On the Growth of Meromorphic Solutions of a type of Systems of Complex Algebraic Differential Equations*

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Abstract This paper is concerned with the growth of meromorphic solutions of a class of systems of complex algebraic differential equations. A general estimate the growth order of solutions of the systems of differential equation is obtained by Zalacman Lemma. We also take an example to show that the result is right.

Keywords normal family; order; systems of complex algebraic differential equations meromorphic function.

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1 Introduction and Main Results

We use the standard notation of the Nevanlinna theory of meromorphic functions (see e.g.[1, 2]).

In 1998, W. Bergweiler [3] considered the order of the solutions of complex differential equation $(f')^{n} = P[f]$, where $P[f](z) = \sum_{r \in I} a_{r}(z, f)(f')^{r_{1}} \cdots (f^{(n)})^{r_{n}}$, $a_{r}(z, f)$ is a rational function in $z$ and $f$, $f$ is a finite index set.

He proved the following result

Theorem A [2] Let $w(z)$ be any meromorphic solution of algebraic differential equation (2), $n > u$, then the growth order $\sigma(w)$ of $w(z)$ are finite.

In 2008, Su X.F. and Gao L.Y. [4] investigated the order of the solutions of a type of the systems of higher-order complex algebraic differential equations as follows:

\begin{align*}
\left\{ \begin{array}{l}
(w_{2}^{(n)})^{m_{1}} = a(w_{1} + c(z))^{p}, \\
(w_{1}^{(n)})^{m_{2}} = \Omega_{k}(w_{2}),
\end{array} \right.
\end{align*}

where $\Omega_{k}(w_{2}) = \sum_{j=0}^{q} b_{j}(z)(w_{2})^{j_{0}}(w_{2}^{(n)})^{j_{1}} \cdots (w_{2}^{(n)})^{j_{n}}$ is a differential polynomial, $b_{j}(z)$ ($j = 0, 1, 2, \cdots , n$) is a polynomial, $\Omega_{k}(w_{2}) = (w_{2}^{(k)} \cdots (w_{2}^{(n)})^{k_{n}}$ is a differential monomial, $m_{1}, m_{2}, p$ and $q$ are nonnegative integers.

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1 INTRODUCTION AND MAIN RESULTS

**Definition 1.1** For (1.1), write $u_j = j_1 + 2j_2 + \cdots + nj_n$, $j \in I$, $u = \max_{j \in I}\{u_j\}$, $v = k_1 + 2k_2 + \cdots + nk_n$.

**Definition 1.2** Let $w = (w_1, w_2)$ be a solution of (1.1), the order $\rho_w$ of the solution $w$ of system of higher-order complex algebraic differential equations $w = (w_1, w_2)$ is defined by $\rho_w = \max_{i=1,2}\{\rho_{w_i}\}$, where denote the order of $w_i (i = 1, 2)$.

Let $F$ be a family of meromorphic functions defined on $D$, $F$ is said to be normal on $D$ if every sequence $f_n \in F$, there exists a subsequence $\{f_{n_i}\}$, such that $\{f_{n_i}\}$ uniformly converges in every point on $D$, conversely, $F$ is not normal on $D$.

They obtained

**Theorem B** Let $w = (w_1, w_2)$ be a non-polynomial meromorphic solution of (1.1). If $c^{(n)}(z) = 0, nm_1m_2 + vp > n^2m_2p + up$, then $\rho_w < \infty$.

A recent paper Yuan et al.[4] established a general estimate of growth order of $w(z)$, the result may be stated as follows:

**Theorem C** Let $w(z)$ be meromorphic in complex plane, $n \in N$, $\Omega[w]$ be a differential polynomial with the form (2), $n > u$. If $w(z)$ satisfies the differential equation $[w'(z)]^n = \Omega[w]$, then the growth order $\sigma(w)$ of $w(z)$ satisfies

$$\sigma(w) \leq 2 + \frac{2\deg_{\infty} a}{n-u}.$$  

It is natural to ask whether get more the precise estimate of growth of solutions of the system of differential equations (1.1)? we get following theorem:

**Theorem 1.1** Let $w = (w_1, w_2)$ be a non-polynomial meromorphic solution of (1.1). If $c^{(n)}(z) = 0, nm_1m_2 + vp > n^2m_2p + up$, then $\rho_w \leq 2\alpha + 2$, where

$$\alpha = \frac{pq}{(m_1 - np)nm_2 + vp - up}.$$  

First we quote the following Lemma.

