The Goldbach Conjecture

Ricardo G. Barca

Abstract

The binary Goldbach conjecture asserts that every even integer greater than 4 is the sum of two primes. In order to prove this statement, we start by defining a kind of double sieve of Eratosthenes as follows. Given a positive even integer x, we sift out from $[1, x]$ all those elements that are congruents to 0 modulo p, or congruents to x modulo p, where p is a prime less than $\sqrt{x}$. So, any integer in the interval $[\sqrt{x}, x]$ that remains unsifted is a prime p for which either $x - p = 1$ or $x - p$ is also a prime. Then, we introduce a new way to formulate this sieve, which we call the sequence of k-tuples of remainders. Using this tool, we obtain a lower bound for the number of elements in $[1, x]$ that survives the sifting process. We prove, for every even number $x > p_{35}^2$, that there exist at least 3 integers in the interval $[1, x]$ that remains unsifted. This proves the binary Goldbach conjecture for every even number $x > p_{35}^2$, which is our main result.

1 Introduction

1.1 The sieve method and the Goldbach's problem

In the year 1742 Goldbach wrote a letter to his friend Euler telling him about a conjecture involving prime numbers. *Goldbach's conjecture: Every even number greater than* 4 *is the sum of two primes.* The Goldbach Conjecture is one of the oldest unsolved problems in number theory [7]. This conjecture was verified many times with powerful computers, but could not be proven. In March 30, 2012, T. Oliveira e Silva verified the conjecture for $n \leq 36 \times 10^{17}$ [11]. Mathematicians had achieved some partial results in their efforts to prove this conjecture. Vinogradov proved, in 1937, that every sufficiently large odd number is the sum of three primes [13]. Later, in 1973, J.R. Chen showed that every sufficiently large even number can be written as the sum of either two primes or a prime and the product of two primes [6]. In 1975, H. Montgomery and R.C. Vaughan showed that 'most' even numbers were expressible as the sum of two primes [10]. Recently, a proof of the related ternary Goldbach conjecture, that every odd integer greater than 5 is the sum of 3 primes, has been given by Harald Helfgott [14].

In this paper we prove (Main Theorem, Section 8) the following:

(a) Every even integer greater than $149^2 = 22,201$ is the sum of two odd primes.

(b) As n runs through the positive even integers, the number of Goldbach partitions of n tends to infinity.

It is well known that one of the principal ways of attacking the problem of the Goldbach's conjecture has been through the use of sieve methods. Viggo Brun [5] was the first to obtain a result, as an approximation to Goldbach's conjecture: *Every sufficiently large even integer is a sum of two integers, each having at most nine prime factors.* Later, other mathematicians in the area of sieve theory have improved this initial result.

In the context of sieve theory, the sieve method consist in removing elements of a list of integers, according to a set of rules; for instance, given a finite sequence A of integers, we could remove from A those members which lie in a given collection of arithmetic progressions. In the original sieve of Eratosthenes, we start with the integers in the interval $[1, x]$, where x is a positive real number, and sift out all those which are divisible by the primes $p < \sqrt{x}$. Therefore, any integer that remains unsifted is a prime in the interval $[\sqrt{x}, x]$.

We begin by describing formally the sieve method; we use, as far as possible, the concepts and notation of the book by Cojocaru and Ram Murty [3], chapters 2 and 5. Let $\mathscr{A}$ be a finite set of integers and let $\mathscr{P}$ be the sequence of all primes; let $z \geq 2$ be a positive real number. Furthermore, to each $p \in \mathscr{P}$, $p < z$ we have associated a subset $\mathscr{A}_p$ of $\mathscr{A}$. The *sieve problem* is to estimate, from above and below, the size of the set

$$\mathscr{A} \setminus \bigcup_{\substack{p \in \mathscr{P} \\ p < z}} \mathscr{A}_p,$$

which consists of the elements of the set $\mathscr{A}$ after removing the elements of all the subsets $\mathscr{A}_p$. We call the procedure of removing the elements of the subsets $\mathscr{A}_p$ from the set $\mathscr{A}$ the *sifting process*. The *sifting* function $S(\mathscr{A}, \mathscr{P}, z)$ is defined by the equation

$$S(\mathscr{A}, \mathscr{P}, z) = \left| \mathscr{A} \setminus \bigcup_{\substack{p \in \mathscr{P} \\ p < z}} \mathscr{A}_p \right|,$$

and counts the elements of $\mathscr{A}$ that have survived the sifting process. Now, let $\mathscr{P}_z$ be the set of primes $p \in \mathscr{P}, p < z$; and for each subset I of $\mathscr{P}_z$, denote by

$$\mathscr{A}_I = \bigcap_{p \in I} \mathscr{A}_p.$$

Then, the inclusion–exclusion principle gives us

$$S(\mathscr{A}, \mathscr{P}, z) = \sum_{I \subseteq \mathscr{P}_z} (-1)^{|I|} \left| \mathscr{A}_I \right|,$$

where for the empty set $\emptyset$ we have $\mathscr{A}_\emptyset = \mathscr{A}$. We often take $\mathscr{A}$ to be a finite set of positive integers, and $\mathscr{A}_p$ to be the subset of $\mathscr{A}$ consisting of elements lying in some congruence classes modulo p.

Using this notation, we can now define formally the sieve of Eratosthenes. Let $\mathscr{A} = \{n \in \mathbb{Z}_+ : n \leq x\}$, where $x \in \mathbb{R}, x > 0$, and let $\mathscr{P}$ be the sequence of all primes; let $z = \sqrt{x}$. Now, to each $p \in \mathscr{P},\ p < z$, we associate the subset $\mathscr{A}_p$ of $\mathscr{A}$, defined as follows: $\mathscr{A}_p = \{n \in \mathscr{A} : n \equiv 0 \pmod p\}$. Then, when we sift out from $\mathscr{A}$ all those elements of every set $\mathscr{A}_p$, the unsifted members of $\mathscr{A}$ in the interval $[\sqrt{x}, x]$ are the integers that are not divisible by primes of $\mathscr{P}$ less than z; that is to say, any integer remaining in $[\sqrt{x}, x]$ is a prime. Let

$$P(z) = \prod_{\substack{p \in \mathscr{P} \\ p < z}} p.$$

Furthermore, if d is a squarefree integer such that $d|P(z)$, we define the set

$$\mathscr{A}_d = \bigcap_{p|d} \mathscr{A}_p.$$

So, from the inclusion–exclusion principle we obtain

$$S(\mathscr{A}, \mathscr{P}, z) = \sum_{d|P(z)} \mu(d) \left| \mathscr{A}_d \right|, \tag{1}$$

where $\mu(d)$ is the Möbius function; and from (1) it can be derived the well-known formula of Legendre

$$S(\mathscr{A}, \mathscr{P}, z) = \sum_{d|P(z)} \mu(d) \left| \mathscr{A}_d \right| = \sum_{d|P(z)} \mu(d) \left\lfloor \frac{x}{d} \right\rfloor.$$

In a first instance, the sieve of Eratosthenes is very useful for finding the prime numbers between $\sqrt{x}$ and x. However, from a theoretical point of view, the experts in sieve theory are interested in estimating for every x the number of integers remaining after the sifting process has been performed.

The use of the Möbius function is a simple way to approach a sieve problem; however, satisfactory results are rather hard to achieve, unless z is very small. We shall illustrate this with the application to the sieve of Eratosthenes-Legendre, given in the book by Halberstam and Richert [2], Chapter 1, Section 5.

As usual in sieve theory, instead of $|\mathscr{A}|$ we can use a close approximation X to $|\mathscr{A}|$. Furthermore, for each prime p we choose a multiplicative function $w(p)$ so that $(w(p)/p)X$ approximates to $|\mathscr{A}_p|$. Then, for each squarefree integer d we have that $(w(d)/d)X$ approximates to $|\mathscr{A}_d|$, and we can write

$$|\mathscr{A}_d| = \frac{w(d)}{d} X + R_d.$$

Then, substituting this into (1),

$$S(\mathscr{A},\mathscr{P},z)=\sum_{d|P(z)}\mu\left(d\right)\left(\frac{w(d)}{d}X+R_d\right)=\sum_{d|P(z)}\mu\left(d\right)\frac{w(d)}{d}X+\sum_{d|P(z)}\mu\left(d\right)R_d. \tag{2}$$

On the other hand, since w is a multiplicative function,

$$\sum_{d|P(z)}\mu\left(d\right)\frac{w(d)}{d}=\prod_{\substack{p\in\mathscr{P}\\p<z}}\left(1-\frac{w\left(p\right)}{p}\right)=W\left(z\right),$$

and, substituting this into (2),

$$S(\mathscr{A},\mathscr{P},z)=XW\left(z\right)+\sum_{d|P(z)}\mu\left(d\right)R_d.$$

Hence, we can write

$$S(\mathscr{A},\mathscr{P},z)=XW\left(z\right)+\theta\sum_{d|P(z)}|R_d|\qquad(\theta\leq 1).$$

Furthermore, if $|R_d|\leq w(d)$ and $w(p)\leq A_0$, for some constant $A_0\geq 1$, we get

$$S(\mathscr{A},\mathscr{P},z)=XW\left(z\right)+\theta\left(1+A_0\right)^z.$$

See [2, Theorem 1.1] for details. In the case of the sieve of Eratosthenes-Legendre, taking $X=x$, $w(p)=1$, $A_0=1$, we obtain

$$S(\mathscr{A},\mathscr{P},z)\leq x\prod_{\substack{p\in\mathscr{P}\\p<z}}\left(1-\frac{1}{p}\right)+2^z. \tag{3}$$

From this we can see that the error term will be very large provided that z is not sufficiently small compared with x. In spite of this, taking $z=\log x$, the formula in (3) can be used to obtain an elementary upper bound for $\pi(x)$. See [2, Ch. 1, (5.8)].

Now, suppose that we express the Goldbach's problem as a sieve problem; it is clear that in order to prove this conjecture what we require is a lower bound for the sifting function. However, there is a well-known phenomenon in sieve theory, called the 'parity barrier' or the 'parity problem', that was first clarified by Selberg (see [16]). It appears that sieve methods cannot distinguish between numbers with an even number of prime factors and an odd number of prime factors. The parity problem was described briefly by Terence Tao [15] as follows: 'If A is a set whose elements are all products of an odd number of primes (or are all products of an even number of primes), then (without injecting additional ingredients), sieve theory is unable to provide non-trivial lower bounds on the size of A.' This means that in order to solve the Goldbach's problem we should be able to define a suitable sieve, and furthermore we should find a way to circumvent the parity problem.

1.2 A sieve for the Goldbach's problem

Let $\mathscr{P}$ be the sequence of all primes; and given $p_k\in\mathscr{P}$, let $m_k=p_1p_2p_3\cdots p_k$. From now on, and throughout this paper, for convenience, we take x to be an even integer greater than $p_4^2=49$. Note that if p_k is the greatest prime less than $\sqrt{x}$, every even number $x>49$ satisfies $p_k^2<x<p_{k+1}^2<m_k$; this fact is very important for our purposes, as we shall see later.

Now, how can we construct a sieve to tackle the Goldbach's problem? Given a positive even integer x, as we have seen in the previous subsection, using the sieve of Eratosthenes we can get the primes between $\sqrt{x}$ and x. Assume that among the primes between $\sqrt{x}$ and x there is at least a prime q such that $x-q$ is also a prime. Then, to attack the Goldbach's problem we need a sieve that sift out all the integers in the interval $[1,x]$ which are divisible by the primes $p<\sqrt{x}$, as the sieve of Eratosthenes does, and that additionally sift out, from the primes p remaining in $[\sqrt{x},x]$, all those such that $x-p$ is not a prime.

Then, in order to construct such a sieve, we propose to modify the sieve of Eratosthenes as follows: First, we sift out all those integers n in the interval $[1,x]$ such that $n\equiv 0 \pmod p$, where $p<\sqrt{x}$; thus, any integer that remains unsifted is a prime in the interval $[\sqrt{x},x]$. Next, we sift out all those integers n that remains in $[\sqrt{x},x]$ such that

$n \equiv x \pmod{p}$. It is easy to see that any number that remains unsifted in $[\sqrt{x}, x]$ is a prime q such that $x - q$ is not divisible by the primes $p < \sqrt{x}$; so, either $x - q = 1$ or $x - q$ is a prime.

Let us define formally this sieve, which we call the Sieve associated with x, or alternatively the *Sieve* I. Let $\mathscr{A} = \{n \in \mathbb{Z}_+ : n \leq x\}$. Let $\mathscr{P}$ be the sequence of all primes; and let $z = \sqrt{x}$. Let

$$P(z) = \prod_{\substack{p \in \mathscr{P} \\ p < z}} p = m_k.$$

Now, to each $p \in \mathscr{P}$, $p < z$, we associate the subset $\mathscr{A}_p$ of $\mathscr{A}$, defined as follows: $\mathscr{A}_p = \{n \in \mathscr{A} : n \equiv 0 \pmod{p} \text{ or } n \equiv x \pmod{p}\}$. Furthermore, if d is a squarefree integer such that $d | P(z)$, we define the set

$$\mathscr{A}_d = \bigcap_{p | d} \mathscr{A}_p.$$

In this case, the sifting function

$$S(\mathscr{A}, \mathscr{P}, z) = \left| \mathscr{A} \setminus \bigcup_{\substack{p \in \mathscr{P} \\ p < z}} \mathscr{A}_p \right|$$

counts the primes q in the interval $[\sqrt{x}, x]$, such that either $x - q = 1$ or $x - q$ is a prime. As in the case of the sieve of Eratosthenes-Legendre, the inclusion–exclusion principle gives us

$$S(\mathscr{A}, \mathscr{P}, z) = \sum_{d | P(z)} \mu(d) |\mathscr{A}_d|.$$

Now, $S(\mathscr{A}, \mathscr{P}, z) > 2$ implies that x is the sum of two primes; and if this is proved for all x, the Goldbach's conjecture would be proved. Then, the solution of the Goldbach's problem depends on establishing a positive lower bound for the sifting function. However, we can not hope to find a suitable lower bound using alone the usual sieve methods, due to the parity problem, which was already mentioned in this Introduction. So far, all attempts to solve the Goldbach's problem by the usual sieve techniques did not have the expected success. For these reasons, the strategy used in this paper differs quite a lot from the usual approach in sieve theory. In the next subsection we shall begin by introducing another way of formulating a sieve problem.

1.3 The sequence of k-tuples of remainders

In this paper we propose to use another formulation for this kind of sieves, which is able to show all the details of the sifting process, and will allow us to obtain a lower bound for the number of elements that remain unsifted. For this purpose, we begin by introducing the notion of sequence of k-tuples of remainders. Let $\{p_1, p_2, p_3, \ldots, p_k\}$ be the ordered set of the first k prime numbers. Suppose that for every natural number n we form a k-tuple, the elements of which are the remainders of dividing n by $p_1, p_2, p_3, \ldots, p_k$; so, we have a sequence of k-tuples of remainders. If we arrange these k-tuples from top to bottom, the sequence of k-tuples of remainders can be seen as a matrix formed by k columns and infinitely many rows, where each column is a periodic sequence of remainders modulo $p_h \in \{p_1, p_2, p_3, \ldots, p_k\}$. It is easy to prove that the sequence of k-tuples of remainders is periodic, and the period is equal to $m_k = p_1 p_2 p_3 \cdots p_k$.

Suppose that within the periods of every sequence of remainders modulo p_h (a given column of the matrix), we define some (not all) of the remainders as *selected* remainders, no matter the criterion for selecting the remainders. Consequently, some k-tuples have one or more selected remainders, and other k-tuples do not have any selected remainder. If a given k-tuple has one or more selected remainders, we say that it is a *prohibited* k-tuple; otherwise we say that it is a *permitted* k-tuple.

Now, in a general context, a sieve is a tool or device that separates, for instance, coarser from finer particles. Then, given a sieve device we can define a 'sieve problem', for instance, to count the number of finer particles that pass through the sieve device. We can think of a sequence of k-tuples as a 'sieve device', in the sense that when a set of integers is 'fed' into the sieve device (the sequence of k-tuples), it separates the integers associated to permitted k-tuples from integers associated to prohibited k-tuples. The sieve problem, in this case, is to estimate the number of integers that 'pass through' the sieve device; that is, to estimate the number of permitted k-tuples attached to some of the integers in the original input set.

Given an even integer $x > 49$, we formulate the Sieve I (the Sieve associated with x) by means of a sequence of k-tuples as follows. Let $\mathscr{P}$ be the sequence of all primes; let $z = \sqrt{x}$, and let p_k be the greatest prime less than z. With the index k corresponding to the prime p_k, we construct the sequence of k-tuples of remainders, where the rules for selecting remainders are the following: In the k-tuples of the sequence, if there are any zeroes, or any of the remainders of dividing x by $p_1, p_2, p_3, \ldots, p_k$, these remainders are defined as selected remainders. So, within the periods of every sequence of remainders modulo p_h (a given column of the matrix), the remainder 0 is always a selected remainder, and besides, if p_h does not divide x, the resulting remainder is a second selected remainder. Let $\mathscr{A}$ be the set consisting of the indices n of the sequence of k-tuples that lie in the interval $[1, x]$. For each $p \in \mathscr{P}$, $p < z$, the set $\mathscr{A}_p \subset \mathscr{A}$ consists of the indices n for which the corresponding element in the sequence of remainders modulo p is a selected remainder. Then, the indices of the prohibited k-tuples lying in $\mathscr{A}$ are sifted out; and the indices of the permitted k-tuples lying in $\mathscr{A}$ remain unsifted. The *sifting* function is given by the the number of permitted k-tuples whose indices lie in the interval $\mathscr{A}$. We shall define more formally the Sieve I in Section 8.

Remark 1.1. Note that given a k-tuple whose index is $n < x$, if $n \equiv 0 \pmod p$ or $n \equiv x \pmod p$ for at least one $p < \sqrt{x}$, then it is a prohibited k-tuple; and if $n \not\equiv 0 \pmod p$ and $n \not\equiv x \pmod p$ for every $p < \sqrt{x}$, then it is a permitted k-tuple.

Therefore, given an even integer $x \geq 49$ ($p_k^2 < x < p_{k+1}^2$), it is easy to see that the ordered set of k-tuples whose indices lie in the interval $[1, x]$ of the sequence is only an alternative formulation of the Sieve associated with x (the Sieve I), which was described before by using the usual sieve theory notation. It follows that the indices (greater than 1) of the permitted k-tuples lying within $[1, x]$ are primes p such that either $x - p$ is a prime or $x - p = 1$. Note that this form of the sieve gives us a detailed picture of the sifting process; other reasons for using this formulation for sieves based on a sequence of k-tuples will be explained later.

n	**2**	**3**	**5**	**7**	n	**2**	**3**	**5**	**7**
1	1	1	1	1	31	1	1	1	3
2	0	2	2	2	32	0	2	2	4
3	1	0	3	3	33	1	0	3	5
4	0	1	4	4	34	0	1	4	6
5	1	2	0	5	35	1	2	0	0
6	0	0	1	6	36	0	0	1	1
7	1	1	2	0	37	1	1	2	2
8	0	2	3	1	38	0	2	3	3
9	1	0	4	2	39	1	0	4	4
10	0	1	0	3	40	0	1	0	5
11	1	2	1	4	41	1	2	1	6
12	0	0	2	5	42	0	0	2	0
13	1	1	3	6	43	1	1	3	1
14	0	2	4	0	44	0	2	4	2
15	1	0	0	1	45	1	0	0	3
16	0	1	1	2	46	0	1	1	4
17	1	2	2	3	47	1	2	2	5
18	0	0	3	4	48	0	0	3	6
19	1	1	4	5	49	1	1	4	0
20	0	2	0	6	50	0	2	0	1
21	1	0	1	0	51	1	0	1	2
22	0	1	2	1	52	0	1	2	3
23	1	2	3	2					
24	0	0	4	3					
25	1	1	0	4					
26	0	2	1	5					
27	1	0	2	6					
28	0	1	3	0					
29	1	2	4	1					
30	0	0	0	2					

$11 + 41 = 52$

$23 + 29 = 52$

Figure 1

Example 1.1. Figure 1 illustrates how the Sieve I can be used to find some Goldbach partitions for the even number $x = 52$. We proceed as follows:

1. We make a list of the primes less than $\sqrt{52}$. We obtain $\{2, 3, 5, 7\}$.
2. We compute the remainders of dividing $x = 52$ by the prime moduli of the list. We obtain $\{0, 1, 2, 3\}$.
3. In every k-tuple we select each 0, and also the elements $\{1, 2, 3\}$, corresponding to the moduli $\{3, 5, 7\}$, respectively. (The selected remainders are circled.)
4. Now, we colour gray the permitted k-tuples. The arrows show the corresponding Goldbach partitions. Note that there is no permitted k-tuple for the partition $47 + 5$. (The relation between permitted k-tuples and Goldbach partitions is given in Section 8.)

1.4 The auxiliary Sieve II

To prove part *(a)* of the Main Theorem, we need to prove for the Sieve I that for every even number $x > 149^2$, the sifting function is greater than or equal to 3. (Note that the index 1 could be associated with a permitted k-tuple.) To prove part *(b)*, it suffices to prove that the sifting function tends to ∞, as $x \to \infty$.

However, we can see that, no matter the formulation, the Sieve I is a 'static' sieve; that is, given an even number x, we can formulate a specific Sieve I for this even number x. For our purposes, we need a 'dinamic' sieve, which is able of working as $x \to \infty$. Suppose that given $x > 49$ and using the Sieve I we have a way to compute the number of permitted k-tuples whose indices lie in $[1, x]$; then, we could prove the Main theorem by constructing a sequence of sieves associated with every even number $x > 49$. That is, we could construct a sequence where the elements are sequences of k-tuples, each one for every even number $x > 49$, and compute the number of permitted k-tuples whose indices lie in the interval $[1, x]$ of each sequence of k-tuples.

Now, using the Sieve I, the implementation of this idea finds some difficulties. For instance, if $x = 50$ the Sieve I can be described as follows: Since the greatest prime less than $\sqrt{50}$ is $p_4 = 7$, we have $k = 4$; so, we construct the sequence of 4-tuples of remainders. In the 4-tuples of the sequence, the selected remainders are the zeroes, or the remainders of dividing x by p_1, p_2, p_3, p_4. Let $\mathscr{A}$ be the set consisting of the indices of the sequence of 4-tuples that lie in the interval $[1, 50]$.

Suppose that we go to the next even integer $x = 52$. In this case, we have again $k = 4$, and the sequence of 4-tuples of remainders is the same as before, but now the set $\mathscr{A}$ consists of the indices that lie in $[1, 52]$, and the selected remainders take specific values for $x = 52$. In addition, as x runs through the even numbers, when $x > 121$ we have $k > 4$, because the greatest prime less than $\sqrt{x}$ will be $p_k > p_4 = 7$. The difficulty resides in the handling of all these variables as x runs through all the even numbers. On the other hand, if x is divisible by any of the primes $p_1, p_2, p_3, \ldots, p_k$, the remainder of dividing x by p_h $(1 \le h \le k)$ is 0; therefore, in each sequence of remainders modulo p_h $(1 \le h \le k)$ that form the sequence of k-tuples, there could exist one or two selected remainders within the period of the sequence (if there is only one selected remainder, it is always 0). This is an additional serious difficulty in order to derive a formula for computing the sifting function.

