But such a successful resolution is not presented here.

The question of using (6.0.16) is also itself rather arbitrary and in no way presents us with an ideal consequence in any case.

Assuming that WCC can not be used to obtain the desired components in the rest of the terms, which is now also shown, we drop WCC as insufficient.

RCC on the other hand has been brought close to a sufficient order of magnitude, but only by assuming (6.0.16), and will therefore continue to be used.

Further problems and doubts arise with respect to (6.0.16) and RCC however, due to a simple observation of the necessary term: it is a derivative of another term in the Kaluza direction. Thus we would expect it to be precisely zero if tracing the term around any loop we arrived back at where we started. It is thus a particularly unconvincing term to have in such a critical role. It therefore needs further investigation, and ideally comparison with a superior alternative:

Torsion is added at the end of the calculation to the significant Christoffel symbols, the reason for using torsion is readily seen - it is an alternative resolution that does not require RCC and (6.0.16), and provides the required degrees of freedom in a far simpler and more convincing way.

$$\partial_4 \hat{\Gamma}^c_{4c} = +\hat{\Gamma}^c_{44}\hat{\Gamma}^d_{cd} + \hat{\Gamma}^4_{44}\hat{\Gamma}^d_{4d} - \hat{\Gamma}^c_{4d}\hat{\Gamma}^d_{4c} - \hat{\Gamma}^4_{4d}\hat{\Gamma}^d_{44} + 2\hat{\Gamma}^c_{44}\hat{S}_{c4}{}^4 \tag{8.0.10}$$

Expanding each term, discounting insignificant contributions (but not yet selecting RCC over WCC), and before considering torsion we have:

$\hat{\Gamma}^c_{44} = g^{cd}\partial_4 A_d - \frac{1}{2}g^{cd}\partial_d\phi^2$ which is in any case $<<<< O(f+)$
$\hat{\Gamma}^4_{44} = \frac{1}{2}\partial_4\phi^2 - A^d\partial_4 A_d + \frac{1}{2}A^d\partial_d\phi^2$
$2\hat{\Gamma}^d_{cd} = 2\Gamma^d_{cd} + g^{de}(\partial_c A_d A_e) - A^d(\partial_c A_d + \partial_d A_c - \partial_4 g_{cd})$
$2\hat{\Gamma}^4_{4d} = -A^e(\partial_4 g_{de} + \partial_4 A_d A_e + \partial_d A_e - \partial_e A_d) + \hat{g}^{44}\partial_d\phi^2$
$2\hat{\Gamma}^c_{4c} = g^{cd}(\partial_4 g_{cd} + \partial_4 A_c A_d + \partial_c A_d - \partial_d A_c) + \hat{g}^{c4}\partial_c\phi^2$
$2\hat{\Gamma}^a_{4c} = g^{ad}(\partial_4 g_{cd} + \partial_4 A_c A_d + \partial_c A_d - \partial_d A_c) + \hat{g}^{a4}\partial_c\phi^2$

This reduces to the two middle terms after applying distributivity to the first term. Both the first and the last pure Christoffel term (and therefore also the torsion containing term) then being bounded by $<< O(f+)O(f+)$.

The second term is, under WCC: ([the lowest of $O(s+)$ or $O(\delta+)$] $+O(f)O(\delta+)+O(f)O(s+)$) times $(O(\delta+) + O(f+) + O(f)O(s+))$ which is $<< O(f+)O(f+)$ as required to discard it provided $O(s+) \leq O(f+)$.

The second term is, under RCC: $(O(s+)/B + O(f)O(f+)/B + O(f)O(s+))$ times $(O(v+)/B+O(f)O(f+)/B+O(f+)+O(f)O(s+))$. This is likewise small.

The significant term remains, relative to which the others are small, and so this becomes the constraint:

$$\partial_4 \hat{\Gamma}^c_{4c} = -\hat{\Gamma}^c_{4d}\hat{\Gamma}^d_{4c}$$

$$\partial_4 \hat{\Gamma}^c_{4c} = (g^{ce}(\partial_4 g_{de}+\partial_4 A_d A_e+\partial_d A_e-\partial_e A_d)+\hat{g}^{a4}\partial_d\phi^2)(g^{df}(\partial_4 g_{cf}+\partial_4 A_c A_f+\partial_c A_f - \partial_f A_c) + \hat{g}^{a4}\partial_c\phi^2)$$