**Lemma 1.1** (Zalcman [6]) Let $F$ be a family of meromorphic functions in unit disc $\Delta$, then $F$ is not normal in $D$ if and only if there exist

1. a real number $0 < r < 1$,
2. a sequence of complex number $\{z_n\}, |z_n| < r$ ,
3. a sequence of functions $f_n \in F$ and
4. a sequence of positive numbers $\rho_n \to 0^+$,

such that $g_n(\xi) = f_n(z_n + \rho_n \xi)$ converges locally uniformly (with respect to the spherical metric) to a non-constant meromorphic function $g_n(\xi)$ for any compact subset on $C$, where $\rho_n = \frac{1}{\rho'(z_n)}$ and $\rho'$ denote the spherical derivative of $f$.

**Lemma 1.2** Let $f$ be a meromorphic function in complex plane, $\sigma := \sigma(f)$. then for each $0 < \rho < \frac{\sigma-2}{2}$, there exist points $a_n \to \infty (n \to \infty)$, such that

$$\lim_{n \to \infty} \frac{f_{\rho}^n(a_n)}{|a_n|^{\rho}} = +\infty$$  

(1.2)
2 Proof of Theorem 1.1

For the systems of complex differential equations (1.1), differentiating the first of equation, we get
\[
\left(\frac{w_2^{(n)}}{a}ight)^{m_1} \left(\frac{w_2^{(n+1)}}{w_2^{(n)}} - \frac{a'}{a}\right)^p = p^p (w_1' + c'(z))^p,
\]
that is
\[
(w_2^{(n)})^{m_1-p} \left(\frac{m_1 a w_2^{(n+1)} - a' w_2^{(n)}}{a}\right)^p = p^p (w_1' + c'(z))^p,
\]
In general, we have
\[
(w_2^{(n)})^{m_1-np} \left(\frac{Q_n(z, w_2)}{a^n}\right)^p = p^np (w_1^{(n)} + c^{(n)}(z))^p, \tag{2.1}
\]
where
\[
Q_1(z, w_2) = m_1 a w_2^{(n+1)} - a' w_2^{(n)},
\]
\[
Q_{n+1}(z, w_2) = (1 - np) a w_2^{(n+1)} - (np + 1) a' w_2^{(n)} Q_n(z, w_2) + npQ_n'(z, w_2).
\]

\(Q_n(z, w_2)\) is a polynomials of \(w_2^{(n)}, w_2^{(n+1)}, \ldots, w_2^{(2n)}\) and \(a, a', \ldots, a^{(n)}\) homogenous of degree \(n\) with respect to \(w_2^{(n)}, w_2^{(n+1)}, \ldots, w_2^{(2n)}\) and \(a, a', \ldots, a^{(n)}\) respectively.

By (2.1) and the second equation of the systems (1.1), we obtain
\[
\left(\frac{(w_2^{(n)})^{m_1-np} (Q_n(z, w_2))^{p}}{a^{np+1} p^{np}}\right)^m z = \left(\frac{\Omega(w_2)}{\Omega_k(w_2)}\right)^p, \tag{2.2}
\]
We suppose the growth \(\alpha < \frac{np}{2} - 1\), by Lemma 1.2, we have a sequence \(\{z_k\}, z_k \to \infty, \frac{w_2(z_k)}{z_k^d} \to \infty\).

Where \(w_2^d\) denote the spherical derivative of \(w_2\). It show that functional family is not normal at \(z = 0\). By Lemma 1, we have both sequence \(\{c_k\}\) and \(\{\rho_k\}\), they satisfy
\[
|c_k - z_k| < 1, \rho_k \to 0, \text{ meanwhile } h_k(z) = w_2(c_k + \rho_k z) \text{ is local convergence to nontrivial meromeric function } h.
\]
By the proof of Lemma 1, we can suppose \(\rho_k = \frac{1}{w_2(z_k)}\) and \(w_2^d(c_k) \leq w_2^d(z_k)\), such that \(c_k^d \rho_k \to 0 (k \to \infty)\) for any constant \(d\).