For all these reasons it is preferable to work with a more general kind of sieve, for which the sequence of k-tuples is more 'homogeneous' than that corresponding to the Sieve I, in the sense that in each sequence of remainders modulo p_h $(1 < h \le k)$ that form the sequence of k-tuples of this new sieve there exist always two selected remainders in every period of the sequence. Then, we introduce another sieve, which we call simply the *Sieve* II. We describe the Sieve II in the form proposed before, by means of a sequence of k-tuples, as follows. Let $\mathscr{P}$ be the sequence of all primes; and let p_k $(k \ge 4)$ be a prime of the sequence. With the index k corresponding to the prime p_k, we construct the sequence of k-tuples of remainders, where the rules for selecting remainders are the following: In every sequence of remainders modulo p_h $(1 < h \le k)$ that form the sequence of k-tuples there are always two selected remainders r and r' modulo p_h; in the sequence of remainders modulo $p_1 = 2$ there is only one selected remainder r modulo p_1. Let $\mathscr{B}$ be the set consisting of the indices of the sequence of k-tuples that lie in the interval $[1, y]$, where y is an integer that satisfies $y > p_k$. For each $p \in \mathscr{P},\ p \le p_k$, the set $\mathscr{B}_p \subset \mathscr{B}$ consists of the indices n for which the corresponding element in the sequence of remainders modulo p is a selected remainder. The indices of the prohibited k-tuples lying in $\mathscr{B}$ are sifted out; and the indices of the permitted k-tuples lying in $\mathscr{B}$ remain unsifted. The sifting function is defined by the equation

$$T(\mathscr{B}, \mathscr{P}, p_k) = \left| \mathscr{B} \setminus \bigcup_{\substack{p \in \mathscr{P} \\ p \le p_k}} \mathscr{B}_p \right|,$$

and counts the number of permitted k-tuples whose indices lie in $\mathscr{B}$. We shall define more formally the Sieve II in Section 2.

Remark 1.2. In this case, given a k-tuple whose index is n, if $n \equiv r \pmod p$ or $n \equiv r' \pmod p$ for at least one $p \le p_k$, where r, r' are the selected remainders modulo p, then it is a prohibited k-tuple; and if $n \not\equiv r \pmod p$ and $n \not\equiv r' \pmod p$ for every $p \le p_k$, then it is a permitted k-tuple.

Note that the unsifted elements in $\mathscr{B}$ may be or may be not prime numbers; indeed, the Sieve II is a collection of sieves, one for each particular choice of the selected remainders.

n	**2**	**3**	**5**	**7**
1	1	1	1	1
2	0	2	2	2
3	1	0	3	3
4	0	1	4	4
5	1	2	0	5
6	0	0	1	6
7	1	1	2	0
8	0	2	3	1
9	1	0	4	2
10	0	1	0	3
11	1	2	1	4
12	0	0	2	5
13	1	1	3	6
14	0	2	4	0
15	1	0	0	1
16	0	1	1	2
17	1	2	2	3
18	0	0	3	4
19	1	1	4	5
20	0	2	0	6
21	1	0	1	0
22	0	1	2	1
23	1	2	3	2
24	0	0	4	3
25	1	1	0	4
26	0	2	1	5
27	1	0	2	6
28	0	1	3	0
29	1	2	4	1
30	0	0	0	2
31	1	1	1	3
32	0	2	2	4
33	1	0	3	5
34	0	1	4	6
35	1	2	0	0
·	·	·	·	·

n	**2**	**3**	**5**	**7**
·	·	·	·	·
176	0	2	1	1
177	1	0	2	2
178	0	1	3	3
179	1	2	4	4
180	0	0	0	5
181	1	1	1	6
182	0	2	2	0
183	1	0	3	1
184	0	1	4	2
185	1	2	0	3
186	0	0	1	4
187	1	1	2	5
188	0	2	3	6
189	1	0	4	0
190	0	1	0	1
191	1	2	1	2
192	0	0	2	3
193	1	1	3	4
194	0	2	4	5
195	1	0	0	6
196	0	1	1	0
197	1	2	2	1
198	0	0	3	2
199	1	1	4	3
200	0	2	0	4
201	1	0	1	5
202	0	1	2	6
203	1	2	3	0
204	0	0	4	1
205	1	1	0	2
206	0	2	1	3
207	1	0	2	4
208	0	1	3	5
209	1	2	4	6
210	0	0	0	0

Figure 2

Now, suppose that in the Sieve II we take $\mathscr{B} = \{n : 1 \le n \le p_k^2\}$. Then, given $x > 49$ an even number that satisfies $p_k^2 < x < p_{k+1}^2$, we can construct the sequence of k-tuples associated to the Sieve I; and using the same k, we can construct the sequence of k-tuples associated to the Sieve II. So, we can compare for every even number $x > 49$ the sifting function of the Sieve I with the sifting function of the attached Sieve II. That is, we can compare the number of permitted k-tuples whose indices lie in the interval $[1, x]$ of the sequence of k-tuples corresponding to the Sieve I, with the number of permitted k-tuples whose indices lie in the interval $[1, p_k^2]$ of the sequence of k-tuples corresponding to the Sieve II. We shall prove later (Lemma 8.2) for every even number $x > 49$ that, under the given conditions, the value of the sifting function corresponding to the Sieve I, is greater than or equal to the minimum value of the sifting function corresponding to the Sieve II.

Example 1.2. For $k = 4$ ($p_k = 7$), the period of the sequence of k-tuples is equal to 210. The first 35, and the last 35 4-tuples in the interval $[1, 210]$ (the first period of the sequence), are pictured in Figure 2, for a given choice of selected remainders. The Sieve II is given by the k-tuples whose indices lie in $[1, 7^2]$.

Using the Sieve II, we can now construct a sequence of sieves indexed by k (a sequence of sequences of k-tuples), where each Sieve II of the sequence is the ordered set of k-tuples whose indices lie in the interval $[1, p_k^2]$ of the sequence of k-tuples.

Now, let $x > 49$ be an even number such that $p_k^2 < x < p_{k+1}^2$. Suppose that for this sequence of Sieves II we prove that for $k \geq 35$ the sifting function is greater than or equal to 3. In this case, for the Sieve associated with x (the Sieve I) the sifting function is also greater than or equal to 3, for every even number x such that $p_k^2 < x < p_{k+1}^2$, where $k \geq 35$; this implies part *(a)* of the Main Theorem. On the other hand, suppose that for the sequence of Sieves II we prove that the sifting function tends to ∞, as $k \to \infty$. In this case, it is easy to see that for the Sieve associated with x (the Sieve I), the sifting function also tends to ∞, as $x \to \infty$; this implies part *(b)* of the Main Theorem.

So, our problem now is, given the Sieve II, how to compute the number of permitted k-tuples whose indices lie within $[1, p_k^2]$. We shall see how the study of the sequences of k-tuples reveals the way to derive a formula for computing this number.

1.5 Computing the number of permitted k-tuples in a period of the sequence of k-tuples of the Sieve II

Usually, the sieve method consist in operate on the formula given by the inclusion-exclusion principle to obtain bounds for the sifting function, as we have illustrated in the case of the sieve of Eratosthenes-Legendre. In our approach, the starting point is also the inclusion-exclusion principle, but only as a first step towards obtaining a lower bound for the sifting function of the Sieve II. That is, from the formula given by this principle we shall compute the exact number of permitted k-tuples within a period of the corresponding sequence of k-tuples, as follows.

Let us consider again the Sieve II, but now we take $\mathscr{B} = \{n : 1 \leq n \leq m_k\}$; that is, $\mathscr{B}$ is now the set of the indices corresponding to the first period of the sequence of k-tuples. Given $p \in \mathscr{P}, 2 < p \leq p_k$, we have $|\mathscr{B}_p| = 2m_k/p$, since $p|m_k$ and there are two selected remainders for each modulus $p > 2$. Furthermore, given a squarefree integer d such that $d|m_k, 2 \nmid d$, the set $\mathscr{B}_d$ is the intersection of the subsets $\mathscr{B}_p$ such that $p|d, 2 < p \leq p_k$. Hence,

$$|\mathscr{B}_d| = \frac{2^{\nu(d)}}{d} m_k \qquad (d|m_k, 2 \nmid d),$$

where $\nu(d)$ is the number of distinct prime divisors of d. Furthermore, we have the identity

$$\sum_{\substack{d|m_k \\ 2\nmid d}} \mu(d) \frac{2^{\nu(d)}}{d} = \prod_{\substack{2<p\leq p_k \\ p\in\mathscr{P}}} \left(1 - \frac{2}{p}\right). \tag{4}$$

On the other hand, the subset $\mathscr{B}_{p_1}$ consist of the integers $n \in \mathscr{B}$ such that $n \equiv r \pmod{p_1}$, where r is the selected remainder for the modulus p_1 in the sequence of k-tuples of the Sieve II. Then $|\mathscr{B}_{p_1}| = m_k/p_1$, since $p_1|m_k$ and there is one selected remainder for the modulus p_1. Furthermore, given a squarefree integer d such that $d|m_k, 2 \mid d$, the set $\mathscr{B}_d$ is now the intersection of the subsets $\mathscr{B}_p$ such that $p|d$. Hence,

$$|\mathscr{B}_d| = \frac{2^{\nu(d)-1}}{d} m_k \qquad (d|m_k, 2 \mid d).$$

Then, by the inclusion-exclusion principle,

$$\begin{aligned} T(\{n : 1 \leq n \leq m_k\}, \mathscr{P}, p_k) &= \sum_{d|m_k} \mu(d) |\mathscr{B}_d| = \sum_{\substack{d|m_k \\ 2\nmid d}} \mu(d) \frac{2^{\nu(d)}}{d} m_k + \sum_{\substack{d|m_k \\ 2|d}} \mu(d) \frac{2^{\nu(d)-1}}{d} m_k = \\ &= \sum_{\substack{d|m_k \\ 2\nmid d}} \mu(d) \frac{2^{\nu(d)}}{d} m_k - \frac{1}{2} \sum_{\substack{d|m_k \\ 2\nmid d}} \mu(d) \frac{2^{\nu(d)}}{d} m_k = \frac{1}{2} \sum_{\substack{d|m_k \\ 2\nmid d}} \mu(d) \frac{2^{\nu(d)}}{d} m_k. \end{aligned}$$

So, using (4) we can see that the number of permitted k-tuples whose indices lie in the interval $[1, m_k]$ (the first period of the sequence of k-tuples associated to the Sieve II) is given by

$$T(\{n : 1 \leq n \leq m_k\}, \mathscr{P}, p_k) = \frac{1}{2} m_k \prod_{\substack{2 < p \leq p_k \\ p \in \mathscr{P}}} \left(1 - \frac{2}{p}\right), \tag{5}$$

whatever the remainders $r, r' \pmod p$, for every $p \in \mathscr{P},\ p \leq p_k$.

1.6 The structure of the first period of the sequence of k-tuples of remainders

Until now, we have arranged the elements of each k-tuple horizontally, from left to right; and we have arranged the k-tuples of the sequence vertically, from top to bottom. Hence, the first period of the sequence of k-tuples can be seen as a matrix, with columns from $h = 1$ to $h = k$, and $m_k = p_1 p_2 p_3 \cdots p_k$ rows. Note that for each $h\,(1 \leq h \leq k)$, we also have a sequence of h-tuples with period $m_h = p_1 p_2 p_3 \cdots p_h$, which fits into the period m_k a whole number of times.

k

7	1	2	3	4	5	6	0	1	2	3	4	5	6	0	1	2	3	4	5	6	0	1	2	3	···
5	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	···
3	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	···
2	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	···
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	···

···	4	5	6	0	1	2	3	4	5	6	0	1	2	3	4	5	6	0	1	2	3	4	5	6	0
···	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0
···	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0
···	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
···	186	187	188	189	190	191	192	193	194	195	196	197	198	199	200	201	202	203	204	205	206	207	208	209	210

n

Figure 3

Suppose that we rotate (for convenience) the entire sequence 90 degrees counterclockwise. Then, the index n of the sequence of k-tuples increases from left to right, and the index k of the elements of each k-tuple increases from the bottom up. Consequently, we can think of the first period of the sequence of k-tuples as a matrix formed by k rows and m_k columns. Each row of this matrix, from $h = 1$ to $h = k$, is formed by the remainders of dividing the integers from $n = 1$ to $n = m_k$ by the modulus p_h. For every n $(1 \leq n \leq m_k)$, the corresponding column matrix is the k-tuple of the remainders of dividing n by the moduli $p_1, p_2, \ldots, p_k$.

Note that if we let $k \to \infty$, the period of the sequence and the size of the involved k-tuples grow simultaneously.

Example 1.3. Figure 3 illustrates the first period of the sequence of 4-tuples pictured in Figure 2, but now arranged horizontally from left to right.

The sequences of k-tuples in general shall be defined more formally in Section 2, but now we need the following definition:

Definition 1.1. Given a sequence of k-tuples, and using the order relation given by the index n, we define an interval of k-tuples, denoted by $I[m, n]_k$, to be the set of consecutive k-tuples associated with an integer interval $[m, n] \cap \mathbb{Z}_+$, where m is the index of the first k-tuple, and n is the index of the last k-tuple. We also use the notation $I[m, n] = I[m, n]_k$ for this interval. We define the *size* of $I[m, n]$ by the equation $|I[m, n]| = n - m + 1$; and we use the notation $I[\,]_k$, or alternatively $I[\,]$, to denote the empty interval.

In particular, let us consider the sequence of k-tuples associated to the Sieve II. Since this sequence is periodic, it suffices to consider its first period, between $n = 1$ and $n = m_k$ (the interval $I[1, m_k]$). Note that for $p_k \geq 7$ ($k \geq 4$), the interval $I[1, p_k^2]$ is completely included within the first period of the sequence of k-tuples. Although this is the interval that interests us, in order to understand the properties of the sequence of k-tuples, and its behaviour as $k \to \infty$, it is necessary to study the whole fundamental period of the sequence, not just the interval $I[1, p_k^2]$.

The following step in our approach consists of dividing into two parts the first period of the sequence of k-tuples, as follows: the left interval $I[1, p_k^2]$, and the right interval $I[p_k^2 + 1, m_k]$. So, since for every h ($1 \leq h \leq k$) there is a sequence of h-tuples of remainders, the interval $I[1, m_k]_h$ of each sequence turns out subdivided into two intervals: the left interval $I[1, p_k^2]_h$, and the right interval $I[p_k^2 + 1, m_k]_h$. If we think of the first period of the sequence of k-tuples as a matrix, we can see that this matrix has been now partitioned into two blocks: the left block, formed by the columns from $n = 1$ to $n = p_k^2$; and the right block, formed by the columns from $n = p_k^2 + 1$ to $n = m_k$. Each row of the left block is formed by the remainders of dividing the integers from $n = 1$ to $n = p_k^2$ by the modulus p_h ($1 \leq h \leq k$); and each row of the right block is formed by the remainders of dividing the integers from $n = p_k^2 + 1$ to $n = m_k$ by the modulus p_h.

Recall that within the first period of the sequence of k-tuples (the interval $I[1, m_k]$), the exact number of permitted k-tuples is given by (5). Furthermore, for every h such that $1 \leq h < k$, since the number of permitted h-tuples in a period of the sequence of h-tuples is given also by (5), and the period m_h divides m_k, we can compute precisely the number of permitted h-tuples in each interval $I[1, m_k]_h$ as well. We can see that for every h ($1 \leq h \leq k$), the number of permitted h-tuples in each interval $I[1, m_k]_h$ is the same, whatever the choice of the selected remainders in the sequence of h-tuples. However, within both the left interval $I[1, p_k^2]_h$ and the right interval $I[p_k^2 + 1, m_k]_h$, the number of permitted h-tuples change when the selected remainders in the sequence of h-tuples are changed, because the positions of the permitted h-tuples along the interval $I[1, m_k]_h$ are modified.

Note that for every sequence of h-tuples of remainders ($1 \leq h \leq k$), the intervals $I[1, m_k]_h$, $I[1, p_k^2]_h$ and $I[p_k^2 + 1, m_k]_h$ are itself sieve devices, that separate prohibited h-tuples from permitted h-tuples.

On the other hand, attached to the first period of the sequence of k-tuples there is a $k \times 2$ matrix, where for every h ($1 \leq h \leq k$), the entry in the row h and first column is the number of permitted h-tuples in $I[1, p_k^2]_h$, and the entry in the row h and second column is the number of permitted h-tuples in $I[p_k^2 + 1, m_k]_h$. Of course, the entries in the matrix depends on the choice of the selected remainders in the sequence of k-tuples. Note that if we take $y = p_k^2$ in the Sieve II, the sifting function is the entry in the first row and first column of this matrix; that is, to compute the sifting function for the Sieve II we ought to be able to compute this entry in the matrix. Note that this quantity is related to the entry in the first row and second column of the matrix, since the sum of both entries is given by (5).

A question may have already occurred to the reader at this point: What is the advantage of the formulation of the sieves based in a sequence of k-tuples of remainders? We shall explain the principal reason in what follows.

Let us consider the sequence of k-tuples of the Sieve II, in horizontal position, where $k \geq 4$. For a given choice of selected remainders, the interval $I[1, m_k]$ of this sequence is a sieve device, that sift out the prohibited k-tuples that lie in $I[1, m_k]$, and allows to survive the permitted k-tuples in this interval. Furthermore, for every h ($1 \leq h < k$) there is a sequence of h-tuples of remainders as well. And the interval $I[1, m_k]_h$ of every sequence of h-tuples is also a sieve device, that sift out the prohibited h-tuples and allows to survive the permitted h-tuples in $I[1, m_k]_h$. So, we have decomposed the sifting process into several stages, from $h = 1$ to $h = k$, where each 'partial' sieve device contributes to the whole sifting process. Hence, we can study the behaviour of this partial sieve devices to determine the behaviour of the whole sieve; the advantage of this perspective will become apparent in the rest of this section. Note that as h goes from 1 to k, the number of permitted h-tuples decreases, as a result of the sifting process in each stage of the sifting process. Of course, there is also a similar structure in the left block and the right block of the first period of the sequence of k-tuples.

1.7 The density of permitted k-tuples

In the Sieve II we have taken first the set $\mathscr{B} = \{n : 1 \leq n \leq p_k^2\}$, and so, the sifting function $T(\{n : 1 \leq n \leq p_k^2\}, \mathscr{P}, p_k)$ is equal to the number of permitted k-tuples in the interval $I[1, p_k^2]$ of the sequence of k-tuples associated to the Sieve II. Note that the evaluation of this sifting function is what we need to solve the Goldbach problem. However, this sifting function depends on the choice of the selected remainders in the sequence of k-tuples associated to the Sieve II, and we can not compute it exactly. The obtaining of a lower bound for this sifting function is the main task that we must perform in this paper.

On the other hand, in the Sieve II we have next taken the set $\mathscr{B} = \{n : 1 \leq n \leq m_k\}$; now, the sifting function $T(\{n : 1 \leq n \leq m_k\}, \mathscr{P}, p_k)$ is equal to the number of permitted k-tuples in the interval $I[1, m_k]$ of the sequence of k-tuples associated to the Sieve II. In this case, the sifting function does not depend on the choice of the selected remainders in the sequence of k-tuples, and it can be computed precisely using (5).

Then, a natural question arises: How we can take advantage of the exact computation of $T(\{n : 1 \leq n \leq m_k\}, \mathscr{P}, p_k)$ for obtaining an estimate of $T(\{n : 1 \leq n \leq p_k^2\}, \mathscr{P}, p_k)$?

Let us consider the interval $I[1, m_k]$ (first period of the sequence of k-tuples of the Sieve II); furthermore, consider the intervals $I[1, p_k^2]$ and $I[p_k^2 + 1, m_k]$. We can see that, for a given choice of selected remainders in the sequence of

k-tuples, if the proportion of permitted k-tuples in $I[1, p_k^2]$ is less than the proportion in $I[1, m_k]$, the proportion of permitted k-tuples in $I[p_k^2 + 1, m_k]$ must be greater than the proportion in $I[1, m_k]$; and vice versa. Note that if we multiply the proportion of permitted k-tuples in $I[1, p_k^2]$ by p_k^2 we obtain the number of permitted k-tuples within this interval.

Suppose that the proportion of permitted k-tuples in the interval $I[1, p_k^2]$ were equal to the proportion of permitted k-tuples in the interval $I[1, m_k]$. In this case, we could compute at once the exact number of permitted k-tuples in the interval $I[1, p_k^2]$, since we know this quantity for the interval $I[1, m_k]$, by (5). Certainly, it is unlikely that our assumption on the proportion of permitted k-tuples in these intervals could be true; however, we could say that in some sense this assumption is 'aproximately' true. This suggests the possibility of working with this amount (the proportion of permitted k-tuples in a given interval) to obtain the expected results.

Now, assume that for every k the proportion of permitted k-tuples in $I[1, p_k^2]$ were greater than some constant $C > 0$; in this case, the number of permitted k-tuples within this interval would be greater than Cp_k^2; this would imply that the number of permitted k-tuples within $I[1, p_k^2]$ tends to infinity with k; and, indeed, this is the result that we wish to prove.

However, it seems unlikely that this constant exists. To see this, suppose that for k large enough, and whatever the choice of the selected remainders, the permitted k-tuples were distributed following a pattern more or less regular along the fundamental period $I[1, m_k]$. In this case, for both intervals $I[1, p_k^2]$ and $I[p_k^2 + 1, m_k]$, the proportion of permitted k-tuples would be approximately the same as that for the interval $I[1, m_k]$. Even so, from (5) it follows that the proportion of permitted k-tuples in the interval $I[1, m_k]$ is given by

$$\frac{1}{2} \prod_{h=2}^{k} \left(1 - \frac{2}{p_h}\right), \tag{6}$$

which tends slowly to 0, as $k \to \infty$. This fact makes extremely doubtful that such constant C exists for the interval $I[1, p_k^2]$, and makes working with this amount (the proportion of permitted k-tuples) not very useful.

For this reason it is more convenient to work with a new quantity, that we call the density of permitted k-tuples, or simply the k-density, which is defined formally in Section 3. It is defined for a given interval as the quotient of the number of permitted k-tuples within this interval and the number of subintervals of size p_k. That is, for a given interval, is the average number of permitted k-tuples within the subintervals of size p_k. Since the number of permitted k-tuples within the period does not depend on the choice of the selected remainders in the sequence of k-tuples, neither does the k-density within the period depend on that choice. We shall prove later (Theorem 3.4) that the density of permitted k-tuples in the period $I[1, m_k]$ tends to infinity as $k \to \infty$. For some values of k, Table 1 gives the number of permitted k-tuples ($npkt$), the ratio $npkt/m_k$, and the density of permitted k-tuples ($dpkt$) within the period of the sequence of k-tuples. We shall define later the appropriate notation for the number of permitted k-tuples, and the density of permitted k-tuples within a given interval of the sequence of k-tuples.