Whether simplified by WCC or RCC, we get the constraint:

$$\partial_4 \hat{\Gamma}^c_{4c} = -g^{ce}(F_{de})g^{df}(F_{cf}) = F_{ab}F^{ab} \tag{8.0.11}$$

Which is the definition of null solutions when the LHS is 0. This is the final check required before discarding WCC, for which indeed the LHS is insignificant in magnitude. Whether the LHS is able to be $O(f+)O(f+)$ is the issue for RCC, in this way non-null solutions may be permitted. The nature of the key term suggests not. RCC should most likely be discarded, it does not lead to a satisfactory resolution, unless errors of the correct order of magnitude can be found in the Lorentz force law experimentally - an unlikely prospect.

At this point we can put the torsion tensor back in (with respect to the significant orders of magnitude) as follows:

$$
\begin{aligned}
&0 = \partial_4 \hat{\Gamma}^c_{4c} = \partial_4 \widehat{\{{}^{\,c}_{4c}\}} - \partial_4 \hat{K}_{4c}{}^c = -[g^{ce}(F_{de}) - \hat{K}_{4d}{}^c][g^{df}(F_{cf}) - \hat{K}_{4c}{}^d] \\
&= -[(F^c_d) - \hat{K}_{4d}{}^c][(F^d_c) - \hat{K}_{4c}{}^d] \\
&= -F_{cd}F^{cd} + 2\hat{K}_{4d}{}^c F^d_c - \hat{K}_{4d}{}^c \hat{K}_{4c}{}^d \quad \text{where all terms are O(f+)O(f+)}
\end{aligned}
$$

([SEM2b] 8.0.12)

Regardless of WCC or RCC, or even KCC, torsion potentially breaks the $[SEM2]$8.0.6 constraint that forces Kaluza theory solutions to be null. This breaking of $[SEM2]$8.0.6 occurs regardless of the total antisymmetry of the torsion tensor. It is also consistent with (7.1.6).

9 Experimental Considerations And Postulates

From the discussion previously two possible ways to allow for non-null electromagnetic solutions were apparent: firstly using RCC and the arbitrary constraint (6.0.16), along with ignoring doubts concerning the feasibility of the term in question. Secondly, simply the introduction of torsion into the scheme.

Of the two possibilities the second might be prefered on grounds of elegance alone, in particular the ω-consistency of torsion in general relativity is philosophically encouraging, but perhaps alone not sufficient to convince. The difficulty of finding a convincing term is not in itself proof that the RCC approach has failed, though perhaps a more astute analysis might eventually provide it. The apparent arbitrariness of (6.0.16) does not preclude the possibility that there is a totally good explanation for it, and that it is in fact correct. Nevertheless, although it has not been provided here, a convincing theoretical reason to discard this possibility may well exist. The brief consistency check when introducing torsion, for example, also provides a bound on $O(\delta+)$ and $O(s+)$ that may be useful in this.

In addition to purely deductive methods, however, experimental validity must also be investigated. They are quicker, easier and often more reliable than a very theoretical argument, even though purely theoretical arguments may have the potential to be correct. The Lorentz force law has been well tested. In particular it has been tested via Coulomb's law, applicable in the case of electric fields which interest us here as being non-null. It has been tested to very great accuracy.

We can see from the discussion accompanying the derivation of (6.0.7) that B is needed to be sufficiently large, and therefore that A must be sufficiently large, otherwise the $O(\delta+)$ term mentioned, $g^{cd}\partial_4\hat{g}_{bd}$, will start to be significant and affect the Lorentz force law. And this will happen quite quickly. For an error in the Lorentz force law of 1 in 1000, we would need an $O(A) \approx 1000$, but we would not be concerned for many more orders of magnitude than that. Thus accuracy in the testing of the Lorentz force law, furnishes directly a lower bound on A.

On the other hand the discussion of $[SEM2]$8.0.6 requires a small A. That is, it provides an upper bound: $O(A^2) \approx 1/O(v)$. This is to make the term $g^{cd}\partial_4\hat{g}_{cd}$ large enough. An estimate of $1/\sqrt{O(v)} \approx 10^{4.5}$ (see 4.2.1) gives us just such an upper bound for A, and it is apparently well below lower bounds for A given by experimental accuracy known for the Lorentz force law. Though it is not the purpose here to go into experimental details. We have, within the limits of the methods used, a contradiction.