When \(c_k + \rho_k z\) replace \(z\) in (2.2), we obtain
\[
\left(\frac{(w_2^{(n)}(c_k + \rho_k z))^{m_1-np} Q_n^p(c_k + \rho_k z, w_2(c_k + \rho_k z))}{a^{np+1} p^{np}}\right)^m z = \left(\frac{\Omega(c_k + \rho_k z, w_2(c_k + \rho_k z))}{\Omega_k(c_k + \rho_k z, w_2(c_k + \rho_k z))}\right)^p,
\]
that is
\[
\left(\frac{(w_2^{(n)}(c_k + \rho_k z))^{m_1-np} Q_n^p(c_k + \rho_k z, h_k(z))}{a^{np+1} p^{np}}\right)^m z = \left(\frac{\Omega(c_k + \rho_k z, h_k(z))}{\Omega_k(c_k + \rho_k z, h_k(z))}\right)^p. \tag{2.3}
\]
Meanwhile, we have
\[
(w_2^{(n)}(c_k + \rho_k z) = \frac{p^{-n} h_k^{(n)}(z)}{\rho_k^m} z = \frac{p^{-n} h_k^{(n)}(z)}{\rho_k^m} \tag{2.4}
\]
Using (2.3) and (2.4), we obtain
\[
\left(\frac{(h_k^{(n)}(z))^{m_1-np} \rho_k^{-n(m_1-np)} Q_n^p(c_k + \rho_k z, h_k(z))}{a^{np+1} p^{np}}\right)^m z = \left(\frac{\Omega(h_k(z))}{\Omega_k(h_k(z))}\right)^p.
\]
\[ |(h_k^{(n)}(z))^{(m_1-np)m_2}| \leq |a|^{m_2(np+1)}p^{m_2np} \frac{\sum_{j=0}^{q} |b_j(c_k + \rho_k z)(h_k(z))^{j_0} P_j(h_k(z))|^p \rho_k^{(m_1-np)m_2+vp-u_jp}}{|\Omega_k(h_k(z))|^p|Q_k^p(h_k(z))|^{m_2}}, \]

where \( u_j = j_1 + 2j_2 + \cdots + nj_n \), \( P_j(h_k(z)) = (h_k'(z))^{j_1} \cdots (h_k^{(n)}(z))^{j_n} \).

For every fixed \( |z| \), \( |b_j(c_k + \rho_k z)h_k^{(j)}(z)c_k^{-q}|^p \) is bound, there maybe outside a set of finite measure as \( k \to \infty \).

Because of \( nm_1m_2 + vp > n^2m_2p + u_jp \), we have \( |c_k|^{pq} \rho_k^{(m_1-np)m_2+vp-u_jp} \to 0 \)
i.e. \( |c_k|^{(m_1-np)m_2+vp-u_jp} \rho_k \to 0 \).

Because \( \rho_k \) is a polynomial with respect to \( z \) by (2.5), it is a contradiction to condition of theorem 1.1. Therefore \( \rho_{w_2} \leq 2\alpha + 2 \).

From the first equation of (1.1), we have \( \rho_{w_1} \leq 2\alpha + 2. \) Hence \( \rho_w \leq 2\alpha + 2. \)

The proof of Theorem 1.1 is complete.

3 Example

Example 3.1 For solutions \( w_1(z) = e^{z^2} \), \( w_2(z) = e^{z^2} \) satisfies the following system of algebraic differential equations

\[
\begin{align*}
\left\{ 
\begin{array}{l}
(w_2'(z))^2 = 4z^2w_1^2(z), \\
(w_1'(z))^3 = \frac{8z^3w_2^3(z)(z^2 + 32z^2w_2^2(z))}{w_2^2(z)},
\end{array}
\right.
\end{align*}
\] (3.1)

where \( n = p = u = q = 1, vp = 2, m_1 = 2, m_2 = 3, a(z) = 4z^2, c(z) = 0, \) we have \( \alpha = \frac{1}{4} \). It is easy to get \( \rho(w) = 2 < 2 + \frac{1}{2} \), which show that our result is right.

References