Note that given the proportion of permitted k-tuples within a given interval, multiplying by p_k we obtain the density of permitted k-tuples within this interval. So, assuming as before that for k large enough the permitted k-tuples were distributed following a pattern more or less regular along the interval $I[1, m_k]$, for both intervals $I[1, p_k^2]$ and $I[p_k^2 + 1, m_k]$ the density of permitted k-tuples would be approximately the same as that for the interval $I[1, m_k]$.

Now, suppose that the minimum value of the density of permitted k-tuples within the left interval $I[1, p_k^2]$ approach the density of permitted k-tuples within the interval $I[1, m_k]$, as $k \to \infty$. This means that the minimum value of the density of permitted k-tuples in $I[1, p_k^2]$ also tends to infinity as $k \to \infty$. And this, in turn, implies that as $k \to \infty$, the number of permitted k-tuples in $I[1, p_k^2]$ tends to infinity as well, since the number of subintervals of size p_k in the interval $I[1, p_k^2]$ is equal to p_k. So, the sifting function of the Sieve II tends to infinity, as $k \to \infty$; and this is the result that we need in order to prove the Main theorem.

Attached to the first period of the sequence of k-tuples there is now another $k \times 2$ matrix, where for every h $(1 \leq h \leq k)$, the entry in the row h and first column is the density of permitted h-tuples in $I[1, p_k^2]_h$, and the entry in the row h and second column is the density of permitted h-tuples in $I[p_k^2 + 1, m_k]_h$. The entries in this matrix also depends on the choice of the selected remainders in the sequence of k-tuples. Note that the entry in the first row and first column of this new $k \times 2$ matrix, multiplied by p_k, is equal to the entry in the first row and first column of the $k \times 2$ matrix described in the preceding subsection. The relationships in the matrix of h-densities, between the elements of the rows, and between the elements of the columns are very important for our purposes, as we shall see later.

1.8 The density of permitted k-tuples in the interval $I[1, p_k^2]$ (discussion)

An obvious question arises: What are the reasons for expecting that the minimum value of the k-density in the interval $I[1, p_k^2]$ tends to infinity with k? We have two reasons; we describe the first as follows. Suppose that for k large enough we divide the interval $I[1, m_k]$ of the sequence of k-tuples in subintervals of size p_k. Then, counting in every subinterval of size p_k the number of permitted k-tuples, we have a 'population' of these values. The arithmetic mean of this population is the k-density for the interval $I[1, m_k]$.

Table 1: Quotient $npkt/m_k$ and density of permitted k-tuples ($dpkt$).

k	p_k	m_k	$npkt$	$npkt/m_k$	$dpkt$
4	7	210	15	0.071	0.500
5	11	2310	135	0.058	0.643
6	13	30030	1485	0.049	0.643
7	17	510510	22275	0.044	0.742
8	19	9699690	378675	0.039	0.742
9	23	223092870	7952175	0.036	0.820
10	29	-	-	0.033	0.962
11	31	-	-	0.031	0.962
12	37	-	-	0.029	1.087
13	41	-	-	0.028	1.145
14	43	-	-	0.027	1.145
15	47	-	-	0.025	1.199
16	53	-	-	0.024	1.301
17	59	-	-	0.024	1.399
18	61	-	-	0.023	1.399
19	67	-	-	0.022	1.490
20	71	-	-	0.022	1.535
21	73	-	-	0.021	1.535
22	79	-	-	0.020	1.619
23	83	-	-	0.020	1.660
24	89	-	-	0.019	1.740
.	.	.	.	.	.

On the other hand, the interval $I[1, p_k^2]$ is the union of p_k subintervals, each one of size p_k. So, counting the number of permitted k-tuples in every subinterval of size p_k in $I[1, p_k^2]$ we obtain a set of values that can be seen as a 'sample' of size p_k drawn from the population. The arithmetic mean of this sample is the k-density for the interval $I[1, p_k^2]$. Then, we can assume that the values of the k-density for the interval $I[1, p_k^2]$ (for all the possible choices of selected remainders in the sequence of k-tuples) are spread round the mean of the population. Since the k-density for the interval $I[1, m_k]$ increases and tends to infinity as $k \to \infty$, it seems reasonably to expect that the minimum value of the k-density for the interval $I[1, p_k^2]$ also increases and tends to infinity with k.

We explain now the second reason. Suppose that, on the contrary, there exist $C > 0$ such that for every k the k-density for the interval $I[1, p_k^2]$ of the sequence of k-tuples (no matter the choice of the selected remainders) is bounded above by C. We know that as $k \to \infty$ the size of $I[p_k^2 + 1, m_k]$ becomes vastly bigger than the size of $I[1, p_k^2]$, since $p_k^2 = \mathrm{o}(m_k)$ (Lemma 2.5). So, since the k-density in the interval $I[1, m_k]$ tends to infinity as $k \to \infty$, we deduce that the k-density in the interval $I[p_k^2 + 1, m_k]$ has the same behaviour. This would mean that while the density of permitted k-tuples in $I[p_k^2 + 1, m_k]$ tends to infinity as $k \to \infty$, the k-density in $I[1, p_k^2]$ remains below C for every k. However, it seems that there is no satisfactory reason for such a different behaviour of the k-density in these intervals. So, we suspect that there is no such upper bound C, and the behaviour of the k-density in $I[1, p_k^2]$ as $k \to \infty$ is the same as that in $I[1, m_k]$.

1.9 Computing the sifting function of the Sieve II (short outline)

At this point, the next step in our strategy is to derive a formula for the minimum value of the k-density in the interval $I[1, p_k^2]$ of the sequence of k-tuples of the Sieve II. Before giving a short outline of the way to proceed, we shall make some further remarks.

Remark 1.3. Given a sequence of k-tuples of remainders, recall that for each $h\,(1 \leq h \leq k)$, we have a periodic sequence of h-tuples, with period $m_h = p_1 p_2 p_3 \cdots p_h$. As we have already seen, we can compute precisely the number of permitted h-tuples within the period m_h; it follows that we can also compute precisely the density of permitted h-tuples within the period m_h. Then, given h and $n > 1$, if we let $n \to \infty$, the size of the interval $I[1, n]_h$ of the sequence of h-tuples increases, and the number of periods m_h that fits in this interval increases as well. Hence, the number of permitted h-tuples within the interval $I[1, n]_h$ of the sequence of h-tuples tends to ∞ as $n \to \infty$, but the h-density in this interval tends to the h-density within the period m_h. This behaviour is very important for our purposes, as we shall see later.

Remark 1.4. Consider again the interval $I[1, m_k]$ (the first period) of the sequence of k-tuples. Given the partition of the period into two blocks, for each $h\,(1 \leq h \leq k)$ we have a left interval $I[1, p_k^2]_h$ and a right interval $I[p_k^2 + 1, m_k]_h$. Taking into account all the choices of selected remainders, there is a set of h-densities for the left interval, and a set of h-densities for the right interval. Since (5) gives us the number of permitted h-tuples within the period m_h, and since m_k is a multiple of m_h, for each $h\,(1 \le h \le k)$ we can compute precisely the number of permitted h-tuples in the interval $I[1, m_k]_h$ of the corresponding sequence of h-tuples. Consequently, if we know the number of permitted h-tuples within the left interval $I[1, p_k^2]_h$, we can compute the number of permitted h-tuples within the right interval $I[p_k^2 + 1, m_k]_h$; and vice versa. So, we can define a bijective function between the set of the h-densities for the left interval, and the set of the h-densities for the right interval. Note that if the h-density in the left interval $I[1, p_k^2]_h$ is less than the h-density in the interval $I[1, m_k]_h$, the h-density in the right interval $I[p_k^2 + 1, m_k]_h$ must be greater than the h-density in $I[1, m_k]_h$, and vice versa.

Remark 1.5. Given $h\,(1 \le h \le k)$, since m_h fits into $I[1, m_k]_h$ a whole number of times, the density of permitted h-tuples within the interval $I[1, m_k]_h$ of the sequence of h-tuples is equal to the h-density within the period m_h, which can be exactly computed. Now, let k be large enough that the size of the left interval $I[1, p_k^2]$ is negligible compared to the size of $I[1, m_k]$. So, considering the right block of the partition, we can see that for every h, most of the permitted h-tuples will be in the right interval $I[p_k^2 + 1, m_k]_h$. Thus, for every h, the h-density in the right interval $I[p_k^2 + 1, m_k]_h$ of the sequence of h-tuples will be very close to the h-density within the interval $I[1, m_k]_h$. Hence, the behaviour of the maximum values of the h-density in every right interval from $h = 1$ to $h = k$ (right block of the partition), will be approximately proportional to the h-density in $I[1, m_k]_h$. This behaviour shall allow us to derive a proportionality formula (see (29) and (30), Section 7) within the right block of the partition for k very large such that, given the maximum value of the density of permitted 4-tuples for $h = 4$ (chosen by convenience), we can compute the maximum value of the density of permitted k-tuples for $h = k$. This formula includes a coefficient which tends to 1 as $k \to \infty$, as we shall prove later (Lemma 7.6).

Recall that attached to the first period of the sequence of k-tuples there is a $k \times 2$ matrix of densities, where the entries depend on the choice of the selected remainders in the sequence of k-tuples. Now, in order to obtain the formula for the minimum value of the k-density in $I[1, p_k^2]$, we need to introduce one more $k \times 2$ matrix, which is described as follows. Taken into account all the possible choices of the selected remainders in the sequence of k-tuples, for every h $(1 \le h \le k)$, the entry in the row h and first column is the minimum value of the h-density in $I[1, p_k^2]_h$, and the entry in the row h and second column is the maximum value of the h-density in $I[p_k^2 + 1, m_k]_h$. That is, the first column of this matrix contains the minimum values of the h-density in the left block of the first period of the sequence of k-tuples, and the second column contains the maximum values of the h-density in the right block. This matrix is associated to the matrix of h-densities, but, of course, the entries in this new matrix does not depend on the choice of the selected remainders in the sequence of k-tuples.

Now, on the one hand, for every h $(1 \le h \le k)$, given the entry in the row h and first column of this matrix we could obtain the entry in the row h and second column, using the bijection described in Remark 1.4; and vice versa. On the other hand, for k large enough, from what we explained in Remark 1.5, we can see that from $h = 1$ to $h = k$, the behaviour of the values in the second column of the matrix (that is, the maximum values of the h-density in $I[p_k^2 + 1, m_k]_h$) must be approximately proportional to the h-density in $I[1, m_k]_h$ $(1 \le h \le k)$.

Suppose that we choose $h = 4$ (for convenience). The relation between the entry in the row $h = 4$, first column and the entry in the row $h = 4$, second column is given by the bijection (for $h = 4$); the relation between the entry in the row $h = 4$, second column and the entry in the row $h = k$, second column is given by the proportionality formula; and the relation between the entry in the row $h = k$, second column and the entry in the row $h = k$, first column is given by the bijection (for $h = k$). Therefore, this last $k \times 2$ matrix is an implicit formula that gives us the relationship between the entries in the row $h = 4$ and row $h = k$, in the first column.

To derive the explicit formula, we proceed as follows. Given the minimum value of the density of permitted 4-tuples within the left interval $I[1, p_k^2]_4$, we use the bijection for $h = 4$ to compute the maximum value of the density of permitted 4-tuples within the right interval $I[p_k^2+1, m_k]_4$ (right block of the partition). Then, using the proportionality formula for the right block of the partition, we compute the maximum value of the density of permitted k-tuples for $h = k$ within the right interval $I[p_k^2 + 1, m_k]$. Next, we use the bijection for $h = k$ to compute the minimum value of the density of permitted k-tuples within the left interval $I[1, p_k^2]$ (returning to the left block of the partition). So, we have derived an explicit formula for the left block of the partition (for k very large) such that, given the minimum value of the density of permitted 4-tuples for $h = 4$, it allows us to compute the minimum value of the density of permitted k-tuples for $h = k$. In summary, to determine the behaviour of the density of permitted h-tuples in the left block of the partition, it was necessary 'to make a detour' through the right block of the partition.

At this time, if we have a lower bound for the density of permitted 4-tuples in the interval $I[1, p_k^2]_4$, using the formula derived before we can obtain a lower bound for the density of permitted k-tuples in the interval $I[1, p_k^2]$. Nevertheless, how could we obtain a lower bound for the 4-density in $I[1, p_k^2]_4$? To do this, we may argue as follows: For $h = 4$, as k increases, the size of the interval $I[1, p_k^2]_4$ of the sequence of 4-tuples also increases, and the number of periods m_4 that fits in this interval increases as well. Thus, the 4-density in the interval $I[1, p_k^2]_4$ approaches the 4-density within the period m_4, by Remark 1.3; so, for $h = 4$ and k large enough, we can obtain a lower bound for the 4-density in the interval $I[1, p_k^2]_4$.

Using this lower bound, the formula derived before allow us to compute a lower bound for the k-density in the interval $I[1, p_k^2]$ of the sequence of k-tuples, where k is very large. Since the number of subintervals of size p_k in the interval $I[1, p_k^2]$ is equal to p_k, multiplying this lower bound by p_k, we obtain a lower bound for the number of permitted k-tuples in this interval; that is, we get a lower bound for the sifting function corresponding to the Sieve II. Hence, we have a lower bound for the sifting function corresponding to the Sieve associated with x (Sieve I), where x is an even number such that $p_k^2 < x < p_{k+1}^2$. This result will allow us to prove part (a) of the Main Theorem.

Furthermore, with the same formula we can show that the k-density in $I[1, p_k^2]$ tends to infinity as $k \to \infty$; so, as $k \to \infty$, the sifting function corresponding to the Sieve II tends to infinity as well. From this it follows that the sifting function corresponding to the Sieve associated with x tends to infinity as $x \to \infty$. This suffices to prove part (b) of the Main Theorem.

2 Periodic sequences of k-tuples

General Notation. We write (a, b) for the greatest common divisor of a and b, if no confusion will arise. In addition, lcm is used as an abbreviation for the least common multiple. Given a set A, we denote by $|A|$ the cardinality of A. For each $a \in \mathbb{R}$, the symbol $\lfloor a \rfloor$ denotes the floor function, and the symbol $\lceil a \rceil$ denotes the ceiling function.

In the Introduction we began by describing a first kind of sieve to attack the Goldbach's problem, which we call the Sieve associated with x (or Sieve I); then, we have introduced the notion of sequence of k tuples of remainders as a new formulation for sieves in general, and for this sieve in particular. The Sieve associated with x (Sieve I) is directly related to the Goldbach's problem; we defer to the last section of the paper the formal definition of this sieve. On the other hand, we have also described in the Introduction a second sieve more general, which we call the Sieve II. As we have seen in the Introduction, the sequence of k-tuples corresponding to the Sieve II is more homogeneous than that corresponding to the Sieve I, in the sense that in every sequence of remainders modulo p_h $(1 < h \leq k)$ there are always two selected remainders. This fact is very important in order to compute the minimum value of the sifting function of the Sieve II.

The Sieve II is not directly related to the Goldbach's problem, but the minimum number of permitted k-tuples in the interval $I[1, p_k^2]$ of the sequence of k-tuples corresponding to the Sieve II (the minimum value of the sifting function of the Sieve II), is a lower bound for the number of permitted k-tuples in the interval $I[1, x]$ of the sequence corresponding to the Sieve I (the sifting function of the Sieve I), where $p_k^2 < x < p_{k+1}^2$; we shall prove this fact in Section 8. In this section we define formally the Sieve II; and we shall deal with the properties of this sieve until Section 7.

We begin by defining the concepts of sequence of remainders and sequence of k-tuples of remainders, and other associated concepts.

Definition 2.1. Let $\mathscr{P}$ be the sequence of all primes, and consider the subset $\{p_1, p_2, p_3, \ldots, p_k\}$ of the first k primes.

(1) Given p_h $(1 \leq h \leq k)$, we define the periodic sequence $\{r_n\}$, where r_n denotes the remainder of dividing n by the modulus p_h. We denote the sequence $\{r_n\}$ by the symbol s_h. The period of the sequence is equal to p_h. See Example 2.1.

(2) We define the sequence $\{(r_1, r_2, r_3, \ldots, r_k)_n\}$, the elements of which are k-tuples of remainders obtained by dividing n by the moduli $p_1, p_2, p_3, \ldots, p_k$. We arrange the sequence of k-tuples of remainders vertically; we usually omit the comma separators in the k-tuples. Then, the sequence of k-tuples can be seen as a matrix formed by k columns and infinitely many rows, where each column of the matrix is a periodic sequence s_h $(1 \leq h \leq k)$. The index k will be called the level of the sequence of k-tuples of remainders. See Example 2.2.

Example 2.1. For the modulus $p_3 = 5$ we have $s_3 = \{1, 2, 3, 4, 0, 1, 2, 3, \ldots\}$.

Example 2.2. Table 2 shows the sequence of 5-tuples of the remainders of dividing n by $\{2, 3, 5, 7, 11\}$.

Definition 2.2. Given a sequence $\{r_n\}$ with prime modulus p_k we assign to the remainders r_n one of the two following states: *selected state* or *not selected state.*

Definition 2.3. Given a sequence of k-tuples of remainders, we define a k-tuple to be *prohibited* if it has one or more selected remainders, and we define it to be *permitted* if it contains no selected remainders.

Definition 2.4. We denote by m_k the product $p_1 p_2 p_3 \cdots p_k$.

Proposition 2.1. *The sequence of k-tuples of remainders is periodic, and its fundamental period is equal to $m_k = p_1 p_2 p_3 \cdots p_k$.*

Proof. Let s_h $(1 \leq h \leq k)$ be the sequences of remainders modulo p_h that form a given sequence of k-tuples. The period of every sequence s_h is equal to $p_h \in \{p_1, p_2, p_3, \ldots, p_k\}$. Since $p_1, p_2, p_3, \ldots, p_k$ are primes, the product m_k is the lcm. Consequently, the fundamental period of the sequence of k-tuples is equal to m_k. ∎

Table 2: Sequence of 5-tuples of remainders.

n	2	3	5	7	11
1	1	1	1	1	1
2	0	2	2	2	2
3	1	0	3	3	3
4	0	1	4	4	4
5	1	2	0	5	5
6	0	0	1	6	6
7	1	1	2	0	7
8	0	2	3	1	8
9	1	0	4	2	9
10	0	1	0	3	10
11	1	2	1	4	0
12	0	0	2	5	1
13	1	1	3	6	2
14	0	2	4	0	3
15	1	0	0	1	4
16	0	1	1	2	5
17	1	2	2	3	6
18	0	0	3	4	7
.	.	.	.	.	.

So far, we have defined the sequence of k-tuples of remainders without defining any rules for selecting remainders; note that without selected remainders, the sequence of k-tuples does not work as a sieve. Before defining these rules, we shall consider another important question concerning the sequence of k-tuples of remainders. As we have seen in the Introduction, in the case of the Sieve II it will be necessary to deal with the behaviour of the sequence of k-tuples as k increases indefinitely. Consequently, we need two more definitions before defining the rules for selecting remainders.

Definition 2.5. Sum of sequences.

Let $\{p_1, p_2, p_3, \ldots, p_k\}$ be the set of the first k primes. Let $\{(r_1\ r_2\ r_3 \ldots r_k)_n\}$ be the sequence of k-tuples of the remainders of dividing n by the k prime moduli $\{p_1, p_2, p_3, \ldots, p_k\}$, and let $\{(r_{k+1})_n\}$ be the sequence of the remainders of dividing n by the prime modulus p_{k+1}. We define the *sum* $\{(r_1\ r_2\ r_3 \ldots r_k)_n\} + \{(r_{k+1})_n\}$, of the sequence $\{(r_1\ r_2\ r_3 \ldots r_k)_n\}$ and the sequence $\{(r_{k+1})_n\}$, to be the sequence of $(k+1)$-tuples given by the equation

$$\{(r_1\ r_2\ r_3 \ldots r_k)_n\} + \{(r_{k+1})_n\} = \{(r_1\ r_2\ r_3 \ldots r_k\ r_{k+1})_n\},$$

and formed by the ordered juxtaposition of each k-tuple of the first sequence with each element (index n modulo p_{k+1}) of the second sequence.

Definition 2.6. Let $\mathscr{P}$ be the sequence of all primes, and let $p_k \in \mathscr{P}$. Let s_k be the sequence of the remainders of dividing n by the modulus p_k. Let $\{s_k\}$ be the sequence of sequences s_k. We define the series denoted by $\sum s_k$ to be the sequence $\{S_k\}$, where S_k denotes the partial sum:

$$\begin{array}{lllllllllll}
S_1 & = & s_1, & & & & & & & & \\
S_2 & = & s_1 & + & s_2, & & & & & & \\
S_3 & = & s_1 & + & s_2 & + & s_3, & & & & \\
. & . & . & . & . & . & . & . & . & . & . \\
S_k & = & s_1 & + & s_2 & + & s_3 & + & s_4 & + \cdots + & s_k,
\end{array}$$

and the symbol $\sum$ refers to the formal addition of sequences. In each partial sum S_k, the greatest prime modulus p_k will be called the characteristic prime modulus of the partial sum S_k. The index k will be called the level, and we shall say that S_k is the partial sum of level k.

Example 2.3. Table 3 shows the partial sum S_4 and the formal addition of the sequence of remainders s_5 to obtain the partial sum S_5.

Table 3: Partial sums S_4 and S_5.

	S_4					s_5		S_5				
n	2	3	5	7		11		2	3	5	7	11
1	1	1	1	1		1		1	1	1	1	1
2	0	2	2	2		2		0	2	2	2	2
3	1	0	3	3		3		1	0	3	3	3
4	0	1	4	4		4		0	1	4	4	4
5	1	2	0	5		5		1	2	0	5	5
6	0	0	1	6		6		0	0	1	6	6
7	1	1	2	0		7		1	1	2	0	7
8	0	2	3	1		8		0	2	3	1	8
9	1	0	4	2	$+$	9	$=$	1	0	4	2	9
10	0	1	0	3		10		0	1	0	3	10
11	1	2	1	4		0		1	2	1	4	0
12	0	0	2	5		1		0	0	2	5	1
13	1	1	3	6		2		1	1	3	6	2
14	0	2	4	0		3		0	2	4	0	3
15	1	0	0	1		4		1	0	0	1	4
16	0	1	1	2		5		0	1	1	2	5
17	1	2	2	3		6		1	2	2	3	6
18	0	0	3	4		7		0	0	3	4	7
.	.	.	.	.		.		.	.	.	.	.

On the one hand, we can look at a given partial sum S_k as a sequence indexed by n, of the k-tuples of remainders obtained by dividing n by the moduli $p_1, p_2, p_3, \ldots, p_k$. On the other hand, the partial sum S_k can be seen as a finite sequence indexed by the set $\{1, \ldots, k\}$ ($k \in \mathbb{Z}_+$), of sequences of remainders modulo $p_h \in \{p_1, p_2, p_3, \ldots, p_k\}$, where the indices $\{1, \ldots, k\}$ increase from left to right. And the series $\sum s_k$ is the sequence indexed by k, of the partial sums S_k.