Thus we have actually shown that neither relaxing cylindricity, nor allowing for scalar field oscillations readily furnishes us with the required non-null solutions, nor are they necessary even as contributory factors. But rather it has been shown that torsion is both likely necessary and is in any case sufficient to resolve $[SEM2]$8.0.6 and extend the range of Kaluza theory solutions to include such essentials as static electric fields - notwithstanding further constraints from $[SEM3]$8.0.7 to be investigated shortly.

What has been presented then is an argument to suggest the use of torsion in Kaluza theory as a key feature to resolve some of its foundational issues. We have argued for discarding attempts at using weaker cylindricity and/or oscillations of the scalar field, despite the fact that investigating this was the original intent of the work.

10 Summary of Postulates

We have arrived at a 5D Kaluza theory with torsion, that is, where the Ricci flat part of Kaluza space includes the torsion tensor in the defining connection, and total antisymmetry of torsion is assumed. The following order of magnitude constraints are imposed on various terms to provide an electromagnetic/usual classical limit:

$O(\delta+) <<<< O(f+)$
$O(s+) <<<< O(f+)$
$O(\hat{S}_{AB}^{\ \ C}) \leq O(\hat{\Gamma}^C_{AB} + \hat{K}_{AB}^{\ \ C}) \qquad \forall A, B$ and $C \in$ {a,b,c,4}

That is: (6.0.14),(6.0.15) and (7.1.6)

Other broad assumptions were made and are still needed: the Geodesic Assumption, causality (and the existence of a hypersurface), suitable topology and so on. But these, although needed for the limit of general relativity, are not part of the theory per se. Similalry loosening the above constraints may yield empirically testable hypotheses.

Despite the length and complexity of the argument we have in fact arrived at straight forward postulates, and require none of the messiness previously entertained. The froth has boiled down to a few simple ideas. Nor indeed, it has been argued, are there other clear ways forward. An exhaustive a search as possible has been attempted. The above, or substantially similar postulates, appear to be the most natural selection of postulates. We can interpret the total antisymmetry of torsion ansatz as simply defining a limit without torsion effects on the Lorentz force law, and as yet uncalculated (presumably small) effects on the details of the field equations.

11 Non-Null Solutions And Degrees of Freedom

In order to check that we do indeed have enough degrees of freedom to allow for non-null fields, $[SEM3]8.0.7$ will now be investigated. The postulates previously stated make the calculation far easier.

$$\hat{R}_{a4} = \partial_C \hat{\Gamma}^C_{a4} - \partial_4 \hat{\Gamma}^C_{aC} + \hat{\Gamma}^C_{a4}\hat{\Gamma}^D_{CD} - \hat{\Gamma}^C_{aD}\hat{\Gamma}^D_{4C} = 0 \qquad \text{([SEM3] 11.0.1)}$$

Torsion terms will be kept, terms strictly less than $O(f{+}{+})$ or $O(f{+})O(v{+})$ will be discarded.

$$\begin{aligned}\hat{R}_{a4} &= \partial_c\hat{\Gamma}^c_{a4} + \partial_4\hat{\Gamma}^4_{a4} - \partial_4\hat{\Gamma}^c_{ac} - \partial_4\hat{\Gamma}^4_{a4} + \hat{\Gamma}^c_{a4}\hat{\Gamma}^D_{cD} + \hat{\Gamma}^4_{a4}\hat{\Gamma}^D_{4D} - \hat{\Gamma}^c_{aD}\hat{\Gamma}^D_{4c} - \hat{\Gamma}^4_{aD}\hat{\Gamma}^D_{44}\\ &= \partial_c\hat{\Gamma}^c_{a4} - \partial_4\hat{\Gamma}^c_{ac} + \hat{\Gamma}^c_{a4}\hat{\Gamma}^d_{cd} + \hat{\Gamma}^c_{a4}\hat{\Gamma}^4_{c4} + \hat{\Gamma}^4_{a4}\hat{\Gamma}^d_{4d} - \hat{\Gamma}^c_{ad}\hat{\Gamma}^d_{4c} - \hat{\Gamma}^c_{a4}\hat{\Gamma}^4_{4c} - \hat{\Gamma}^4_{ad}\hat{\Gamma}^d_{44}\\ &= \partial_c\hat{\Gamma}^c_{a4} - \partial_4\hat{\Gamma}^c_{ac} + \hat{\Gamma}^c_{a4}\hat{\Gamma}^d_{cd} + \hat{\Gamma}^4_{a4}\hat{\Gamma}^d_{4d} - \hat{\Gamma}^c_{ad}\hat{\Gamma}^d_{4c} - \hat{\Gamma}^4_{ad}\hat{\Gamma}^d_{44} + \hat{\Gamma}^c_{a4}(\hat{K}_{4c}{}^4 - \hat{K}_{c4}{}^4)\end{aligned}$$