Now we are ready to define the rules for selecting remainders in the sequences s_h ($1 \leq h \leq k$) that make up every partial sum S_k of the series $\sum s_k$.

Definition 2.7. Let s_h ($1 \leq h \leq k$) be one of the sequences of remainders that form the partial sum S_k.

Rule 1. If $h = 1$, in the sequence of remainders s_1 there will be selected one remainder, the same one in every period of the sequence.

Rule 2. If $1 < h \leq k$, in every sequence of remainders s_h, there will be selected two remainders, the same two in every period of the sequence.

Example 2.4. Table 4 shows the partial sum of level $k = 4$, where the selected remainders can be seen marked between the square brackets [ ]. Note that the 4-tuples 1 and 7 are permitted k-tuples.

Properly speaking, a given partial sum S_k is a sequence of k-tuples of remainders. However, from now on, when we refer to a given partial sum S_k, we mean S_k together with the selected remainders, unless we specifically state otherwise. Now we are ready to define formally the Sieve II.

Definition 2.8. Let $\mathscr{P}$ be the sequence of all primes; and let p_k ($k \geq 4$) be a prime of the sequence. Let $\mathscr{B}$ be the set consisting of the indices of the partial sum S_k that lie in the interval $[1, y]$, where y is an integer that satisfies $y > p_k$. For each $p = p_h \in \mathscr{P}$ ($1 \leq h \leq k$), the subset $\mathscr{B}_p$ of $\mathscr{B}$ consists of the indices whose remainder modulo $p = p_h$ is one of the selected remainders r or r'. The indices of the prohibited k-tuples lying in $\mathscr{B}$ are sifted out; and the indices of the permitted k-tuples lying in $\mathscr{B}$ remain unsifted. See Remark 1.1. The *sifting* function

$$T(\mathscr{B}, \mathscr{P}, p_k) = \left| \mathscr{B} \setminus \bigcup_{\substack{p \in \mathscr{P} \\ p \leq p_k}} \mathscr{B}_p \right|,$$

is given by the the number of permitted k-tuples whose indices lie in the interval $\mathscr{B}$.

Table 4: Partial sum S_4 with selected remainders.

n	2	3	5	7
1	1	1	1	1
2	[0]	[2]	2	2
3	1	[0]	[3]	[3]
4	[0]	1	4	4
5	1	[2]	[0]	[5]
6	[0]	[0]	1	6
7	1	1	2	0
8	[0]	[2]	[3]	1
9	1	[0]	4	2
10	[0]	1	[0]	[3]
11	1	[2]	1	4
12	[0]	[0]	2	[5]
13	1	1	[3]	6
14	[0]	[2]	4	0
15	1	[0]	[0]	1
16	[0]	1	1	2
.	.	.	.	.

Hereafter until the end of the paper, we take $\mathscr{B} = \{n : 1 \leq n \leq p_k^2\}$.

In the following theorems we prove some other properties of the partial sums of the series $\sum s_k$, which will be used throughout this paper.

Proposition 2.2. *Let S_k be a given partial sum. Let s_{k+1} be the sequence of remainders modulo p_{k+1}. Let r $(0 \leq r < p_{k+1})$ be one of the remainders modulo p_{k+1} of the sequence s_{k+1}. Let $n \in \mathbb{Z}_+$ be the index of a given k-tuple of S_k. Then, when we juxtapose the elements of the sequence s_{k+1} to the right of each k-tuple of S_k, we have the following.*

(1) If the k-tuple at position n is prohibited, then the $(k+1)$-tuple of S_{k+1} at position n will be prohibited as well.

(2) If the k-tuple at position n is permitted and $n \equiv r \pmod{p_{k+1}}$, then:

(a) The $(k+1)$-tuple of S_{k+1} at position n is prohibited if and only if r is a selected remainder;

(b) The $(k+1)$-tuple of S_{k+1} at position n is permitted if and only if r is not a selected remainder.

Proof. By definition, a given k-tuple is prohibited if it has one or more selected remainders; if it has no selected remainder, the k-tuple is permitted. The proof is immediate. ∎

Definition 2.9. For a given partial sum S_k, we denote by c_k the number of permitted k-tuples within a period of S_k.

Proposition 2.3. *Let S_k be a given partial sum. We have: $c_k = (p_1 - 1)(p_2 - 2)(p_3 - 2) \cdots (p_k - 2)$.*

Proof. It follows from (5), by simplifying the expression. ∎

Lemma 2.4. *Let S_k be a given partial sum. Let p_k be the characteristic prime modulus of the partial sum S_k. Let c_k be the number of permitted k-tuples within the period of S_k. We have $p_k^2 = \mathrm{o}(c_k)$.*

Proof. Using Proposition 2.3, we have

$$\begin{aligned}\frac{p_k^2}{c_k} &= \frac{p_k^2}{(p_1-1)(p_2-2)(p_3-2)\cdots(p_k-2)} = \\ &= \left(\frac{1}{(p_1-1)(p_2-2)(p_3-2)\cdots(p_{k-2}-2)}\right)\left(\frac{p_k}{p_{k-1}-2}\right)\left(\frac{p_k}{p_k-2}\right).\end{aligned} \tag{7}$$

Let g_{k-1} denote the gap $p_k - p_{k-1}$; so, $p_k/(p_{k-1}-2) = (p_{k-1}+g_{k-1})/(p_{k-1}-2)$. By the Bertrand-Chebyshev theorem, we have $g_{k-1} < p_{k-1} \implies (p_{k-1}+g_{k-1})/(p_{k-1}-2) < 2p_{k-1}/(p_{k-1}-2)$. It follows that $\lim_{k\to\infty} p_k/(p_{k-1}-2) < \lim_{k\to\infty} 2p_{k-1}/(p_{k-1}-2) = 2$. Since $\lim_{k\to\infty} p_k/(p_k-2) = 1$, returning to (7), it is easy to see that $\lim_{k\to\infty} p_k^2/c_k = 0$. ∎

Lemma 2.5. *Let S_k be a given partial sum. Let p_k be the characteristic prime modulus of the partial sum S_k. Let m_k be the period of S_k. We have $p_k^2 = \mathrm{o}(m_k)$.*

Proof. Since $m_k = p_1p_2p_3\cdots p_k$ by definition, the proof follows at once from Proposition 2.3 and Lemma 2.4. ∎

Proposition 2.6. *The Construction Procedure*

Let S_k and S_{k+1} be consecutive partial sums of the series $\sum s_k$. Let m_k and m_{k+1} be the periods of S_k and S_{k+1}, respectively. Consider the following procedure. First we take p_{k+1} periods of the partial sum S_k. Next we juxtapose the remainders of the sequence s_{k+1} to the right of each k-tuple of S_k (that is to say, we perform the operation $S_k + s_{k+1}$). This produces a whole period of the partial sum S_{k+1}.

Proof. By Proposition 2.1, the period m_k of the partial sum S_k is equal to $m_k = p_1p_2p_3\cdots p_k$. If we repeat p_{k+1} times the period of the partial sum S_k, the total number of k-tuples will be $m_kp_{k+1} = p_1p_2p_3\cdots p_kp_{k+1} = m_{k+1}$. Thus, when we add the sequence s_{k+1}, the number of $(k+1)$-tuples of S_{k+1} that we obtain is equal to m_{k+1}, that is to say, a period of S_{k+1}. ∎

By the Construction Procedure, to get a period of the partial sum S_{k+1}, we first take p_{k+1} periods of the partial sum S_k. The following proposition shows that the distribution of the permitted k-tuples that are within the p_{k+1} periods of the partial sum S_k over the residue classes modulo p_{k+1} is uniform.

Proposition 2.7. *The permitted k-tuples within the first p_{k+1} periods of the partial sum S_k are uniformly distributed over the residue classes modulo p_{k+1}.*

Proof. Let c_k be the number of permitted k-tuples within a period of S_k. Let $[y] = [0], [1], [2], \ldots, [p_{k+1}-1]$ be the residue classes modulo p_{k+1}. Let $n \in \mathbb{Z}_+$ be the index of a given permitted k-tuple within the first period of the partial sum S_k. Thus, within p_{k+1} periods of the partial sum S_k there are p_{k+1} permitted k-tuples with indices $n' = m_kx + n$, where $x = 0, 1, 2, 3, \ldots, p_{k+1}-1$ represents each period. Because $(m_k, p_{k+1}) = 1$, for each residue class $[y]$ the congruence $m_kx + n \equiv y \pmod{p_{k+1}}$ has a unique solution x. Therefore, since there are c_k permitted k-tuples within the period of S_k, it follows that there are c_k permitted k-tuples within each residue class modulo p_{k+1}, and the resulting distribution is uniform. ∎

Corollary 2.8. *If there are m' consecutive periods of the partial sum S_k (including the first), where m' is a multiple of p_{k+1}, the permitted k-tuples within these m' periods are also uniformly distributed over the residue classes modulo p_{k+1}.*

3 Definition and properties of the density of permitted k-tuples

In this section, we define more formally the concept of the density of permitted k-tuples, and we prove that the density of permitted k-tuples within a period of the partial sum S_k is increasing and tends to ∞ as $k \to \infty$.

Definition 3.1. Let S_k be a given partial sum of the series $\sum s_k$; let $I[m,n]$ be a given interval of k-tuples. We denote by $c_k^{I[m,n]}$ the number of permitted k-tuples within $I[m,n]$. By abuse of notation, we normally omit specific mention of the integer interval $[m,n]\cap\mathbb{Z}_+$ and write c_k^I instead of $c_k^{I[m,n]}$ if no confusion will arise.

Definition 3.2. Let S_k be a partial sum of the series $\sum s_k$; let $I[m,n]$ be a given interval of k-tuples. The number of subintervals of size p_k in this interval is equal to $|I[m,n]|/p_k$. We define the *density of permitted k-tuples* in the interval $I[m,n]$ (or simply the *k-density*) by

$$\delta_k^{I[m,n]} = \frac{c_k^I}{|I\,[m,n]|\,/p_k}.$$

For the empty interval we define $\delta_k^{I[]} = 0$. By abuse of notation, we often omit specific mention of the integer interval $[m,n]\cap\mathbb{Z}_+$ and write δ_k^I instead of $\delta_k^{I[m,n]}$.

Remark 3.1. The density of permitted k-tuples is the average number of permitted k-tuples inside subintervals of size p_k.

Definition 3.3. Let S_k be a given partial sum of the series $\sum s_k$; let m_k be the period of S_k. Recall that we have used the notation $c_k = c_k^{I[1,m_k]}$ for the number of permitted k-tuples within the interval $I[1, m_k]$ (the first period of S_k). We normally use the notation $\delta_k = \delta_k^{I[1,m_k]}$ for the density of permitted k-tuples within the interval $I[1, m_k]$. Since m_k/p_k is the number of subintervals of size p_k within a period of S_k, by definition, we have

$$\delta_k = \frac{c_k}{m_k/p_k}.$$

By Proposition 2.3, the number of permitted k-tuples within the interval $I[1, m_k]$ (the first period of S_k), does not depend on which are the selected remainders in the sequences of remainders that form S_k. Therefore, we think of $I[1, m_k]$ as being a special interval, and this explains why we use the special notation c_k for the number of permitted k-tuples within $I[1, m_k]$, and δ_k for the density of permitted k-tuples within $I[1, m_k]$.

Example 3.1. The period of the partial sum S_4 is equal to $m_4 = 2 \times 3 \times 5 \times 7 = 30 \times 7 = 210$, and the number of permitted 4-tuples within the period is equal to $c_4 = (2-1)(3-2)(5-2)(7-2) = 15$. Then

$$\delta_4 = \frac{c_4}{m_4/p_4} = \frac{15}{30} = 0.5.$$

The following lemma gives a formula for computing δ_k.

Lemma 3.1. *We have*

$$\delta_k = \left(\frac{p_1 - 1}{p_1}\right)\left(\frac{p_2 - 2}{p_2}\right)\left(\frac{p_3 - 2}{p_3}\right)\cdots\left(\frac{p_{k-1} - 2}{p_{k-1}}\right)(p_k - 2).$$

Proof. Within a period of the partial sum S_k, the total number of k-tuples is equal to $m_k = p_1 p_2 p_3 p_4 \cdots p_{k-1} p_k$ (see Proposition 2.1). Therefore, the number of intervals of size p_k is equal to $(p_1 p_2 p_3 p_4 \cdots p_{k-1} p_k)/p_k = p_1 p_2 p_3 p_4 \cdots p_{k-1}$. On the other hand, the number of permitted k-tuples within a period of S_k is equal to $c_k = (p_1 - 1)(p_2 - 2)(p_3 - 2) \cdots (p_{k-1} - 2)(p_k - 2)$, by Proposition 2.3. Consequently, by definition, we obtain

$$\begin{aligned}\delta_k &= \frac{(p_1 - 1)(p_2 - 2)(p_3 - 2)\cdots(p_{k-1} - 2)(p_k - 2)}{p_1 p_2 p_3 p_4 \cdots p_{k-1}} = \\ &= \left(\frac{p_1 - 1}{p_1}\right)\left(\frac{p_2 - 2}{p_2}\right)\left(\frac{p_3 - 2}{p_3}\right)\cdots\left(\frac{p_{k-1} - 2}{p_{k-1}}\right)(p_k - 2).\end{aligned}$$

∎

The next lemma proves that δ_k is increasing if $k > 1$.

Lemma 3.2. *Let S_k and S_{k+1} be consecutive partial sums of the series $\sum s_k$. If δ_k denotes the density of permitted k-tuples within a period of S_k, and δ_{k+1} denotes the density of permitted $(k+1)$-tuples within a period of S_{k+1}, then*

$$\delta_{k+1} = \delta_k \left(\frac{p_{k+1} - 2}{p_k}\right).$$

Proof. By Lemma 3.1,

$$\delta_k = \left(\frac{p_1 - 1}{p_1}\right)\left(\frac{p_2 - 2}{p_2}\right)\left(\frac{p_3 - 2}{p_3}\right)\cdots\left(\frac{p_{k-1} - 2}{p_{k-1}}\right)(p_k - 2)$$

and

$$\delta_{k+1} = \left(\frac{p_1 - 1}{p_1}\right)\left(\frac{p_2 - 2}{p_2}\right)\left(\frac{p_3 - 2}{p_3}\right)\cdots\left(\frac{p_{k-1} - 2}{p_{k-1}}\right)\left(\frac{p_k - 2}{p_k}\right)(p_{k+1} - 2).$$

Taking the quotient and simplifying yields

$$\frac{\delta_{k+1}}{\delta_k} = \frac{p_{k+1} - 2}{p_k} \Longrightarrow \delta_{k+1} = \delta_k \left(\frac{p_{k+1} - 2}{p_k}\right).$$

∎

Corollary 3.3. *By Lemma 3.2,*

1. $p_{k+1} - p_k < 2 \Longrightarrow \delta_{k+1} < \delta_k$.
2. $p_{k+1} - p_k = 2 \Longrightarrow \delta_{k+1} = \delta_k$.
3. $p_{k+1} - p_k > 2 \Longrightarrow \delta_{k+1} > \delta_k$.

Example 3.2. The characteristic prime moduli of the partial sums S_4 and S_5 are $p_4 = 7$ and $p_5 = 11$. The period of the partial sum S_4 is $m_4 = 2 \times 3 \times 5 \times 7 = 30 \times 7 = 210$, and the number of permitted 4-tuples is $c_4 = (2-1)(3-2)(5-2)(7-2) = 15$. Then $\delta_4 = 15/30 = 0.500$. On the other hand, the period of the partial sum S_5 is $m_5 = 2 \times 3 \times 5 \times 7 \times 11 = 210 \times 11 = 2310$, and the number of permitted 5-tuples is $c_5 = (2-1)(3-2)(5-2)(7-2)(11-2) = 135$. Then $\delta_5 = 135/210 \approx 0.643$. Note that since 7 and 11 are not twin primes, $\delta_5 > \delta_4$ (see Corollary 3.3).

Now we prove that $\delta_k \to \infty$ as $k \to \infty$. First, we make a definition.

Definition 3.4. Let $p_k > 2$ and p_{k+1} be consecutive primes. We denote by θ_k the difference $p_{k+1} - p_k - 2$.

Theorem 3.4. *Let S_k be a given partial sum. Let δ_k be the density of permitted k-tuples within a period of S_k. As $k \to \infty$, we have $\delta_k \to \infty$.*

Proof. Lemma 3.1 implies

$$\delta_k = \left(\frac{p_1 - 1}{p_1}\right)\left(\frac{p_2 - 2}{p_2}\right)\left(\frac{p_3 - 2}{p_3}\right)\left(\frac{p_4 - 2}{p_4}\right)\left(\frac{p_5 - 2}{p_5}\right)\cdots\left(\frac{p_{k-1} - 2}{p_{k-1}}\right)(p_k - 2).$$

If we shift denominators to the right, we obtain

$$\delta_k = (p_1 - 1)\left(\frac{p_2 - 2}{p_1}\right)\left(\frac{p_3 - 2}{p_2}\right)\left(\frac{p_4 - 2}{p_3}\right)\left(\frac{p_5 - 2}{p_4}\right)\cdots\left(\frac{p_{k-1} - 2}{p_{k-2}}\right)\left(\frac{p_k - 2}{p_{k-1}}\right).$$

By definition, $\theta_k = p_{k+1} - p_k - 2 \Longrightarrow p_{k+1} - 2 = p_k + \theta_k$. Consequently, we can write the expression of δ_k as

$$\begin{aligned}
\delta_k &= \frac{1}{2}\left(\frac{p_2 + \theta_2}{p_2}\right)\left(\frac{p_3 + \theta_3}{p_3}\right)\left(\frac{p_4 + \theta_4}{p_4}\right)\cdots\left(\frac{p_{k-2} + \theta_{k-2}}{p_{k-2}}\right)\left(\frac{p_{k-1} + \theta_{k-1}}{p_{k-1}}\right) = \\
&= \frac{1}{2}\left(1 + \frac{\theta_2}{p_2}\right)\left(1 + \frac{\theta_3}{p_3}\right)\left(1 + \frac{\theta_4}{p_4}\right)\cdots\left(1 + \frac{\theta_{k-2}}{p_{k-2}}\right)\left(1 + \frac{\theta_{k-1}}{p_{k-1}}\right) = \\
&= \frac{1}{3}\left[\left(1 + \frac{1}{p_1}\right)\left(1 + \frac{\theta_2}{p_2}\right)\left(1 + \frac{\theta_3}{p_3}\right)\left(1 + \frac{\theta_4}{p_4}\right)\cdots\left(1 + \frac{\theta_{k-1}}{p_{k-1}}\right)\left(1 + \frac{\theta_k}{p_k}\right)\right]\frac{p_k}{p_k + \theta_k}.
\end{aligned}$$

Then

$$\lim_{k\to\infty} \delta_k = \frac{1}{3}\left[\left(1 + \frac{1}{p_1}\right)\prod_{k=2}^{\infty}\left(1 + \frac{\theta_k}{p_k}\right)\right]\lim_{k\to\infty}\frac{p_k}{p_k + \theta_k}. \tag{8}$$

The infinite product between square brackets diverges if the series

$$\frac{1}{p_1} + \sum_{k=2}^{\infty}\frac{\theta_k}{p_k} \tag{9}$$

diverges. In the series (9), if p_k is the first of a pair of twin primes, by definition we have $\theta_k = 0$; otherwise we have $\theta_k \geq 2$. Let $\sum_{j=1}^{\infty} 1/q_j$ denote the series where every prime q_j is the first of a pair of twin primes. Since the series of reciprocals of the twin primes converges [4], the series $\sum_{j=1}^{\infty} 1/q_j$ also converges. Therefore, the series $\sum_{k=1}^{\infty} 1/p_k - \sum_{j=1}^{\infty} 1/q_j$ diverges, because $\sum_{k=1}^{\infty} 1/p_k$ diverges. By comparison with the series $\sum_{k=1}^{\infty} 1/p_k - \sum_{j=1}^{\infty} 1/q_j$, it follows that the series (9) diverges, because $\theta_k/p_k > 1/p_k$ for the terms where $\theta_k > 0$. Thus, the infinite product in (8) tends to ∞ as well. On the other hand, by the Bertrand-Chebyshev theorem, $p_k < p_{k+1} < 2p_k \Longrightarrow \theta_k < p_k \Longrightarrow p_k/(p_k + \theta_k) > 1/2$. Consequently, $\delta_k \to \infty$ as $k \to \infty$.

∎

4 The average density of permitted k-tuples within a given interval $I[m, n]$

Let S_k be a given partial sum of the series $\sum s_k$. In Section 3 we showed that, for the interval $I[1, m_k]$ of the partial sum S_k (the first period), the density of permitted k-tuples does not depend on the choice of the selected remainders in the sequences s_h $(1 \leq h \leq k)$ that form the partial sum S_k (see Lemma 3.1). However, it is easy to see that this assertion does not hold for all the intervals $I[m, n]$ of the partial sum S_k. In this section we prove that, within a given interval $I[m, n]$ of the partial sum S_k, the average of the values of the k-density for all the possible choices of the selected remainders is equal to δ_k. First, we make some definitions.

Definition 4.1. Let s_h $(1 \leq h \leq k)$ be the sequences of remainders that form the partial sum S_k. A given choice of the selected remainders within the period of one of the sequences s_h, or within the periods of all the sequences s_h $(1 \leq h \leq k)$, will be called a *combination* of selected remainders. We denote by ν_h the number of combinations of selected remainders within the period of a given sequence s_h. Since, by definition, for the sequences s_h $(1 < h \leq k)$ there are two selected remainders within the period p_h,

$$\nu_h = \binom{p_h}{2}. \tag{10}$$

In the sequence s_1 there is only one selected remainder within the period; then, $p_1 = 2 \Longrightarrow \nu_1 = 2$. We denote by N_k the number of combinations of selected remainders within the periods of all the sequences s_h $(1 \leq h \leq k)$. Then

$$N_k = \binom{p_1}{1}\binom{p_2}{2}\binom{p_3}{2}\cdots\binom{p_k}{2}. \tag{11}$$

Convention. From now on, when we refer to the average density of permitted k-tuples within a given interval $I[m, n]$ of the partial sum S_k, we mean that this average is computed taking into account all the combinations of selected remainders in the sequences s_h that form the partial sum S_k. We use the same convention when we refer to the average number of permitted k-tuples.

Definition 4.2. The operation of Type A.

Let s_h $(1 \leq h \leq k)$ be the sequences of remainders that form the partial sum S_k. For $h > 1$, let $r, r' \pmod{p_h}$ be the selected remainders within a period p_h of the sequence s_h. We define the operation that changes the selected remainders $r, r' \pmod{p_h}$ to $r+1, r'+1 \pmod{p_h}$ to be the *Type A* operation.

For the sequence s_1, we also define the operation of changing the selected remainder $r \pmod{p_1}$ to $r+1 \pmod{p_1}$ to be the operation of Type A.