Removing small terms due to Postulates:

$$\hat{R}_{a4} = 0 = \partial_c\hat{\Gamma}^c_{a4} + \hat{\Gamma}^c_{a4}\hat{\Gamma}^d_{cd} - \hat{\Gamma}^c_{ad}\hat{\Gamma}^d_{4c}$$

$$\begin{aligned}\hat{\Gamma}^c_{a4} &= \tfrac{1}{2}[\hat{g}^{cd}(\partial_4 g_{ad} + \partial_4\phi^2 A_a A_d + \partial_a\phi^2 A_d - \partial_d\phi^2 A_a) + \hat{g}^{c4}\partial_a\phi^2] - \hat{K}_{a4}{}^c\\ \hat{\Gamma}^d_{4c} &= \tfrac{1}{2}[\hat{g}^{de}(\partial_4 g_{ce} + \partial_4\phi^2 A_c A_e + \partial_c\phi^2 A_e - \partial_e\phi^2 A_c) + \hat{g}^{d4}\partial_c\phi^2] - \hat{K}_{4c}{}^d\end{aligned}$$

$$\begin{aligned}\partial_c\hat{\Gamma}^c_{a4} &= \tfrac{1}{2}\partial_c F^c_a - \partial_c\hat{K}_{a4}{}^c\\ \hat{\Gamma}^c_{a4}\hat{\Gamma}^d_{cd} &= [\tfrac{1}{2}F^c_a - \hat{K}_{a4}{}^c][\Gamma^d_{cd}]\\ -\hat{\Gamma}^c_{ad}\hat{\Gamma}^d_{4c} &= -[\Gamma^c_{ad}][\tfrac{1}{2}F^d_c - \hat{K}_{c4}{}^d]\end{aligned}$$

$$0 = \frac{1}{2}\partial_c F^c_a - \partial_c\hat{K}_{a4}{}^c + [\frac{1}{2}F^c_a - \hat{K}_{a4}{}^c][\Gamma^d_{cd}] - [\Gamma^c_{ad}][\frac{1}{2}F^d_c - \hat{K}_{c4}{}^d] \qquad \text{([SEM3b] 11.0.2)}$$

Setting $\hat{K}_{a4}{}^{c}$ to 0 or to $\frac{1}{2}F_a^c$ does nothing but rederive null solutions via $[SEM2b]$8.0.12 and antisymmetry. These are not the solutions we are looking for. That there should be other solutions is determined by the fact that via $[SEM2b]$8.0.12 and $[SEM3]$8.0.7 there are only 5 constraints on $\hat{K}_{a4}{}^{c}$, yet being antisymmetric we have 6 degrees of freedom. This provides room to look for explicit non-null solutions. There is no reason here to think that non-null solutions are prohibited.

We do however have, when torsion is included, the 16 constraints associated with setting the 5D Ricci tensor to 0, that is the constraints associated with $[SEM1]$8.0.5. 10 of which can be assigned to delimiting the 4D metric and 5 of the other 6 to the remaining components of the 5D metric, which although it includes the electromagnetic fields does not include the torsion which, via the preceding, determines nullity and non-nullity. Here we see that we are left with 1 extra constraint. This could be used to constrain any component of the torsion tensor, leaving no particular constraint on $\hat{K}_{a4}{}^{c}$ and thus allowing enough degrees of freedom, exactly 1 degree of freedom, for non-null electromagnetic fields, relative to the relevant constraints: $[SEM2b]$8.0.12 and $[SEM3b]$11.0.2. The 6 degrees of freedom might equally be paired off with torsion terms that do not depend on nullity/non-nullity, i.e the $\hat{K}_{ab}{}^{c}$ terms. Either way we have enough degrees of freedom available thanks to the torsion.

The outstanding issue is the Charge Ansatz. Until this point no attempt to justify this ansatz has been made, apart from the fact that its assumption has been necessary. This will now be investigated.

12 The Nature Of Charge

In general relativity at a weak field limit the conservation of momentum-energy can be given in terms of the stress-energy tensor as follows [9]. Energy:

$$\frac{\partial \hat{T}^{00}}{\partial t} + \frac{\partial \hat{T}^{i0}}{\partial x^i} = 0 \tag{12.0.1}$$

Momentum in the j direction:

$$\frac{\partial \hat{T}^{0j}}{\partial t} + \frac{\partial \hat{T}^{ij}}{\partial x^i} = 0 \tag{12.0.2}$$

This can be applied equally to Kaluza theory (with matter models that are not Ricci flat in the Kaluza space). It needs to be applied to the underlying (not here Ricci flat) Kaluza space. We have a description of conservation of momentum in the 5th dimension as follows:

$$\frac{\partial \hat{T}^{04}}{\partial t} + \frac{\partial \hat{T}^{i4}}{\partial x^i} = 0 \tag{12.0.3}$$

This is accurate at the weak field limit, and so is valid at the usual classical scale. We also have by the postulates the situation where i=4 can be treated as small. Thus the conservation of 'charge' becomes the property of a 4-vector current at the usual classical scale, which we know to be conserved:

$$V = (\hat{T}^{04}, \hat{T}^{14}, \hat{T}^{24}, \hat{T}^{34}) \tag{12.0.4}$$

$$\partial_0 \hat{T}^{04} + \partial_1 \hat{T}^{14} + \partial_2 \hat{T}^{24} + \partial_3 \hat{T}^{34} = 0 \tag{12.0.5}$$

(V is a vector for the same reasons as the vector potential is a vector.)

At this point we can calculate this 'current' in terms of the metric and Ricci tensor, noting that even with torsion present the original Einstein tensor constructed from the torsion free connection and metric can still be used. We have two connections on one manifold. And two possible Einstein tensors. The Einstein-Cartan stress-energy tensor $\hat{P}$ with torsion may be asymmetric.

For the moment we shall ignore torsion, and revert to the Einstein tensor $\hat{G}$ and torsion free connection.

We can identify the current with the following components of the 5D (torsion free) Einstein tensor by discounting small terms, and thus consistently with (6.0.5) and the previous derivations of the Lorentz force law:

$$V \approx (\hat{R}^{04} - 1/2\hat{g}^{04}\hat{R}, \hat{R}^{14} - 1/2\hat{g}^{14}\hat{R}, \hat{R}^{24} - 1/2\hat{g}^{24}\hat{R}, \hat{R}^{34} - 1/2\hat{g}^{34}\hat{R}) \tag{12.0.6}$$

$$V \approx (\hat{R}^{04}, \hat{R}^{14}, \hat{R}^{24}, \hat{R}^{34}) \text{ due to O(f) terms in the metric.} \tag{12.0.7}$$

The following parts of the Ricci tensor will be looked at due to its significance $O(f++)$:

$$\hat{X}^{a4} = \partial_C \hat{\Gamma}^{Ca4} - \partial^4 \hat{\Gamma}^{Ca}{}_C \tag{12.0.8}$$

Regardless of torsion (if we repeat this calculation for the torsion bearing Einstein tensor $\hat{P}$) the other part of the Ricci tensor will have a significance $<< O(f++)$ since, being always compounded of two Christoffel symbols, it starts off bounded by $O(v+)O(v+)$. We need only one of the Christoffel symbols to be O(f+) and we have already an order of significance less than $O(f++)$ via distributivity. That there is always such a term follows from the fact that at least one of each pair of Christoffel symbols in the remaining part of the Ricci tensor will have an index that is 4. This then also makes insignificant any contribution from the corresponding torsion tensors. The terms are simplified and discarded using the Postulates and distributivity, and by comparing relative significances.

$2V \approx 2\hat{X}^{a4} = \partial_C[\hat{g}^C{}_D(\partial^a \hat{g}^{D4} + \partial^4 \hat{g}^{Da} - \partial^D \hat{g}^{a4})] - \partial^4[\hat{g}_{CD}(\partial^a \hat{g}^{DC} + \partial^C \hat{g}^{Da} - \partial^D \hat{g}^{aC})]$

$$
\begin{aligned}
&\approx \partial_C[\hat{g}^C{}_D(\partial^a \hat{g}^{D4} + \partial^4 \hat{g}^{Da} - \partial^D \hat{g}^{a4})] \\
&\approx \partial_C[\hat{g}^C{}_D(\partial^a \hat{g}^{D4} - \partial^D \hat{g}^{a4})] \\
&\approx \partial_c[\hat{g}^c{}_d(\partial^a \hat{g}^{d4} - \partial^d \hat{g}^{a4})] \\
&\approx \partial_c(\partial^a \hat{g}^{c4} - \partial^c \hat{g}^{a4}) \\
&\approx -\partial_c(\partial^a A^c - \partial^c A^a) \\
&\approx -\partial_c F^{ac}
\end{aligned}
\tag{12.0.9}
$$