Example 4.1. Table 5 shows the first period of the sequence of remainders s_4 $(p_4 = 7)$, where initially we select the remainders [1] and [3] and then we apply successively the Type A operation.

Table 5: First period of the sequence of remainders s_4.

n							
1	[1]	1	1	1	1	[1]	1
2	2	[2]	2	2	2	2	[2]
3	[3]	3	[3]	3	3	3	3
4	4	[4]	4	[4]	4	4	4
5	5	5	[5]	5	[5]	5	5
6	6	6	6	[6]	6	[6]	6
7	0	0	0	0	[0]	0	[0]

Definition 4.3. The operation of Type B.

Let s_h $(1 < h \leq k)$ be the sequences of remainders that form the partial sum S_k. Let $r, r' \pmod{p_h}$ be the selected remainders (in that order), within a period p_h of the sequence s_h. We define the *Type B* operation as follows:

1) The remainder r holds selected.

2) We change the other selected remainder $r' \pmod{p_h}$ to $r'+1 \pmod{p_h}$, $r \neq r'+1$.

Example 4.2. Table 6 shows the first period of the sequence of remainders s_4 $(p_4 = 7)$, where initially we selected the remainders [1] and [2], and then we applied successively the Type B operation.

Table 6: First period of the sequence of remainders s_4.

n						
1	[1]	[1]	[1]	[1]	[1]	[1]
2	[2]	2	2	2	2	2
3	3	[3]	3	3	3	3
4	4	4	[4]	4	4	4
5	5	5	5	[5]	5	5
6	6	6	6	6	[6]	6
7	0	0	0	0	0	[0]

Definition 4.4. Let s_h $(1 \leq h \leq k)$ be a given sequence of remainders modulo p_h. We define ν_h^A by $\nu_h^A = p_h$, and we define ν_h^B $(h > 1)$ by $\nu_h^B = (p_h - 1)/2$.

Remark 4.1. Suppose that we choose two consecutive selected remainders r, r' within the period of the sequence s_h $(1 < h \leq k)$. Then we have one out of ν_h combinations of selected remainders. Repeating the Type A operation $\nu_h^A - 1$ times, we obtain $\nu_h^A = p_h$ different combinations of selected remainders. Now, if for each one of these combinations we leave unchanged the selected remainder r, and then we repeat $\nu_h^B - 1$ times the Type B operation, we obtain all the ν_h combinations of selected remainders within the period of the sequence s_h. This is expressed by the equation

$$\nu_h = \binom{p_h}{2} = \frac{p_h!}{2!\,(p_h - 2)!} = p_h \frac{p_h - 1}{2} = \nu_h^A \nu_h^B.$$

Definition 4.5. Let S_k and S_{k+1} be the partial sums of level k and $k+1$. Let s_{k+1} be the sequence of remainders of level $k+1$. Let $I[m,n]_k$ be an interval of k-tuples of S_k, and let $I[m,n]_{k+1}$ be an interval of $(k+1)$-tuples of S_{k+1}, where the indices m, n are the same for both intervals. When we juxtapose the remainders of the sequence s_{k+1} to the right of each k-tuple of S_k, then, by Proposition 2.2, the permitted k-tuples of S_k, whose indices are congruent to a given selected remainder of s_{k+1} modulo p_{k+1}, are converted to prohibited $(k+1)$-tuples of S_{k+1}. We denote by f_{k+1} the fraction of the permitted k-tuples within the interval $I[m,n]_k$ that are converted to prohibited $(k+1)$-tuples within the interval $I[m,n]_{k+1}$. For the partial sum S_1, let f_1 denote the fraction of the prohibited 1-tuples within the interval $I[m,n]_{k=1}$.

We denote by $\overline{f}_{k+1}$ the average of f_{k+1} for all the combinations of selected remainders in the sequence s_{k+1} $(k \geq 1)$. For the partial sum S_1, let $\overline{f}_1$ denote the average of f_1 for the 2 combinations of selected remainders in the sequence s_1.

The following lemma gives a formula for computing the average fraction $\overline{f}_{k+1}$.

Lemma 4.1. *For $k \geq 1$ we have $\overline{f}_{k+1} = 2/p_{k+1}$. For S_1 we have $\overline{f}_1 = 1/p_1$.*

Proof. Let $[0], [1], [2], \ldots, [p_{k+1} - 1]$ be the residue classes modulo p_{k+1}. Let c_k^I be the number of permitted k-tuples within $I[m,n]_k$. We denote by $\eta_0, \eta_1, \eta_2, \ldots, \eta_{p_{k+1}-1}$ the number of permitted k-tuples within $I[m,n]_k$ whose indices belong to the residue classes $[0], [1], [2], \ldots, [p_{k+1} - 1]$ respectively. Therefore, $c_k^I = \eta_0 + \eta_1 + \eta_2 + \cdots + \eta_{p_{k+1}-1}$.

We wish to compute the average fraction of the permitted k-tuples within the interval $I[m,n]_k$ that are converted to prohibited $(k+1)$-tuples within the interval $I[m,n]_{k+1}$, for all the ν_{k+1} combinations of selected remainders in the sequence s_{k+1} $(k \geq 1)$. Now,

$$\nu_{k+1} = \nu_{k+1}^A \nu_{k+1}^B = p_{k+1} \frac{(p_{k+1} - 1)}{2},$$

by Remark 4.1. Consequently, we begin by taking the average over the ν_{k+1}^A combinations obtained by Type A operations, and then we take the average of the previous averages over the ν_{k+1}^B combinations obtained by Type B operations.

Step 1. Suppose that we choose two selected remainders r, r' within the period of the sequence s_{k+1}. By Proposition 2.2, the indices of the permitted k-tuples within the interval $I[m,n]_k$ of S_k that are converted to prohibited $(k+1)$-tuples within the interval $I[m,n]_{k+1}$ of S_{k+1} belong to one of the residue classes $[r]$ or $[r']$. It follows that the fraction of the c_k^I permitted k-tuples within the interval $I[m,n]_k$ of S_k that are converted to prohibited $(k+1)$-tuples within the interval $I[m,n]_{k+1}$ of S_{k+1} is equal to $(\eta_r + \eta_{r'})/c_k^I$. Taking the average over the ν_{k+1}^A combinations of selected remainders obtained by repeated Type A operations, we obtain

$$\frac{\displaystyle\sum_{i=1}^{\nu_{k+1}^A} \frac{\eta_r+\eta_{r'}}{c_k^I}}{\nu_{k+1}^A} = \frac{\displaystyle\sum_{i=1}^{p_{k+1}} \frac{\eta_r+\eta_{r'}}{c_k^I}}{p_{k+1}} = \frac{(1/c_k^I)\left(\displaystyle\sum_{r=0}^{p_{k+1}-1} \eta_r + \sum_{r'=0}^{p_{k+1}-1} \eta_{r'}\right)}{p_{k+1}} = \frac{(1/c_k^I)\left(c_k^I+c_k^I\right)}{p_{k+1}} = \frac{2}{p_{k+1}}.$$

Step 2. Now, if we take the average over the $\nu_{k+1}^B = (p_{k+1}-1)/2$ combinations of selected remainders obtained by repeated Type B operations from each one of the combinations obtained before, we obtain

$$\overline{f}_{k+1} = \frac{\displaystyle\sum_{j=1}^{\nu_{k+1}^B} \frac{2}{p_{k+1}}}{\nu_{k+1}^B} = \frac{2}{p_{k+1}},$$

because p_{k+1} does not depend on the index j $(1 \le j \le \nu_{k+1}^B)$.

For the partial sum S_1, there are two residue classes modulo $p_1 = 2$ and one selected remainder. Therefore, it is easy to see that $\overline{f}_1 = 1/p_1$.

∎

Definition 4.6. It follows from Proposition 2.2 that when we juxtapose the remainders of the sequence s_{k+1} to the right of each k-tuple of S_k, the permitted k-tuples of S_k whose indices are not congruent to any of the two selected remainders of s_{k+1} modulo p_{k+1} are, as $(k+1)$-tuples of S_{k+1}, still permitted. We denote by f'_{k+1} the fraction of permitted k-tuples within the interval $I[m,n]_k$ of S_k that are transferred to the interval $I[m,n]_{k+1}$ of S_{k+1} as permitted $(k+1)$-tuples. For the partial sum S_1, let f'_1 denote the fraction of the permitted 1-tuples within the interval $I[m,n]_{k=1}$.

We denote by $\overline{f}'_{k+1}$ the average of f'_{k+1} for all the combinations of selected remainders in the sequence s_{k+1}. For the partial sum S_1, let $\overline{f}'_1$ denote the average of f'_1 for the 2 combinations of selected remainders in the sequence s_1.

Now, using the preceding lemma, we can calculate the average fraction $\overline{f}'_{k+1}$.

Lemma 4.2. *We have $\overline{f}'_{k+1} = (p_{k+1}-2)/p_{k+1}$. For S_1, we have $\overline{f}'_1 = (p_1-1)/p_1$.*

Proof. By Proposition 2.2, a given permitted k-tuple within the interval $I[m,n]_k$ of S_k can be transferred to the interval $I[m,n]_{k+1}$ of S_{k+1} either as a permitted $(k+1)$-tuple or as a prohibited $(k+1)$-tuple. Consequently, $f_{k+1} + f'_{k+1} = 1$, and so $\overline{f}_{k+1} + \overline{f}'_{k+1} = 1$. Therefore, using Lemma 4.1, we obtain $\overline{f}'_{k+1} = 1 - \overline{f}_{k+1} = 1 - 2/p_{k+1} = (p_{k+1}-2)/p_{k+1}$.

For the partial sum S_1, we have $\overline{f}_1 = 1/p_1 \Longrightarrow \overline{f}'_1 = (p_1-1)/p_1$.

∎

Definition 4.7. Let S_k be the partial sum of level k. Let $I[m,n]$ be an interval of k-tuples of S_k. We denote by $\overline{c}_k^I$ the average number of permitted k-tuples within the interval $I[m,n]$. We denote by $\overline{\delta}_k^I$ the average density of permitted k-tuples within the interval $I[m,n]$.

Finally, using the preceding lemmas, we calculate the average k-density within a given interval $I[m,n]$, and show that it is equal to the k-density within the period of S_k.

Theorem 4.3. *Let δ_k be the density of permitted k-tuples within a period of the partial sum S_k. Then $\overline{\delta}_k^I = \delta_k$.*

Proof. Let s_h $(1 \le h \le k)$ be the sequences of remainders that form S_k. If there were no selected remainders within the sequences s_h, all the k-tuples within the interval $I[m,n]$ would be permitted k-tuples, and then $\overline{c}_k^I = |I[m,n]|$, where $|I[m,n]|$ is the size of the interval $I[m,n]$. However, since we have selected remainders in every sequence s_h $(1 \le h \le k)$, using Lemma 4.2 at each level transition from $h=1$ to $h=k$, we can write

$$\overline{c}_k^I = |I\,[m,n]| \left(\frac{p_1-1}{p_1}\right)\left(\frac{p_2-2}{p_2}\right)\left(\frac{p_3-2}{p_3}\right)\cdots\left(\frac{p_k-2}{p_k}\right). \tag{12}$$

Now, the number of intervals of size p_k within the interval $I[m,n]$ is equal to $|I[m,n]|/p_k$. Consequently, by definition,

$$\overline{\delta}_k^I = \frac{\overline{c}_k^I}{\frac{|I[m,n]|}{p_k}} = \frac{p_k}{|I\,[m,n]|}\overline{c}_k^I. \tag{13}$$

Therefore, substituting (12) for $\overline{c}_k^I$ in (13), and using Lemma 3.1, we obtain

$$\begin{aligned}\overline{\delta}_k^I &= \frac{p_k}{|I\,[m,n]|}\left(|I\,[m,n]|\left(\frac{p_1-1}{p_1}\right)\left(\frac{p_2-2}{p_2}\right)\left(\frac{p_3-2}{p_3}\right)\cdots\left(\frac{p_k-2}{p_k}\right)\right) = \\ &= \left(\frac{p_1-1}{p_1}\right)\left(\frac{p_2-2}{p_2}\right)\left(\frac{p_3-2}{p_3}\right)\cdots(p_k-2) = \delta_k.\end{aligned}$$

∎

5 The density of permitted k-tuples within the interval $I[1,n]$ as $n \to \infty$

Let S_k ($k > 2$) be a partial sum of the series $\sum s_k$. Let p_k be its characteristic prime modulus, and let m_k be its period. Let δ_k be the density of permitted k-tuples within the period of S_k. Let $I[1,n]$ ($n \geq m_k$) be a given interval of k-tuples of the partial sum S_k. We denote by c_k^I the number of permitted k-tuples, and by δ_k^I the k-density in $I[1,n]$. In this section we will show that δ_k^I converges to δ_k as $n \to \infty$. First, we make a definition.

Definition 5.1. Let $\lfloor n/m_k \rfloor$ denote the integer part of n/m_k ($n \geq m_k$). We denote by c_η the number of permitted k-tuples within the interval $I[1, \lfloor n/m_k \rfloor m_k] \subseteq I[1,n]$. If n is not a multiple of m_k, we denote by c_ϵ the number of permitted k-tuples within the interval $I[\lfloor n/m_k \rfloor m_k + 1, n] \subset I[1,n]$; otherwise $c_\epsilon = 0$. We call the interval $I[\lfloor n/m_k \rfloor m_k + 1, n]$ the *incomplete period* of the interval $I[1,n]$.

The following lemma gives us a formula for the k-density in the interval $I[1,n]$.

Lemma 5.1. *We have*

$$\delta_k^I = \frac{\left\lfloor \frac{n}{m_k} \right\rfloor m_k}{n}\delta_k + \frac{p_k c_\epsilon}{n}.$$

Proof. By definition,

$$\delta_k^I = \frac{c_k^I}{\frac{n}{p_k}}.$$

Since $\lfloor n/m_k \rfloor$ is the number of times that the period of S_k fits in the interval $I[1,n]$, the interval $I[1, \lfloor n/m_k \rfloor m_k]$ is that part of $I[1,n]$ whose size is a multiple of m_k, and so $\lfloor n/m_k \rfloor m_k/p_k$ is the number of subintervals of size p_k within this part of the interval $I[1,n]$. Consequently, multiplying by the k-density in the period of S_k, we obtain

$$c_\eta = \frac{\left\lfloor \frac{n}{m_k} \right\rfloor m_k}{p_k}\delta_k.$$

Since c_k^I is the number of permitted k-tuples within $I[1,n]$, we have $c_k^I = c_\eta + c_\epsilon$. Then

$$\delta_k^I = \frac{c_k^I}{\frac{n}{p_k}} = \frac{c_\eta + c_\epsilon}{\frac{n}{p_k}} = \frac{\frac{\left\lfloor \frac{n}{m_k} \right\rfloor m_k}{p_k}\delta_k + c_\epsilon}{\frac{n}{p_k}} = \frac{\left\lfloor \frac{n}{m_k} \right\rfloor m_k}{n}\delta_k + \frac{p_k c_\epsilon}{n}.$$

∎

Remark 5.1. Let c_k be the number of permitted k-tuples within the period of S_k. By definition,

$$\delta_k = \frac{c_k}{m_k/p_k} \implies c_k = \delta_k m_k/p_k.$$

Now, using the formula from the preceding lemma, we find lower and upper bounds for the k-density within the interval $I[1,n]$.

Lemma 5.2. *Let $I[1,n]$ ($n \geq m_k$) be an interval of k-tuples of a given partial sum S_k. For $k > 2$,*

$$\frac{\left\lfloor \frac{n}{m_k} \right\rfloor m_k}{\left\lfloor \frac{n}{m_k} \right\rfloor m_k + (m_k - 1)} \delta_k < \delta_k^I < \frac{\left(\left\lfloor \frac{n}{m_k} \right\rfloor + 1 \right) m_k}{\left\lfloor \frac{n}{m_k} \right\rfloor m_k + 1} \delta_k. \tag{14}$$

Proof. Step 1. We first consider the case where n is not a multiple of m_k. By Lemma 5.1,

$$\delta_k^I = \frac{\left\lfloor \frac{n}{m_k} \right\rfloor m_k}{n} \delta_k + \frac{p_k c_\epsilon}{n}. \tag{15}$$

To obtain bounds for δ_k^I, we proceed as follows. We begin by obtaining bounds for c_ϵ. By Remark 5.1, $\delta_k m_k / p_k$ is the number of permitted k-tuples within the period of S_k. Since by assumption n is not a multiple of m_k, it is easy to see that

$$0 \leq c_\epsilon \leq \delta_k m_k / p_k. \tag{16}$$

Next, we obtain bounds for the denominator in (15); since n is not a multiple of m_k,

$$\lfloor n/m_k \rfloor m_k + 1 \leq n \leq \lfloor n/m_k \rfloor m_k + (m_k - 1)\,. \tag{17}$$

Step 2. We obtain a lower bound for δ_k^I. If we replace the denominator in (15) by the upper bound in (17),

$$\frac{\left\lfloor \frac{n}{m_k} \right\rfloor m_k}{\left\lfloor \frac{n}{m_k} \right\rfloor m_k + (m_k - 1)} \delta_k + \frac{p_k c_\epsilon}{\left\lfloor \frac{n}{m_k} \right\rfloor m_k + (m_k - 1)} \leq \delta_k^I. \tag{18}$$

Note that if n is equal to the upper bound in (17), the size of the incomplete period differs from the period m_k by one. On the other hand, it is easy to check, using Proposition 2.3, that within the period of the partial sum S_k ($k > 2$) there is more than one permitted k-tuple. It follows that if n is equal to the upper bound in (17), then there is at least one permitted k-tuple within the incomplete period of $I[1,n]$, and so $c_\epsilon > 0$. Therefore, if we replace c_ϵ in (18) by the lower bound in (16),

$$\frac{\left\lfloor \frac{n}{m_k} \right\rfloor m_k}{\left\lfloor \frac{n}{m_k} \right\rfloor m_k + (m_k - 1)} \delta_k < \delta_k^I. \tag{19}$$

Step 3. We now obtain an upper bound for δ_k^I. If we replace the denominator in (15) by the lower bound in (17),

$$\delta_k^I \leq \frac{\left\lfloor \frac{n}{m_k} \right\rfloor m_k}{\left\lfloor \frac{n}{m_k} \right\rfloor m_k + 1} \delta_k + \frac{p_k c_\epsilon}{\left\lfloor \frac{n}{m_k} \right\rfloor m_k + 1}. \tag{20}$$

Note that if n is equal to the lower bound in (17), the size of the incomplete period is equal to 1, and so there can not be more than one permitted k-tuple within the incomplete period of $I[1,n]$. On the other hand, we saw in Step 2 that for a level $k > 2$, there is more than one permitted k-tuple within the period of the partial sum S_k. It follows that if n is equal to the lower bound in (17), then $c_\epsilon \leq 1 < \delta_k m_k / p_k$. Therefore, if we replace c_ϵ in (20) by the upper bound in (16),

$$\delta_k^I < \frac{\left\lfloor \frac{n}{m_k} \right\rfloor m_k}{\left\lfloor \frac{n}{m_k} \right\rfloor m_k + 1} \delta_k + \frac{p_k \frac{\delta_k m_k}{p_k}}{\left\lfloor \frac{n}{m_k} \right\rfloor m_k + 1} = \frac{\left(\left\lfloor \frac{n}{m_k} \right\rfloor + 1 \right) m_k}{\left\lfloor \frac{n}{m_k} \right\rfloor m_k + 1} \delta_k. \tag{21}$$

Step 4. Now we complete the proof. Suppose that n is a multiple of m_k. Then $n = \lfloor n/m_k \rfloor m_k$ and $c_\epsilon = 0$, and so it is easy to see that $\delta_k^I = \delta_k$. Since the lower bound in (19) is less than δ_k, and the upper bound in (21) is greater than δ_k, we conclude that for every interval $I[1, n]$ of the partial sum S_k $(k > 2)$, the inequalities (19) and (21) are satisfied, and the lemma is proved.

∎

Remark 5.2. It is easy to check that the upper bound is decreasing, and the lower bound is increasing, in (14).

Finally, we show that the k-density in the interval $I[1, n]$ of a given partial sum S_k tends to δ_k as the size n of the interval increases.

Proposition 5.3. *Let S_k $(k > 2)$ be a given partial sum of the series $\sum s_k$. As $n \to \infty$, the density δ_k^I converges to δ_k, whatever the combination of selected remainders in the sequences s_h that form the partial sum S_k.*

Proof. Using the inequalities of Lemma 5.2, if we take limits as $n \to \infty$,

$$\lim_{n\to\infty} \frac{\left\lfloor \frac{n}{m_k} \right\rfloor m_k}{\left\lfloor \frac{n}{m_k} \right\rfloor m_k + (m_k - 1)} \delta_k < \lim_{n\to\infty} \delta_k^I < \lim_{n\to\infty} \frac{\left(\left\lfloor \frac{n}{m_k} \right\rfloor + 1 \right) m_k}{\left\lfloor \frac{n}{m_k} \right\rfloor m_k + 1} \delta_k.$$

Now, dividing the numerator and denominator by $\lfloor n/m_k \rfloor$, we obtain

$$\lim_{n\to\infty} \frac{m_k}{m_k + \frac{(m_k - 1)}{\left\lfloor \frac{n}{m_k} \right\rfloor}} \delta_k < \lim_{n\to\infty} \delta_k^I < \lim_{n\to\infty} \frac{\left(1 + \frac{1}{\left\lfloor \frac{n}{m_k} \right\rfloor}\right) m_k}{m_k + \frac{1}{\left\lfloor \frac{n}{m_k} \right\rfloor}} \delta_k.$$

Since for a given level k, the values m_k and δ_k are constants, as $n \to \infty$ we have $\lfloor n/m_k \rfloor \to \infty$, and the lower and upper bounds tend to δ_k. This implies that δ_k^I converges to δ_k as $n \to \infty$.

∎

6 The k-density within the intervals $I[1, p_k^2]$ and $I[p_k^2 + 1, m_k]$

Suppose given a partial sum S_k. In this section, we shall subdivide the interval $I[1, m_k]$ of S_k into two parts, and shall establish the relation between the density of permitted k-tuples within one part and the density of permitted k-tuples within the other part. We begin by introducing some terminology and notation.

Definition 6.1. Let S_k and S_{k+1} be consecutive partial sums of the series $\sum s_k$. We use the notation $p_k \to p_{k+1}$ or alternatively $k \to k + 1$ to denote the transition from level k to level $k + 1$. For the level transition $p_k \to p_{k+1}$, we call the difference $p_{k+1} - p_k$ the *order* of the transition.

Definition 6.2. When we juxtapose the remainders of the sequence s_{k+1} to the right of each k-tuple of S_k, by Proposition 2.2, a given permitted k-tuple of S_k, whose index is congruent to a selected remainder of s_{k+1} modulo p_{k+1}, is converted to a prohibited $(k + 1)$-tuple of S_{k+1}. In that case, we say that at the level transition $k \to k + 1$ one permitted k-tuple is *removed.*

Let S_k $(k \geq 4)$ be a given partial sum of the series $\sum s_k$. Let s_h $(1 \leq h \leq k)$ be the periodic sequences of remainders that form the partial sum S_k. Let m_h be the period of every partial sum S_h from level $h = 1$ to level $h = k$. Let c_h be the number of permitted h-tuples, and let δ_h be the h-density within the period of every partial sum S_h $(1 \leq h \leq k)$.