This, as the explicit equation of Maxwellian charge sources (albeit approximate), provides justification for the approximate, but consistent, association of Toth charge with Maxwell charge. It is no longer an ansatz as such at all, but now follows from the Postulates.

However we have here ignored the torsion tensor. This doesn't necessarily matter as the sought for property of zero divergence in the (torsion free) Einstein tensor will still present a conservation law. In particular the kinematic momenta and energies are equivalent for both connections as, as mentioned in section 7 in relation to total antisymmetry (and now also noting the Geodesic Assumption), the geodesic paths followed by particles with momentum and energy are the same using either connection. But it would be better to have it defined in terms of the torsion bearing connection and hence in terms of $\hat{P}$. This can be done via the Belinfante-Rosenfeld tensor $\hat{B}$. Since $\hat{B}$ is equivalent to $\hat{G}$ [12] we immediately have the required result in terms of tensors defined by torsion and the torsion bearing connection.

13 Conclusion

Kaluza's 1921 theory of gravity and electromagnetism using a fifth wrapped-up spatial dimension is at the root of many modern attempts to develop new physical theories. Lacking non-null electromagnetic fields however the theory is incomplete.

This work sought to investigate the issue of non-null electromagnetic solutions in Kaluza variant theories by the method of an order of magnitude analysis under various assumptions, such as variant cylinder conditions, and making the ansatz that charge can be identified with 5D momentum - the initial justification for which was the ease of derivation of the Lorentz force law under limited circumstances.

The non-null solutions were not found by relaxing the cylinder condition or allowing for scalar fields, and further the Lorentz force law was maintained best with respect to Toth charges by maintaining a tight cylinder condition and very limited scalar field oscillations. This was despite a search that tried to be as exhaustive as possible. Attempts at using both the scalar field and various 5th dimensional oscillations proved tantalising but ultimately ineffective. And the program to find non-null solutions as a result of these two factors failed.

The derivation of the Lorentz force law was not, however, impaired by the admission of torsion. And at the same time adding torsion along with reasonable and necessary constraints meant that unhelpful bounds on the field equations were loosened considerably. Further, enough degrees of freedom for the sought for non-null solutions were found. And this was achieved without the help of weakening the cylinder conditions or a significant fluctuating scalar field.

We can therefore conclude that when 5D momentum is to be identified with Maxwellian charge, and when there is no (or very weak) scalar field, that the cylinder condition must be more or less as given by Kaluza, and that torsion is necessary to get non-null electromagnetic fields, such as static electric fields. It was also noted that in any case Einstein-Cartan theory is a natural and necessary extension of general relativity via ω-consistency, thus the use of torsion is really very natural, and all the more so with the resulting resolution to the foundational issues of Kaluza theory.

The identification of 5D momentum and charge is shown to be a consequence of the new Postulates developed through the paper, and in particular the presence of torsion that allows the Postulates to offer the full range of electromagnetic solutions.

Some theorists investigate relativity theory with torsion, and some theorists investigate Kaluza or Kaluza Klein theories. Here it is shown why it makes sense to investigate both together, and in particular why Kaluza theory should have torsion added. This work has resolved foundational issues associated with classical Kaluza theory and provides motivation for further investigation. Further, the coupling such as it is between the torsion tensor and other tensors (primarily the electromagnetic field) means that there are, in principle, testable phenomena, though the effects may be small. What we are really talking about potentially is non-Eisntein Maxwell electrovacuum solutions to general relativity carrying the electromagnetic field, but curved space alternatives. This assumes that the field equations yielded byincluding torsion do not always perfectly reduce to electrovacuum solutions, which in any case is unlikely, but this ultimately awaits definitive proof or disproof via explicit calculation of the variant field equations. The overall purpose of the theory is the same as that of Kaluza, to provide an explanation for all classical electromagnetic phenomena in terms of geometry. The variant theory merges two serious attempts at unifying electromagnetism and gravity. The argument justifies this.

14 Aknowledgements

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