Definition 6.3. If we write the index n of the sequences s_h from top to bottom, and the level k from left to right (see Table 2) we say that the partial sum S_k is in *vertical position.* Now, suppose that the partial sum S_k is in vertical position, and we rotate it 90 degrees counterclockwise. Then, the index n of the sequences s_h increases from left to right, and the level k increases from the bottom up. In this case, we say that the partial sum S_k is in *horizontal position.*

For every partial sum S_h from level $h = 1$ to level $h = k$ in horizontal position, let us consider the interval $I[1, m_k]_h$, whose size is the period m_k of S_k.

Remark 6.1. Using Proposition 2.1, it is easy to check that the period of the partial sum S_1 is equal to $m_1 = p_1 = 2$. On the other hand, by Proposition 2.3, within every period of S_1 we have only one permitted 1-tuple. Therefore, the interval $I[1, m_k]_1$ of the partial sum S_1 is divided into subintervals of size $m_1 = 2$, each one containing one permitted 1-tuple. The position of the permitted 1-tuple is the same within every subinterval, and is determined by the selected remainder in the sequence s_1. Note that the positions of consecutive permitted 1-tuples in the partial sum S_1 differ by two.

Remark 6.2. By the preceding remark, the positions of the permitted 1-tuples show a regular pattern along the interval $I[1, m_k]_1$ of the partial sum S_1. However, when we add the sequences s_h from level $h = 2$ to level $h = k$, the selected remainders in each sequence s_h remove permitted $(h-1)$-tuples from the partial sum S_{h-1}. Consequently, we obtain an interval $I[1, m_k]_k$ where the permitted k-tuples are spread along the interval, in positions that show an irregular pattern. Note that if we change the combination of selected remainders in the sequences s_h $(1 \leq h \leq k)$, within the interval $I[1, m_k]_k$ some permitted k-tuples 'disappear', and other permitted k-tuples 'appear', although the number of permitted k-tuples within the interval $I[1, m_k]_k$ of the partial sum S_k does not change (see Proposition 2.3).

The following lemma gives us the number of permitted h-tuples within the interval $I[1, m_k]_h$ of every partial sum S_h where $h < k$.

Lemma 6.1. *Let S_k $(k \geq 4)$ be a given partial sum of the series $\sum s_k$. Let us consider the interval $I[1, m_k]_h$ in every partial sum S_h, from level $h = 1$ to level $h = k$.*

For any given partial sum S_h $(h < k)$, the number of permitted h-tuples within the interval $I[1, m_k]_h$ is equal to $c_h p_{h+1} p_{h+2} \cdots p_k$.

Proof. Choose a level $h < k$. By definition, we have $m_k = p_1 p_2 p_3 \cdots p_h p_{h+1} p_{h+2} \cdots p_k = m_h p_{h+1} p_{h+2} \cdots p_k$. That is, the size of the interval $I[1, m_k]_h$ of the partial sum S_h is equal to $p_{h+1} p_{h+2} \cdots p_k$ times the period m_h of the partial sum S_h. Consequently, it is easy to see that the number of permitted h-tuples within the interval $I[1, m_k]_h$ is equal to $c_h p_{h+1} p_{h+2} \cdots p_k$.

∎

Remark 6.3. By Proposition 2.3 and Lemma 6.1, the number of permitted h-tuples within the interval $I[1, m_k]_h$ $(1 \leq h \leq k)$ does not depend on the combination of selected remainders in the sequences s_h that form the partial sum S_h; therefore, neither does the density of permitted h-tuples within this interval (see Lemma 3.1). It is easy to see that this h-density is equal to δ_h.

Now, let us denote by c'_h the number of permitted h-tuples within the interval $I[1, m_k]_h$ of every partial sum S_h $(1 \leq h \leq k)$, which is computed using Proposition 2.3 and Lemma 6.1. We have a question at this point: What is the behaviour of c'_h as h goes from level 1 to level k? This behaviour can be described as follows.

Remark 6.4. For every partial sum S_h $(h < k)$, suppose that we juxtapose the remainders of the sequence s_{h+1} to each h-tuple of S_h. By Proposition 2.2, the permitted h-tuples within the interval $I[1, m_k]_h$ whose indices are included in two residue classes modulo p_{h+1} are removed by the selected remainders within the sequence s_{h+1}; and the permitted h-tuples whose indices are not included in these residue classes are transferred to level $h + 1$ as permitted $(h + 1)$-tuples within the interval $I[1, m_k]_{h+1}$ of the partial sum S_{h+1}, whatever the combination of selected remainders in the sequence s_{h+1}. Since for every level $h < k$ the size of the interval $I[1, m_k]_h$ is a multiple of p_{h+1}, by Proposition 2.7 and Corollary 2.8, the permitted h-tuples within the interval $I[1, m_k]_h$ of S_h are distributed uniformly over the residue classes modulo p_{h+1}. Therefore, for each level $h < k$, a fraction $2/p_{h+1}$ of the permitted h-tuples within the interval $I[1, m_k]_h$ of S_h have been removed, and a fraction $(p_{h+1} - 2)/p_{h+1}$ have been transferred to level $h + 1$ as permitted $(h+1)$-tuples within the interval $I[1, m_k]_{h+1}$ of S_{h+1}, whatever the combination of selected remainders in the sequence s_{h+1}.

Let us examine now the behaviour of δ_h as h goes from level 1 to level k. Since the selected remainders of the sequences s_{h+1} remove permitted h-tuples within the interval $I[1, m_k]_h$ of the partial sum S_h, at each level transition $h \to h + 1$, the number of permitted h-tuples decreases as the level increases from $h = 1$ to $h = k$. However, by Lemma 3.2 and Corollary 3.3, the h-density within the interval $I[1, m_k]_h$ of the partial sum S_h grows at each transition $p_h \to p_{h+1}$ of order greater than 2, because to compute the h-density we count the permitted h-tuples within subintervals of size p_h, which grow by more than 2, overcompensating for the permitted h-tuples removed. If $p_h \to p_{h+1}$ is a level transition of order 2, the h-density does not change, because the increase in the size p_h is compensated for by the permitted h-tuples removed. (Note that $p_1 \to p_2$ is the only level transition where δ_h decreases.) Therefore, the h-density increases between $h = 1$ and $h = k$ if we choose k so large that there are a sufficient number of level transitions of order greater than 2 between $h = 1$ and $h = k$.

Now, if we 'cut' the first period of S_k into two parts, between the indices p_k^2 and $p_k^2 + 1$, we obtain a left-hand subinterval and a right-hand subinterval.

Definition 6.4. Let S_k $(k \geq 4)$ be a given partial sum, in horizontal position. We subdivide the interval $I[1, m_k]$ (its first period) into two intervals: $I[1, p_k^2]$, which we call the *Left interval*, and $I[p_k^2 + 1, m_k]$, which we call the *Right interval*. We often denote the Left interval $I[1, p_k^2]$ by the symbol L_k, and the Right interval $I[p_k^2 + 1, m_k]$ by the symbol R_k. For every partial sum S_h from level $h = 1$ to level $h = k - 1$ there is also a Left interval $I[1, p_k^2]_h$, and a Right interval $I[p_k^2 + 1, m_k]_h$. See Figure 4.

So, the first period of the sequence of k-tuples can now be seen as a matrix, with m_k columns and k rows. Each row of this matrix, from $h = 1$ to $h = k$, is formed by the remainders modulo p_h from $n = 1$ to $n = m_k$. In addition, this matrix has been partitioned into two blocks: the *Left block* formed by the columns from $n = 1$ to $n = p_k^2$; and the *Right block* formed by the columns from $n = p_k^2 + 1$ to $n = m_k$. Each row of the Left block is formed by the remainders of dividing the integers from $n = 1$ to $n = p_k^2$ by the modulus p_h; and each row of the Right block is formed by the remainders of dividing the integers from $n = p_k^2 + 1$ to $n = m_k$ by the modulus p_h.

$$\begin{array}{lcccccc}
\textit{Level } h = k & & I\left[1, p_k^2\right]_{h=k} & \cup & I\left[p_k^2 + 1, m_k\right]_{h=k} & = & I\left[1, m_k\right]_{h=k} \\
\vdots & & \vdots & & \vdots & & \vdots \\
\textit{Level } h = h' & & I\left[1, p_k^2\right]_{h=h'} & \cup & I\left[p_k^2 + 1, m_k\right]_{h=h'} & = & I\left[1, m_k\right]_{h=h'} \\
\vdots & & \vdots & & \vdots & & \vdots \\
\textit{Level } h = 1 & & I\left[1, p_k^2\right]_{h=1} & \cup & I\left[p_k^2 + 1, m_k\right]_{h=1} & = & I\left[1, m_k\right]_{h=1}
\end{array}$$

Figure 4: Left and Right intervals

Definition 6.5. For a given partial sum S_h $(1 \le h \le k)$, we use the notation $c_h^{L_k}$ to denote the number of permitted h-tuples within the Left interval $I[1, p_k^2]_h$, and we use the notation $c_h^{R_k}$ to denote the number of permitted h-tuples within the Right interval $I[p_k^2 + 1, m_k]_h$.

Although the number of permitted h-tuples within the interval $I[1, m_k]_h$ of every partial sum S_h $(1 \le h \le k)$ does not change if we choose another set of selected remainders, the positions of the permitted h-tuples along the period of S_h will be changed. Then, it seems reasonable to expect that some permitted h-tuples will be transferred from the Left interval $I[1, p_k^2]_h$ to the Right interval $I[p_k^2 + 1, m_k]_h$, or vice versa, because the size of the Left interval and the size of the Right interval are not a multiple of m_h. Hence, the number of permitted h-tuples within the Left interval $I[1, p_k^2]_h$ and within the Right interval $I[p_k^2 + 1, m_k]_h$ is determined by the combination of selected remainders in the sequences s_h that form the partial sum S_h.

Definition 6.6. For a given partial sum S_h (that is to say, a partial sum where we have a given combination of selected remainders in the sequences that form the partial sum S_h), we use the notation $\delta_h^{L_k}$ to denote the density of permitted h-tuples within the Left interval $I[1, p_k^2]_h$, and we use the notation $\delta_h^{R_k}$ to denote the density of permitted h-tuples within the Right interval $I[p_k^2 + 1, m_k]_h$.

By Remark 6.3, the h-density within the interval $I[1, m_k]_h$ does not depend on the combination of selected remainders in the sequences s_h that form the partial sum S_h. However, the transfer of some permitted h-tuples from the Left interval $I[1, p_k^2]_h$ to the Right interval $I[p_k^2 + 1, m_k]_h$, or in the opposite direction, when we change the combination of selected remainders, brings about changes in the h-density within both intervals. The crossing of some permitted h-tuples from $I[1, p_k^2]_h$ to $I[p_k^2 + 1, m_k]_h$ decreases $\delta_h^{L_k}$ and increases the $\delta_h^{R_k}$, and vice versa. By Theorem 4.3, the average of $\delta_h^{L_k}$ within $I[1, p_k^2]_h$ is equal to δ_h, and the average of $\delta_h^{R_k}$ within $I[p_k^2 + 1, m_k]_h$ is also equal to δ_h. Hence

$$\begin{aligned}
\delta_h^{L_k} > \delta_h &\Longleftrightarrow \delta_h^{R_k} < \delta_h, \\
\delta_h^{L_k} < \delta_h &\Longleftrightarrow \delta_h^{R_k} > \delta_h.
\end{aligned} \tag{22}$$

Definition 6.7. We often call $\delta_h^{L_k}$ $(\delta_h^{R_k})$ the *true* h-density to distinguish it from the average δ_h within the Left interval $I[1, p_k^2]_h$ (the Right interval $I[p_k^2 + 1, m_k]_h$).

The following lemma shows that for every partial sum S_h $(1 \le h \le k)$, the h-density within the Left interval $I[1, p_k^2]_h$ is not equal to the h-density within the Right interval $I[p_k^2 + 1, m_k]_h$.

Lemma 6.2. *Let S_k $(k \ge 4)$ be a given partial sum of the series $\sum s_k$. Let us consider the interval $I[1, m_k]_h$ (whose size is the period m_k of S_k), the Left interval $I[1, p_k^2]_h$ and the Right interval $I[p_k^2 + 1, m_k]_h$, in every partial sum S_h from level $h = 1$ to level $h = k$. Let us denote by m_h the period of the partial sum S_h, and by c_h the number of permitted h-tuples within a period of the partial sum S_h $(1 \le h \le k)$.*

For every partial sum S_h we have $\delta_h^{L_k} \ne \delta_h^{R_k}$.

Proof. Step 1. By Remark 6.1, the positions of consecutive permitted 1-tuples in the partial sum S_1 differ by 2. It follows that the number of permitted h-tuples in every Left interval $I[1, p_k^2]_h$ $(1 \le h \le k)$ is less than the size of the interval. In symbols, $c_h^{L_k} < p_k^2$ $(1 \le h \le k)$.

Step 2. Let us consider a given partial sum S_h, where $1 \leq h < k$. Consider the number of permitted h-tuples in the Left interval $I[1, p_k^2]_h$, denoted by $c_h^{L_k}$, and the number of permitted h-tuples in the Right interval $I[p_k^2+1, m_k]_h$, denoted by $c_h^{R_k}$. By Lemma 6.1, we have $c_h^{L_k} + c_h^{R_k} = c_h p_{h+1} p_{h+2} \cdots p_k$; so, if $c_h^{L_k}$ is a multiple of p_k, then $c_h^{R_k}$ is a multiple of p_k as well. In this case, $(c_h^{L_k}/p_k)$ is a whole number; and so $(c_h^{L_k}/p_k)/p_k$, is a reduced fraction, since $c_h^{L_k} < p_k^2$, by Step 1. Hence, it is easy to check that this fraction is not equal to $(c_h^{R_k}/p_k)/(m_{k-1} - p_k)$, since $(c_h^{R_k}/p_k)$ is also a whole number, and $m_{k-1} - p_k$ is not a multiple of p_k.

On the other hand, if $c_h^{L_k}$ is not a multiple of p_k, then $c_h^{L_k}/p_k^2$ is a reduced fraction; and it is easy to check that this fraction is not equal to $c_h^{R_k}/(m_k - p_k^2)$, since $m_k - p_k^2$ is not a multiple of p_k^2. In either case, we proved that $c_h^{L_k}/p_k^2$ is not equal to $c_h^{R_k}/(m_k - p_k^2)$.

Step 3. Now, let us consider the partial sum S_k. Consider the number of permitted k-tuples in the Left interval $I[1, p_k^2]$, denoted by $c_k^{L_k}$, and the number of permitted k-tuples in the interval $I[1, m_k]$ (complete period of S_k), denoted by c_k. By Proposition 2.3 we have $c_k = (p_1 - 1)(p_2 - 2)(p_3 - 2) \cdots (p_k - 2)$. Now, if $c_k^{L_k}$ is not a multiple of p_k, we can see that $c_k^{L_k}/p_k^2$ is a reduced fraction; and it is easy to check that this fraction can not be equal to c_k/m_k, since m_k is a squarefree integer.

On the other hand, if $c_k^{L_k}$ is a multiple of p_k, then $(c_k^{L_k}/p_k)$ is a whole number; and $(c_k^{L_k}/p_k)/p_k$, is a reduced fraction, since $c_k^{L_k} < p_k^2$, by Step 1. From Proposition 2.1 and Proposition 2.3 follows

$$\begin{aligned}\frac{c_k}{m_k} &= \frac{(p_1-1)(p_2-2)(p_3-2)\cdots(p_{k-1}-2)(p_k-2)}{p_1 p_2 p_3 \cdots p_{k-1} p_k} = \\ &= \left(\frac{p_1-1}{p_1}\right)\left(\frac{p_2-2}{p_2}\right)\left(\frac{p_3-2}{p_3}\right)\cdots\left(\frac{p_{k-1}-2}{p_{k-1}}\right)\left(\frac{p_k-2}{p_k}\right),\end{aligned}$$

and shifting the denominators to the right, we obtain

$$\frac{c_k}{m_k} = \left(\frac{p_2-2}{p_1}\right)\left(\frac{p_3-2}{p_2}\right)\cdots\left(\frac{p_{k-1}-2}{p_{k-2}}\right)\left(\frac{p_k-2}{p_{k-1}}\right)\frac{1}{p_k}.$$

Note that p_1 can not be canceled with any numerator of the fractions in parentheses, since all these are odd integers. Thus, it is easy to check that this fraction is not equal to the reduced fraction $(c_k^{L_k}/p_k)/p_k$.

In either case, the proportion of permitted k-tuples in the interval $I[1, p_k^2]$, given by $c_k^{L_k}/p_k^2$, is not equal to the proportion of permitted k-tuples in the interval $I[1, m_k]$, given by c_k/m_k. Thus, if $c_k^{L_k}/p_k^2 > c_k/m_k$, it must be $c_k^{R_k}/(m_k - p_k^2) < c_k/m_k$; and vice versa; this implies $c_k^{L_k}/p_k^2 \neq c_k^{R_k}/(m_k - p_k^2)$.

Step 4. We prove the lemma. From Steps 2 and 3, for every partial sum S_h $(1 \leq h \leq k)$ it follows that $c_h^{L_k}/p_k^2 \neq c_h^{R_k}/(m_k - p_k^2)$; multiplying by p_h we obtain $p_h c_h^{L_k}/p_k^2 \neq p_h c_h^{R_k}/(m_k - p_k^2)$; and so $\delta_h^{L_k} \neq \delta_h^{R_k}$. ∎

Even though the increase of the number of permitted h-tuples within one interval is equal to the decrease of the number of permitted h-tuples within the other interval, the increase of the h-density within one interval is not equal to the decrease of the h-density within the other interval. This is due to their being more subintervals of size p_h within $I[p_k^2+1, m_k]_h$ than within $I[1, p_k^2]_h$, for $k > 3$. The following lemma gives the relation between the h-density within $I[1, p_k^2]_h$ and the h-density within $I[p_k^2+1, m_k]_h$. First, a definition:

Definition 6.8. Let S_h be the partial sums from level $h = 1$ to level $h = k$ $(k \geq 4)$. Let $I[1, p_k^2]_h$ be the Left interval, and let $I[p_k^2+1, m_k]_h$ be the Right interval, in every partial sum S_h $(1 \leq h \leq k)$. For a given partial sum S_h $(1 \leq h \leq k)$, let $\delta_h^{L_k}$ be the density of permitted h-tuples within the Left interval $I[1, p_k^2]_h$, and let $\delta_h^{R_k}$ be the density of permitted h-tuples within the Right interval $I[p_k^2+1, m_k]_h$. We use the notation $\{\delta_h^{L_k}\}$ to denote the set of values of $\delta_h^{L_k}$, and we use the notation $\{\delta_h^{R_k}\}$ to denote the set of values of $\delta_h^{R_k}$, for all the combinations of selected remainders in the sequences that form the partial sum S_h.

Lemma 6.3. *There is a bijective function* $f_h : \{\delta_h^{L_k}\} \to \{\delta_h^{R_k}\}$ *such that*

$$f_h(x) = \delta_h - (x - \delta_h)\frac{p_k^2}{m_k - p_k^2},$$

and

$$f_h^{-1}(x) = \delta_h + (\delta_h - x)\frac{m_k - p_k^2}{p_k^2}.$$

Proof. For a given level h $(1 \leq h \leq k)$, if we change the combination of selected remainders in the partial sum S_h, some permitted h-tuples will be transferred from the Left interval $I[1, p_k^2]_h$ to the Right interval $I[p_k^2+1, m_k]_h$, or vice versa, as we have seen before. So, it is easy to see that there is a set of values of the number of permitted h-tuples within the Left interval, and a set of values of the number of permitted h-tuples within the Right interval. However, there exists a one-to-one correspondence between both sets, since the number of permitted h-tuples within the interval $I[1, m_k]_h$ is the same, whatever the combination of selected remainders in the sequences s_h that form the partial sum S_h, by Proposition 2.3 and Lemma 6.1. It follows that there is also a one-to-one correspondence between the set of values of $\delta_h^{L_k}$, and the set of values of $\delta_h^{R_k}$. So, for a given level h $(1 \leq h \leq k)$, we can define a bijective function $f_h : \{\delta_h^{L_k}\} \to \{\delta_h^{R_k}\}$.

Now, for the partial sum S_h, assume that the density of permitted h-tuples within $I[1, p_k^2]_h$, and within $I[p_k^2 + 1, m_k]_h$, is equal to the average δ_h. Then, suppose that some permitted h-tuples are transferred from the Right interval to the Left interval. We have an increase $(\delta_h^{L_k} - \delta_h)$ of the h-density within the Left interval, and a decrease $(\delta_h - \delta_h^{R_k})$ of the h-density within the Right interval. See (22). Because within the Left interval $I[1, p_k^2]_h$ we have p_k^2/p_h subintervals of size p_h, by definition, the number of permitted h-tuples entering the Left interval is equal to $(\delta_h^{L_k} - \delta_h)p_k^2/p_h$. In the same way, within the Right interval $I[p_k^2 + 1, m_k]_h$ we have $(m_k - p_k^2)/p_h$ subintervals of size p_h, and then, by definition, the number of permitted h-tuples coming out of the Right interval is equal to $(\delta_h - \delta_h^{R_k})(m_k - p_k^2)/p_h$. Since the number of permitted h-tuples entering the Left interval must be equal to the number of permitted h-tuples coming out of the Right interval,

$$\left(\delta_h^{L_k} - \delta_h\right) \frac{p_k^2}{p_h} = \left(\delta_h - \delta_h^{R_k}\right) \frac{m_k - p_k^2}{p_h} \Longrightarrow \delta_h^{R_k} = \delta_h - \left(\delta_h^{L_k} - \delta_h\right) \frac{p_k^2}{m_k - p_k^2}.$$

Therefore, we have a bijective function $f_h : \{\delta_h^{L_k}\} \to \{\delta_h^{R_k}\}$, such that $f_h(x) = \delta_h - (x - \delta_h)(p_k^2/(m_k - p_k^2))$, and it is easy to check that $f_h^{-1}(x) = \delta_h + (\delta_h - x)(m_k - p_k^2)/p_k^2$. ∎

7 The sifting function of the Sieve II

7.1 The behaviour of the h-density within the Right interval as $k \to \infty$

In this section we establish a lower bound for the minimum value of the sifting function of the Sieve II. Furthermore, we prove that the minimum value of the sifting function of the Sieve II tends to infinity as $k \to \infty$. However, before achieving these results, we need to establish a lower bound for the k-density within the interval $I[1, p_k^2]$ of the partial sum S_k. Now, for reasons that will be clear later, we begin by studying the behaviour of the h-density $(1 \leq h \leq k)$ within the Right block of the partition of the first period of S_k; the following example illustrates this behaviour for a level k not too large.

Example 7.1. Let S_k be a partial sum of the series $\sum s_k$. Suppose that we take first $k = 4$, and then we let $k \to \infty$. We can see that, as the level k increases, for every partial sum S_h from $h = 1$ to $h = k$, the size of the Right interval $I[p_k^2 + 1, m_k]_h$ grows very fast, since $p_k^2 = \mathrm{o}(m_k)$, by Lemma 2.5. Note that for $h = 1$ there is one permitted 1-tuple within every period of size $p_1 = 2$ of the partial sum S_1, by Remark 6.1. So, as the size of the Right interval $I[p_k^2 + 1, m_k]_1$ increases, the number of permitted 1-tuples grows very fast as well. Therefore, we can reach a level k large enough (although not too large) that the distribution of the permitted 1-tuples that are within $I[p_k^2 + 1, m_k]_1$ over the residue classes modulo $p_2 = 3$ is not far from uniform. So, the fraction of permitted 1-tuples within the Right interval of S_1 that are transferred to the Right interval of S_2 as permitted 2-tuples is approximately the average fraction $(p_2 - 2)/p_2$ (see Remark 6.4).

However, as h goes from level 1 to level k, the number of permitted h-tuples that are within the Right interval decreases (see Remark 6.2), and p_h increases; in addition, the number of combinations of selected remainders within the sequences that form every partial sum S_h increases as well. See (10) and (11). Therefore, as h goes from level 1 to level $k - 1$, for some combinations of selected remainders, the distribution of the permitted h-tuples that are within $I[p_k^2 + 1, m_k]_h$ over the residue classes modulo p_{h+1} becomes far from uniform. So, for these combinations of selected remainders, the fraction of permitted h-tuples within the Right interval of S_h that are transferred to the Right interval of S_{h+1} as permitted $(h + 1)$-tuples moves away from the average fraction $(p_{h+1} - 2)/p_{h+1}$, and consequently, the values of $\delta_h^{R_k}$ moves away from δ_h, as h goes from level 1 to level k.

Now, we shall consider the case where k is very large.

Remark 7.1. For every partial sum S_h from level $h = 1$ to level $h = k$ $(k \geq 4)$, let us consider the interval $I[1, m_k]_h$, and the Right interval $I[p_k^2 + 1, m_k]_h$; we denote by c_k the number of permitted k-tuples within the interval $I[1, m_k]_k$; and we denote by c'_h the number of permitted h-tuples within every interval $I[1, m_k]_h$ where $1 \leq h < k$. By Lemma 2.4 we have $p_k^2 = \mathrm{o}(c_k)$, and by Lemma 2.5 we have $p_k^2 = \mathrm{o}(m_k)$. Then, for a level k sufficiently large, most of the permitted h-tuples in every interval $I[1, m_k]_h$ $(1 \leq h \leq k)$ belong to the Right interval $I[p_k^2 + 1, m_k]_h$, and furthermore $c_k \gg p_k$.

Note that in this case, $c'_h \gg p_h$ for every $h < k$, since $p_h < p_k$ and $c'_h > c_k$ (see Proposition 2.3 and Lemma 6.1). Hence, it is easy to see that for each level from $h = 1$ to $h = k - 1$, the distribution of the permitted h-tuples that are within the Right interval over the residue classes modulo p_{h+1} will be almost uniform, whatever the combination of selected remainders in the sequences s_h that form the partial sum S_k; so, the fraction of the permitted h-tuples within the Right interval of S_h that are transferred to the Right interval of S_{h+1} as permitted $(h+1)$-tuples will be approximately the average fraction $(p_{h+1} - 2)/p_{h+1}$ (see Remark 6.4). It follows that for each level from $h = 1$ to $h = k$, the values of $\delta_h^{R_k}$ will be very close to δ_h, whatever the combination of selected remainders in the sequences s_h that form the partial sum S_k.

The following lemma shows that as $k \to \infty$, the true h-density within the Right interval $I[p_k^2 + 1, m_k]_h$ of every partial sum S_h $(1 \le h \le k)$ converges uniformly to the average δ_h.

Lemma 7.1. *Let S_k $(k \ge 4)$ be a partial sum of the series $\sum s_k$. Let us consider the Right interval $I[p_k^2+1, m_k]_h$ in every partial sum S_h, from level $h = 1$ to level $h = k$. For every $\epsilon > 0$, there exists N (depending only on ϵ) such that level $k > N$ implies $|\delta_h^{R_k} - \delta_h| < \epsilon$, for every partial sum S_h from level $h = 1$ to level $h = k$, whatever the combination of selected remainders in the sequences s_h that form every partial sum S_k.*

Proof. Step 1. The size of the Right interval $I[p_k^2 + 1, m_k]_h$ of the partial sum S_h, by definition, is equal to $m_k - p_k^2$, and so the number of subintervals of size p_h within the Right interval is equal to $(m_k - p_k^2)/p_h$ $(1 \le h \le k)$. Denoting by $c_h^{R_k}$ the number of permitted h-tuples within $I[p_k^2 + 1, m_k]_h$, by definition, we have

$$\delta_h^{R_k} = \frac{c_h^{R_k}}{(m_k - p_k^2)\,/p_h} \qquad (1 \le h \le k). \tag{23}$$

Step 2. Let us denote by m_h the period of the partial sum S_h, and by c_h the number of permitted h-tuples within a period of the partial sum S_h. For every level from $h = 1$ to $h = k$, let c'_h be the number of permitted h-tuples within the interval $I[1, m_k]_h$ of the partial sum S_h. Using Lemma 6.1, we obtain

$$\begin{aligned} c'_1 &= c_1 p_2 p_3 \cdots p_k, \\ c'_2 &= c_2 p_3 p_4 \cdots p_k, \\ &\cdots \\ c'_h &= c_h p_{h+1} p_{h+2} \cdots p_k, \\ &\cdots \\ c'_k &= c_k. \end{aligned} \tag{24}$$

Note that c'_h increases as the level decreases from $h = k$ to $h = 1$ (see Proposition 2.3). For every level from $h = 1$ to $h = k$, since $I[1, m_k]_h = I[1, p_k^2]_h \cup I[p_k^2 + 1, m_k]_h$, the number of permitted h-tuples within the Right interval $I[p_k^2 + 1, m_k]_h$ can not be greater than c'_h, and so we have $c_h^{R_k} \le c'_h$. On the other hand, the number of permitted h-tuples within the Left interval $I[1, p_k^2]_h$ of the partial sum S_h can not be greater than the size p_k^2 of the Left interval. Therefore, $c'_h - p_k^2 \le c_h^{R_k}$. Consequently, replacing the numerator in (23) by $c'_h - p_k^2$ and by c'_h, we obtain

$$\frac{c'_h - p_k^2}{(m_k - p_k^2)\,/p_h} \le \delta_h^{R_k} \le \frac{c'_h}{(m_k - p_k^2)\,/p_h} \qquad (1 \le h \le k).$$

Extracting the common factors c'_h and m_k, we obtain

$$\frac{c'_h}{m_k/p_h}\left(\frac{1 - p_k^2/c'_h}{1 - p_k^2/m_k}\right) \le \delta_h^{R_k} \le \frac{c'_h}{m_k/p_h}\left(\frac{1}{1 - p_k^2/m_k}\right).$$

Now, by definition, $m_k = p_1 p_2 p_3 \cdots p_h p_{h+1} p_{h+2} \cdots p_k = m_h p_{h+1} p_{h+2} \cdots p_k$. Then, using (24), we can simplify both sides:

$$\frac{c_h}{m_h/p_h}\left(\frac{1 - p_k^2/c'_h}{1 - p_k^2/m_k}\right) \le \delta_h^{R_k} \le \frac{c_h}{m_h/p_h}\left(\frac{1}{1 - p_k^2/m_k}\right).$$

By definition,

$$\delta_h = \frac{c_h}{m_h/p_h}.$$

Therefore, for every partial sum S_h from level $h = 1$ to level $h = k$, whatever the combination of selected remainders in the sequences s_h that form the partial sum S_k, we have the bounds

$$\delta_h \left(\frac{1 - p_k^2/c'_h}{1 - p_k^2/m_k} \right) \leq \delta_h^{R_k} \leq \delta_h \left(\frac{1}{1 - p_k^2/m_k} \right). \tag{25}$$

Step 3. Now, let $\epsilon > 0$ be a given small number, and let $N \geq 12$. For level $h = k$, from (25), we obtain

$$\delta_k \left(\frac{1 - p_k^2/c'_k}{1 - p_k^2/m_k} \right) \leq \delta_k^{R_k} \leq \delta_k \left(\frac{1}{1 - p_k^2/m_k} \right).$$

On the one hand, $p_k^2 = \mathrm{o}(m_k)$, by Lemma 2.5. On the other hand, $c_k = c'_k$ by (24), and so, by Lemma 2.4, $p_k^2 = \mathrm{o}(c'_k)$. Besides, it follows from Proposition 2.3 that $c'_k < m_k$. Therefore, we can take N large enough that for level $k > N$,

$$\delta_k - \frac{\epsilon}{2} < \delta_k \left(\frac{1 - p_k^2/c'_k}{1 - p_k^2/m_k} \right) \leq \delta_k^{R_k} \leq \delta_k \left(\frac{1}{1 - p_k^2/m_k} \right) < \delta_k + \frac{\epsilon}{2}, \tag{26}$$

at level $h = k$.

Step 4. Now, the rightmost inequality in (26) implies

$$\delta_k \left(\frac{1}{1 - p_k^2/m_k} - 1 \right) < \frac{\epsilon}{2}.$$

For a given level $h < k$, since $k > N \geq 12$ by assumption, it is easy to verify, using Lemma 3.1 and Lemma 3.2, that $\delta_h < \delta_k$. Hence,

$$\delta_h \left(\frac{1}{1 - p_k^2/m_k} - 1 \right) < \frac{\epsilon}{2} \Longrightarrow \delta_h \left(\frac{1}{1 - p_k^2/m_k} \right) < \delta_h + \frac{\epsilon}{2}. \tag{27}$$

Step 5. The leftmost inequality in (26) implies

$$\delta_k \left(1 - \frac{1 - p_k^2/c'_k}{1 - p_k^2/m_k} \right) < \frac{\epsilon}{2}.$$

For a given level $h < k$, since $k > N \geq 12$, we have $\delta_h < \delta_k$, as we have seen in Step 4. On the other hand, $c'_h > c'_k = c_k$ by Proposition 2.3. Hence,

$$\delta_h \left(1 - \frac{1 - p_k^2/c'_h}{1 - p_k^2/m_k} \right) < \frac{\epsilon}{2} \Longrightarrow \delta_h - \frac{\epsilon}{2} < \delta_h \left(\frac{1 - p_k^2/c'_h}{1 - p_k^2/m_k} \right). \tag{28}$$

Step 6. We now prove the lemma. By (25), (26), (27), and (28), for $k > N$ we can write

$$\delta_h - \frac{\epsilon}{2} < \delta_h \left(\frac{1 - p_k^2/c'_h}{1 - p_k^2/m_k} \right) \leq \delta_h^{R_k} \leq \delta_h \left(\frac{1}{1 - p_k^2/m_k} \right) < \delta_h + \frac{\epsilon}{2},$$

for every level from $h = 1$ to $h = k$. This implies $|\delta_h^{R_k} - \delta_h| < \epsilon$ for every level from $h = 1$ to $h = k$ ($k > N$), whatever the combination of selected remainders in the sequences s_h that form every partial sum S_k.

∎

7.2 One formula for the maximum number of permitted k-tuples within the Right interval

Let S_h be the partial sums from level $h=1$ to level $h=k$ ($k \geq 4$). Consider for every S_h the Left interval $I[1,p_k^2]_h$ and the Right interval $I[p_k^2+1,m_k]_h$. Recall the notation $\{\delta_h^{L_k}\}$ to denote the set of values of $\delta_h^{L_k}$, and the notation $\{\delta_h^{R_k}\}$ to denote the set of values of $\delta_h^{R_k}$, for all the combinations of selected remainders in the sequences that form the partial sum S_h ($1 \leq h \leq k$).

Definition 7.1. We use the notation $\hat{\delta}_h^{L_k}$ and $\check{\delta}_h^{L_k}$ to denote, respectively, $\min\{\delta_h^{L_k}\}$ and $\max\{\delta_h^{L_k}\}$, and the notation $\hat{\delta}_h^{R_k}$ and $\check{\delta}_h^{R_k}$ to denote, respectively, $\min\{\delta_h^{R_k}\}$ and $\max\{\delta_h^{R_k}\}$.

Remark 7.2. By Lemma 6.2, for every partial sum S_h ($1 \leq h \leq k$), we have $\delta_h^{L_k} \neq \delta_h^{R_k}$. On the other hand, for any given level h, the average density of permitted h-tuples within the Left interval $I[1,p_k^2]_h$ (the Right interval $I[p_k^2+1,m_k]_h$) is equal to δ_h, by Theorem 4.3. Then, for each level between $h=1$ and $h=k$, we have $\hat{\delta}_h^{L_k} < \delta_h < \check{\delta}_h^{L_k}$ ($\hat{\delta}_h^{R_k} < \delta_h < \check{\delta}_h^{R_k}$). See (22).

Given S_k ($k > 4$), with a given combination of selected remainders, consider for level $h=k$ the Left interval $I[1,p_k^2]_k$ and the Right interval $I[p_k^2+1,m_k]_k$. And for level $h=4$, consider also the Left interval $I[1,p_k^2]_4$ and the Right interval $I[p_k^2+1,m_k]_4$. By Lemma 6.3 there is a bijection between the values of the k-density in $I[1,p_k^2]_k$, and the values of the k-density in $I[p_k^2+1,m_k]_k$, and there is also a bijection between the values of the 4-density in $I[1,p_k^2]_4$, and the values of the 4-density in $I[p_k^2+1,m_k]_4$. However, what is the relation between the k-density within the Left interval $I[1,p_k^2]_k$, and the 4-density within the Left interval $I[1,p_k^2]_4$? And what is the relation between the k-density within the Right interval $I[p_k^2+1,m_k]_k$, and the 4-density within the Right interval $I[p_k^2+1,m_k]_4$? These are the questions in which we are interested.

By now, we know that the h-density within each interval $I[1,m_k]_h$ is equal to δ_h ($1 \leq h \leq k$). Consequently, for a sufficiently large level k, when we subdivide every interval $I[1,m_k]_h$ into a Left interval and Right interval, it seems reasonable to expect that the behaviour of $\delta_h^{L_k}$ and $\delta_h^{R_k}$, between level $h=1$ and level $h=k$, is analogous to the behaviour of δ_h. In particular, with regard to the Right block, the values of $\delta_h^{R_k}$ ($1 \leq h \leq k$) approximate the respective average δ_h more and more closely as the level k becomes large, by Lemma 7.1. Thus, it is easy to see that, from level $h=1$ to level $h=k$, the maximum values of $\delta_h^{R_k}$ within the Right interval $I[p_k^2+1,m_k]_h$, tend to be proportional to the values of δ_h, as $k \to \infty$. We can use this fact to obtain a formula for the maximum value of $\delta_k^{R_k}$. We explain this briefly as follows.

Let S_k be a given partial sum of the series $\sum s_k$, where k is very large; consider the first period of S_k partitioned as we have seen before, in a Left block and a Right block. We shall restrict our attention to the behaviour of $\check{\delta}_h^{R_k}$ (in the Right block of the partition) as h goes from 1 to k; in particular, let us consider the Right interval $I[p_k^2+1,m_k]_k$, and the Right interval $I[p_k^2+1,m_k]_4$. Assume that for the levels $h=4$ and $h=k$, the values of $\check{\delta}_h^{R_k}$ were exactly proportional to the values of δ_h. In this case, we could write

$$\check{\delta}_k^{R_k} = \delta_k + \frac{\delta_k}{\delta_4}\left(\check{\delta}_4^{R_k} - \delta_4\right). \tag{29}$$

On the other hand, as noted above, the values of $\check{\delta}_h^{R_k}$ become approximately proportional to the values of δ_h ($1 \leq h \leq k$), as $k \to \infty$. So, without the preceding assumption, for k large enough we can write

$$\check{\delta}_k^{R_k} = \delta_k + \beta\frac{\delta_k}{\delta_4}\left(\check{\delta}_4^{R_k} - \delta_4\right), \tag{30}$$

for some number $\beta > 1$. We shall derive this crucial formula later (Lemma 7.7), and we shall prove that $\beta \to 1$ as $k \to \infty$ (Lemma 7.6).

The following result is a corollary of Lemma 7.1. First, we make a definition.

Definition 7.2. For every level from $h=1$ to $h=k$, we define the coefficient $\check{\xi}_h^{R_k} > 1$, such that $\check{\delta}_h^{R_k} = \check{\xi}_h^{R_k}\delta_h$.

Lemma 7.2. *Let S_k ($k \geq 4$) be a partial sum of the series $\sum s_k$. For every $\varepsilon > 0$, there exists N (depending only on ε) such that level $k > N$ implies $1 < \check{\xi}_h^{R_k} < 1+\varepsilon$, for every partial sum S_h from level $h=1$ to level $h=k$, whatever the combination of selected remainders in the sequences s_h that form the partial sum S_k.*

Proof. By Lemma 7.1, for every $\varepsilon' > 0$, there exists an N (depending only on ε') such that if the level $k > N$, then $\delta_h - \varepsilon' < \delta_h^{R_k} < \delta_h + \varepsilon'$ for every partial sum S_h from level $h=1$ to level $h=k$, whatever the combination of selected remainders in the sequences s_h that form the partial sum S_k. Therefore, we can write $\delta_h < \check{\delta}_h^{R_k} < \delta_h + \varepsilon'$ ($1 \leq h \leq k$). Then, dividing by δ_h, we obtain

$$1 < \frac{\check{\delta}_h^{R_k}}{\delta_h} = \check{\xi}_h^{R_k} < 1 + \frac{\varepsilon'}{\delta_h} \qquad (1 \le h \le k).$$

Now, using Lemma 3.1, Lemma 3.2, and Corollary 3.3, it is easy to check that the smallest element of δ_h $(1 \le h \le k)$ is δ_2. Hence,

$$1 < \check{\xi}_h^{R_k} < 1 + \frac{\varepsilon'}{\delta_h} \le 1 + \frac{\varepsilon'}{\delta_2} \qquad (1 \le h \le k). \tag{31}$$

Therefore, for every $\varepsilon > 0$, we can take $\varepsilon' = \varepsilon\delta_2$, find the number N, and then, substituting this into (31), we obtain $1 < \check{\xi}_h^{R_k} < 1 + \varepsilon$ $(1 \le h \le k)$.

∎

For a level $k \ge 4$, let us consider again the Right interval $I[p_k^2+1, m_k]_h$, of the partial sums S_h, from level $h = 1$ to level $h = k$. Let $h = i$ and $h = j$ be two given levels $(1 \le i < j \le k)$. The following lemma shows that $\check{\delta}_j^{R_k}$ is greater than the value $\check{\delta}_i^{R_k}\delta_j/\delta_i$, computed by assuming $\check{\delta}_h^{R_k} \propto \delta_h$, between level $h = 1$ and level $h = k$.

Lemma 7.3. *Let S_h be the partial sums from level $h = 1$ to level $h = k$ $(k \ge 4)$. Let us consider the Right interval in every partial sum S_h $(1 \le h \le k)$. Let i, j be two levels such that $1 \le i < j \le k$. Then*

$$\delta_j < \check{\delta}_i^{R_k}\frac{\delta_j}{\delta_i} < \check{\delta}_j^{R_k}. \tag{32}$$

In other words, the value of $\check{\delta}_j^{R_k}$ exceeds the value calculated by assuming $\check{\delta}_h^{R_k} \propto \delta_h$, between level $h = 1$ and level $h = k$.

Proof. By Remark 7.2, for each level between $h = 1$ and $h = k$ we have $\hat{\delta}_h^{R_k} < \delta_h < \check{\delta}_h^{R_k}$. Now, for level $h = i$, there exists one combination of selected remainders in the sequences s_h that form the partial sum S_i, such that the density of permitted i-tuples within the Right interval $I[p_k^2+1, m_k]_i$ is equal to the maximum value $\check{\delta}_i^{R_k}$. Since by definition the size of the Right interval $I[p_k^2+1, m_k]_i$ of the partial sum S_i is equal to $m_k - p_k^2$, the number of subintervals of size p_i within this interval is $(m_k - p_k^2)/p_i$. Consequently, for this particular combination of selected remainders, the number of permitted i-tuples within the Right interval $I[p_k^2+1, m_k]_i$ is equal to $\check{\delta}_i^{R_k}(m_k - p_k^2)/p_i$. Therefore, using Lemma 4.2 at each level transition $h \to h+1$, up to level $h = j$, we obtain

$$\begin{aligned}\left(\check{\delta}_i^{R_k}\left(\frac{m_k - p_k^2}{p_i}\right)\right)\left(\frac{p_{i+1}-2}{p_{i+1}}\right)\left(\frac{p_{i+2}-2}{p_{i+2}}\right)\cdots\left(\frac{p_j-2}{p_j}\right) &= \\ = \check{\delta}_i^{R_k}\left(\frac{p_{i+1}-2}{p_i}\right)\left(\frac{p_{i+2}-2}{p_{i+1}}\right)\cdots\left(\frac{p_j-2}{p_{j-1}}\right)\left(\frac{m_k - p_k^2}{p_j}\right)&,\end{aligned}$$

which is the average number of permitted j-tuples within the Right interval $I[p_k^2+1, m_k]_j$, for all the combinations of selected remainders in the sequences s_h from level $i+1$ to level j, starting with the combination corresponding to the maximum value $\check{\delta}_i^{R_k}$. Now, dividing by $(m_k - p_k^2)/p_j$ (the number of subintervals of size p_j within the Right interval $I[p_k^2+1, m_k]_j$, for level $h = j$), we get

$$\check{\delta}_i^{R_k}\left(\frac{p_{i+1}-2}{p_i}\right)\left(\frac{p_{i+2}-2}{p_{i+1}}\right)\cdots\left(\frac{p_j-2}{p_{j-1}}\right),$$

which is the corresponding average j-density. We can write this expression as

$$\check{\delta}_i^{R_k}\left(\frac{p_{i+1}-2}{p_i}\right)\left(\frac{p_{i+2}-2}{p_{i+1}}\right)\cdots\left(\frac{p_j-2}{p_{j-1}}\right) = \check{\delta}_i^{R_k}\frac{\delta_i\left(\frac{p_{i+1}-2}{p_i}\right)\left(\frac{p_{i+2}-2}{p_{i+1}}\right)\cdots\left(\frac{p_j-2}{p_{j-1}}\right)}{\delta_i},$$

and, using Lemma 3.2, it is easy to see that

$$\check{\delta}_i^{R_k}\left(\frac{p_{i+1}-2}{p_i}\right)\left(\frac{p_{i+2}-2}{p_{i+1}}\right)\cdots\left(\frac{p_j-2}{p_{j-1}}\right) = \check{\delta}_i^{R_k}\frac{\delta_j}{\delta_i}.$$

Therefore, we can see that $\check{\delta}_i^{R_k}\delta_j/\delta_i$ is the average density of permitted j-tuples within the Right interval $I[p_k^2 + 1, m_k]_j$, for all the combinations of selected remainders in the sequences s_h from level $h = i+1$ to level $h = j$ such that the combination of selected remainders in the sequences s_h from level $h = 1$ to level $h = i$ is the one corresponding to the maximum value $\check{\delta}_i^{R_k}$. That is, $\check{\delta}_i^{R_k}\delta_j/\delta_i$ is an average, not a maximum value. Consequently, it is easy to see that $\check{\delta}_j^{R_k}$ must be greater than $\check{\delta}_i^{R_k}\delta_j/\delta_i$, and then we can write

$$\delta_j < \check{\delta}_i^{R_k}\frac{\delta_j}{\delta_i} < \check{\delta}_j^{R_k}.$$

The lemma is proved.

∎

Corollary 7.4. *We have* $\check{\xi}_i^{R_k} < \check{\xi}_j^{R_k}$ $(1 \le i < j \le k)$.

Proof. By definition, $\check{\delta}_i^{\,R_k} = \check{\xi}_i^{R_k}\delta_i$, and $\check{\delta}_j^{\,R_k} = \check{\xi}_j^{R_k}\delta_j$. So, replacing $\check{\delta}_i^{\,R_k}$ by $\check{\xi}_i^{R_k}\delta_i$ and $\check{\delta}_j^{\,R_k}$ by $\check{\xi}_j^{R_k}\delta_j$ in (32), and then simplifying, we obtain $\check{\xi}_i^{R_k} < \check{\xi}_j^{R_k}$.

∎

Now we define the coefficient $\beta_4^{R_k}$, that we shall use later in the formula for $\check{\delta}_k^{R_k}$.

Definition 7.3. For $k > 4$, let S_h be the partial sums from level $h = 1$ to level $h = k$. We define $\beta_4^{R_k}$ by the equation

$$\beta_4^{R_k} = \frac{\left(\check{\xi}_k^{R_k} - 1\right)}{\left(\check{\xi}_4^{R_k} - 1\right)}.$$

Lemma 7.5. *For $k > 4$, let S_h be the partial sums from level $h = 1$ to level $h = k$. For every $k > 4$, we have* $\beta_4^{R_k} > 1$.

Proof. Given a level $k > 4$, by definition, $(\check{\xi}_k^{R_k} - 1) > 0$ and $(\check{\xi}_4^{R_k} - 1) > 0$; besides, by Corollary 7.4, we have $\check{\xi}_k^{R_k} > \check{\xi}_4^{R_k}$. Then

$$\beta_4^{R_k} = \frac{\left(\check{\xi}_k^{R_k} - 1\right)}{\left(\check{\xi}_4^{R_k} - 1\right)} > 1.$$

The lemma is proved.

∎

Now, for the levels 4 and k, the number $\beta_4^{R_k}$ measures the degree of departure of the values of $\check{\delta}_h^{R_k}$ from being exactly proportional to the respective values of δ_h $(1 \le h \le k)$; we explain this as follows.

Let S_h be the partial sums from level $h = 1$ to level $h = k$ $(k > 4)$; for every level h let us consider the Right sinterval $I[p_k^2 + 1, m_k]_h$. Recall that δ_h denotes the density of permitted h-tuples within the period of the partial sum S_h; furthermore, δ_h is the average density of permitted h-tuples within the Right interval $I[p_k^2 + 1, m_k]_h$, by Theorem 4.3.

Remark 7.3. For a level k very large, we know by Lemma 7.1 that $\hat{\delta}_h^{R_k}$ and $\check{\delta}_h^{R_k}$ will be very close to δ_h, for every partial sum S_h from level $h = 1$ to level $h = k$. Consequently, as we have seen before, for a level k sufficiently large, the behaviour of $\check{\delta}_h^{R_k}$ will be approximately proportional to δ_h, between level $h = 1$ and level $h = k$; and this approximation improves as $k \to \infty$, by the same lemma. In the limit as $k \to \infty$, it will be $\hat{\delta}_h^{R_k} = \check{\delta}_h^{R_k} = \delta_h$ for every level h; so, we can say that in the limit $\check{\delta}_h^{R_k}$ is 'exactly proportional' to δ_h for every level h, where the constant of proportionality is 1.

Now, given h, the quotient $\check{\delta}_h^{R_k}/\delta_h$ is the proportion of the maximum density of permitted h-tuples to the average h-density, within the Right interval. Suppose that we take the fixed level $h = 4$; assume that the quotient $\check{\delta}_k^{R_k}/\delta_k$ is equal to the quotient $\check{\delta}_4^{R_k}/\delta_4$; that is, we are assuming that for the levels 4 and k, the values of $\check{\delta}_h^{R_k}$ are proportional to the respective values of δ_h. This means that given the value of $\check{\delta}_4^{R_k}$ for level $h = 4$, we could compute the value of $\check{\delta}_k^{R_k}$ for level $h = k$, since using Lemma 3.1 we can compute δ_h for every level h. This is precisely what is expressed in (29). Now, by Lemma 7.3 we know that this assumption is not true; the only situation where this assumption is true is in the limit as $k \to \infty$, as was explained in Remark 7.3. However, for a level k large enough, the preceding assumption of proportionality becomes approximately true, as we have noted in the same remark.

Recall that we denote by $\check{\xi}_k^{R_k}$ the quotient $\check{\delta}_k^{R_k}/\delta_k$, and we denote by $\check{\xi}_4^{R_k}$ the quotient $\check{\delta}_4^{R_k}/\delta_4$. By definition, $\check{\xi}_k^{R_k} > 1$ and $\check{\xi}_4^{R_k} > 1$; furthermore, $\check{\xi}_k^{R_k} \to 1$ and $\check{\xi}_4^{R_k} \to 1$ as $k \to \infty$, by Lemma 7.2. Hence, the difference $\check{\xi}_k^{R_k} - 1$ measures how far $\check{\delta}_k^{R_k}$ 'deviates' from δ_k, at level $h = k$; and the difference $\check{\xi}_4^{R_k} - 1$ measures how far $\check{\delta}_4^{R_k}$ 'deviates' from δ_4, at level $h = 4$. So, the quotient

$$\frac{\left(\check{\xi}_k^{R_k} - 1\right)}{\left(\check{\xi}_4^{R_k} - 1\right)}, \tag{33}$$

denoted by $\beta_4^{R_k}$, measures how far the values of $\check{\delta}_h^{R_k}$, for the levels 4 and k, 'deviates' from being exactly proportional to the respective values of δ_h. Note that the behaviour of the quotient in (33) as $k \to \infty$ depends on the rates at which $\check{\xi}_k^{R_k}$ and $\check{\xi}_4^{R_k}$ tend to 1.

The next lemma shows that $\beta_4^{R_k}$ tends to 1 as $k \to \infty$.

Lemma 7.6. *For $k > 4$, let S_h be the partial sums from level $h = 1$ to level $h = k$. We have $\beta_4^{R_k} \to 1$, as $k \to \infty$.*

Proof. Step 1. By Lemma 7.1, for each level from $h = 1$ to $h = k$, the density of permitted h-tuples in the Right interval $I[p_k^2+1, m_k]_h$ tends uniformly to the average δ_h, as $k \to \infty$ (see Remark 7.1). It follows that $\check{\delta}_h^{R_k} \to \delta_h$, as $k \to \infty$; and so $\check{\xi}_h^{R_k} \to 1$ as $k \to \infty$ also uniformly (see Lemma 7.2).

Now, in the limit as $k \to \infty$, we have $\check{\delta}_h^{R_k} = \delta_h$, simultaneously for every level between $h = 1$ and $h = k$, by Lemma 7.1; and $\check{\xi}_h^{R_k} = 1$, by Lemma 7.2, also simultaneously for every level between $h = 1$ and $h = k$. Note that $\check{\delta}_h^{R_k}$ can never attain the limit δ_h as $k \to \infty$ (see Remark 7.2).

Step 2. By Lemma 7.5,

$$\beta_4^{R_k} = \frac{\left(\check{\xi}_k^{R_k} - 1\right)}{\left(\check{\xi}_4^{R_k} - 1\right)} > 1. \tag{34}$$

We shall see what happens to the difference quotient in (34) as we let $k \to \infty$. Note that we are comparing the difference $\check{\xi}_4^{R_k} - 1$, at the fixed level $h = 4$, with the difference $\check{\xi}_k^{R_k} - 1$, at level $h = k$, where $k \to \infty$.

By Step 1, in the limit as $k \to \infty$, both $\check{\xi}_k^{R_k}$ and $\check{\xi}_4^{R_k}$ must be equal to 1 simultaneously. Therefore, since $\check{\xi}_4^{R_k} < \check{\xi}_k^{R_k}$ by Corollary 7.4, it must exist k sufficiently large that from this level, letting $k \to \infty$, the rate at which $\check{\xi}_k^{R_k}$ tends to 1 is greater than the rate at which $\check{\xi}_4^{R_k}$ tends to 1, so that they can be 1 in the limit at the same time. We conclude that the difference quotient in (34) tends to 1 from the right, as $k \to \infty$. In symbols,

$$\lim_{k\to\infty} \beta_4^{R_k} = \lim_{k\to\infty} \frac{\left(\check{\xi}_k^{R_k} - 1\right)}{\left(\check{\xi}_4^{R_k} - 1\right)} = 1,$$

and the lemma is proved. ∎

Using simple algebra, the following lemma gives us a formula for the maximum value of $\check{\delta}_k^{R_k}$.

Lemma 7.7. *Let S_h be the partial sums from level $h = 1$ to level $h = k$ ($k \geq 4$). For the level $h = 4$,*

$$\check{\delta}_k^{R_k} = \delta_k + \beta_4^{R_k} \frac{\delta_k}{\delta_4} \left(\check{\delta}_4^{R_k} - \delta_4\right).$$

Proof. By definition, we have $\check{\delta}_4^{R_k} = \check{\xi}_4^{R_k} \delta_4$. Therefore, we can write

$$\frac{\check{\delta}_k^{R_k}}{\check{\delta}_4^{R_k}} = \frac{\check{\xi}_k^{R_k}}{\check{\xi}_4^{R_k}} \frac{\delta_k}{\delta_4}.$$

Then

$$\check{\delta}_k^{R_k} = \frac{\check{\xi}_k^{R_k}}{\check{\xi}_4^{R_k}} \frac{\delta_k}{\delta_4} \check{\delta}_4^{R_k} = \delta_k + \frac{\check{\xi}_k^{R_k}}{\check{\xi}_4^{R_k}} \frac{\delta_k}{\delta_4} \check{\delta}_4^{R_k} - \delta_k = \delta_k + \frac{\check{\xi}_k^{R_k}}{\check{\xi}_4^{R_k}} \frac{\delta_k}{\delta_4} \check{\delta}_4^{R_k} - \frac{\delta_4}{\delta_4} \delta_k =$$

$$= \delta_k + \frac{\delta_k}{\delta_4} \left(\frac{\check{\xi}_k^{R_k}}{\check{\xi}_4^{R_k}} \check{\delta}_4^{R_k} - \delta_4 \right) = \delta_k + \frac{\delta_k}{\delta_4} \frac{\left(\frac{\check{\xi}_k^{R_k}}{\check{\xi}_4^{R_k}} \check{\delta}_4^{R_k} - \delta_4 \right)}{\left(\check{\delta}_4^{R_k} - \delta_4 \right)} \left(\check{\delta}_4^{R_k} - \delta_4 \right),$$

and, by definition,

$$\check{\delta}_k^{R_k} = \delta_k + \frac{\delta_k}{\delta_4} \frac{\left(\frac{\check{\xi}_k^{R_k}}{\check{\xi}_4^{R_k}} \check{\xi}_4^{R_k} \delta_4 - \delta_4 \right)}{\left(\check{\xi}_4^{R_k} \delta_4 - \delta_4 \right)} \left(\check{\delta}_4^{R_k} - \delta_4 \right).$$

Now, cancelling the common factors, we obtain

$$\check{\delta}_k^{R_k} = \delta_k + \frac{\delta_k}{\delta_4} \frac{\left(\check{\xi}_k^{R_k} - 1 \right)}{\left(\check{\xi}_4^{R_k} - 1 \right)} \left(\check{\delta}_4^{R_k} - \delta_4 \right).$$

Finally, by definition,

$$\check{\delta}_k^{R_k} = \delta_k + \beta_4^{R_k} \frac{\delta_k}{\delta_4} \left(\check{\delta}_4^{R_k} - \delta_4 \right).$$

∎

Note that we have derived the formula in (30), where the symbol β is an alternative notation for the coefficient $\beta_4^{R_k}$ in the formula of Lemma 7.7.

7.3 A lower bound for the sifting function of the Sieve II

We recall that the first period of the partial sum S_k ($k \geq 4$), in horizontal position, can be seen as a matrix, with m_k columns and k rows; and we recall also that this matrix was partitioned into two blocks: the Left block formed by the columns from $n = 1$ to $n = p_k^2$; and the Right block formed by the columns from $n = p_k^2 + 1$ to $n = m_k$. The preceding lemma gives us a formula for $\check{\delta}_k^{R_k}$ within the Right block of the partition. However, we need a similar formula for $\hat{\delta}_k^{L_k}$ within the Left block of the partition; in order to derive this formula, we proceed as we have explained in the Introduction. Suppose that for the level 4 we know the minimum value of the density of permitted 4-tuples $\hat{\delta}_4^{L_k}$, within the Left interval $I[1, p_k^2]_4$ (Left block of the partition). Using the function f_4 of Lemma 6.3 we can compute, for the level 4, the maximum value of the density of permitted 4-tuples $\check{\delta}_4^{R_k}$, within the Right interval $I[p_k^2 + 1, m_k]_4$ (Right block of the partition). Then, using the formula given by the preceding lemma, we can compute, for the level k, the maximum value of the density of permitted k-tuples $\check{\delta}_k^{R_k}$, within the Right interval $I[p_k^2 + 1, m_k]_k$ (Right block of the partition). Finally, using the function f_k^{-1} of Lemma 6.3 we can compute, for the level k, the minimum value of the density of permitted k-tuples $\hat{\delta}_k^{L_k}$, within the Left interval $I[1, p_k^2]_k$ (Left block of the partition). This formula is given in the following lemma.

Remark 7.4. Note that for a given level h ($1 \leq h \leq k$), the image of $\hat{\delta}_h^{L_k}$ under the function f_h of Lemma 6.3 is $\check{\delta}_h^{R_k}$, and the image of $\check{\delta}_h^{L_k}$ under f_h is $\hat{\delta}_h^{R_k}$. See (22).

Lemma 7.8. *Let S_k ($k \geq 4$) be a partial sum of the series $\sum s_k$. Within the first period of S_k, consider the Left interval $I[1, p_k^2]_h$ and the Right interval $I[p_k^2 + 1, m_k]_h$ of every partial sum S_h from $h = 1$ to $h = k$ ($k \geq 4$). We have*

$$\hat{\delta}_k^{L_k} = \delta_k - \beta_4^{R_k} \frac{\delta_k}{\delta_4} \left(\delta_4 - \hat{\delta}_4^{L_k} \right).$$

Proof. Step 1. We compute the value of f_4 (see Lemma 6.3) at $x = \hat{\delta}_4^{L_k}$ (the minimum density of permitted 4-tuples within the Left interval $I[1, p_k^2]_4$). We obtain

$$\check{\delta}_4^{R_k} = f_4\left(\hat{\delta}_4^{L_k} \right) = \delta_4 - \left(\hat{\delta}_4^{L_k} - \delta_4 \right) \frac{p_k^2}{m_k - p_k^2}.$$

See Remark 7.4.

Step 2. Next, we take the maximum density of permitted 4-tuples within the Right interval $I[p_k^2+1, m_k]_4$ obtained in the previous step, and using the formula from Lemma 7.7, we get

$$\check{\delta}_k^{R_k} = \delta_k + \beta_4^{R_k}\frac{\delta_k}{\delta_4}\left(\check{\delta}_4^{R_k} - \delta_4\right) = \delta_k + \beta_4^{R_k}\frac{\delta_k}{\delta_4}\left(\left(\delta_4 - \left(\hat{\delta}_4^{L_k} - \delta_4\right)\frac{p_k^2}{m_k - p_k^2}\right) - \delta_4\right).$$

Step 3. Finally, we compute the value of f_k^{-1} (see Lemma 6.3) at $x = \check{\delta}_k^{R_k}$ (the maximum density of permitted k-tuples within the Right interval $I[p_k^2+1, m_k]_k$, obtained in the preceding step). We obtain

$$\begin{aligned}
\hat{\delta}_k^{L_k} = f_k^{-1}\left(\check{\delta}_k^{R_k}\right) &= \delta_k + \left(\delta_k - \check{\delta}_k^{R_k}\right)\frac{m_k - p_k^2}{p_k^2} = \\
&= \delta_k + \left(\delta_k - \left(\delta_k + \beta_4^{R_k}\frac{\delta_k}{\delta_4}\left(\left(\delta_4 - \left(\hat{\delta}_4^{L_k} - \delta_4\right)\frac{p_k^2}{m_k - p_k^2}\right) - \delta_4\right)\right)\right)\frac{m_k - p_k^2}{p_k^2} = \\
&= \delta_k + \left(\delta_k - \left(\delta_k + \beta_4^{R_k}\frac{\delta_k}{\delta_4}\left(-\left(\hat{\delta}_4^{L_k} - \delta_4\right)\frac{p_k^2}{m_k - p_k^2}\right)\right)\right)\frac{m_k - p_k^2}{p_k^2} = \\
&= \delta_k + \left(-\beta_4^{R_k}\frac{\delta_k}{\delta_4}\left(-\left(\hat{\delta}_4^{L_k} - \delta_4\right)\frac{p_k^2}{m_k - p_k^2}\right)\right)\frac{m_k - p_k^2}{p_k^2} = \\
&= \delta_k - \beta_4^{R_k}\frac{\delta_k}{\delta_4}\left(\delta_4 - \hat{\delta}_4^{L_k}\right).
\end{aligned}$$

See Remark 7.4.

∎

The following lemma proves that as $k \to \infty$, the minimum value of the density of permitted k-tuples within the Left interval $I[1, p_k^2]$ of the partial sum S_k tends asymptotically to the average δ_k.

Lemma 7.9. *Let S_k ($k \geq 4$) be a partial sum of the series $\sum s_k$. As $k \to \infty$, we have $\hat{\delta}_k^{L_k} \sim \delta_k$.*

Proof. Let S_h be the partial sums from level $h = 1$ to level $h = k$.

Step 1. Let us pay special attention to the Left interval $I[1, p_k^2]_4$ of the partial sum S_4, as the level k increases. That is, we shall keep the level h fixed at 4, and see what happens to the true density of permitted 4-tuples within $I[1, p_k^2]_4$, denoted by $\delta_4^{L_k}$, as the level k increases. First of all, we observe that as k increases, the size of $I[1, p_k^2]_4$ increases as well. Consequently, by Proposition 5.3, the density $\delta_4^{L_k}$ converges to δ_4 as $k \to \infty$, whatever the combination of selected remainders in the sequences s_h that form the partial sum S_4. It follows that as $k \to \infty$, the minimum value of $\delta_4^{L_k}$ converge to the average δ_4 as well. Consequently, for this level $h = 4$,

$$\lim_{k\to\infty}\left(\delta_4 - \hat{\delta}_4^{L_k}\right) = 0. \tag{35}$$

Step 2. Now, we can prove this lemma. By Lemma 7.8,

$$\hat{\delta}_k^{L_k} = \delta_k - \beta_4^{R_k}\frac{\delta_k}{\delta_4}\left(\delta_4 - \hat{\delta}_4^{L_k}\right).$$

Dividing by δ_k, and taking limits as $k \to \infty$, we obtain

$$\lim_{k\to\infty}\frac{\hat{\delta}_k^{L_k}}{\delta_k} = \lim_{k\to\infty}\left(1 - \beta_4^{R_k}\frac{1}{\delta_4}\left(\delta_4 - \hat{\delta}_4^{L_k}\right)\right).$$

By Lemma 7.6, as $k \to \infty$ we have $\beta_4^{R_k} \to 1$; then, from this result and by Step 1, we can compute the limit, and finally we obtain

$$\lim_{k\to\infty}\frac{\hat{\delta}_k^{L_k}}{\delta_k} = 1.$$

■

Now we are ready to establish a lower bound for the true density of permitted k-tuples within the Left interval $I[1,p_k^2]$ of the partial sum S_k, denoted by $\delta_k^{L_k}$. By Theorem 4.3, the average density of permitted k-tuples within $I[1,p_k^2]$ is equal to δ_k, that is to say, is equal to the k-density within the period of S_k. By Lemma 3.2 and Corollary 3.3, the density δ_k increases at each level transition $p_k \to p_{k+1}$ of order greater than 2, and this implies that for level $k > 4$ we have $\delta_k > \delta_4$. However, what happens to the true density $\delta_k^{L_k}$? By Lemma 7.9, as $k \to \infty$, we have $\hat{\delta}_k^{L_k} \sim \delta_k$; hence $\hat{\delta}_k^{L_k} \to \infty$, by Theorem 3.4. It follows that there exists $K \in \mathbb{Z}_+$ such that $\delta_k^{L_k} > \delta_4$ for every level $k > K$, whatever the combination of selected remainders in the sequences s_h that form every partial sum S_k.

Lemma 7.10. *Let $\delta_k^{L_k}$ be the true density of permitted k-tuples within the Left interval $I[1,p_k^2]$. For $k \geq 35$ ($p_k \geq 149$), we have $\delta_k^{L_k} > \delta_4$, whatever the combination of selected remainders in the sequences s_h that form every partial sum S_k.*

Remark 7.5. Note that between level $h = 4$ and level $h = 35$, there are 22 level transitions of order greater than 2; by Corollary 3.3, the average density δ_h increases at each level transition of order greater than 2.

Proof. Step 1. We begin by finding bounds for the density of permitted 4-tuples within the Left interval $I[1,p_k^2]_4$, for level $k \geq 35$. Let $\epsilon = 0.005$; we compute δ_4 using Lemma 3.1. For level $k \geq 35$ ($p_k \geq p_{35}$) we have $p_k^2 \geq p_{35}^2 = 22201$, and consequently, for the Left interval $I[1,p_k^2]_4$, using Lemma 5.2 and Remark 5.2, it is easy to check that

$$\delta_4 - \epsilon < \delta_4^{L_k} < \delta_4 + \epsilon \qquad (k \geq 35). \tag{36}$$

Step 2. By Lemma 2.5, we can take k large enough that the size of the Left interval $I[1,p_k^2]$ is negligible compared to the size of the interval $I[1,m_k]$. On the other hand, $\beta_4^{R_k} \to 1$ as $k \to \infty$, by Lemma 7.6. From the proof of Lemma 7.6, it is easy to see that this result follows from the fact that as $k \to \infty$, the size of $I[p_k^2+1,m_k]$ becomes larger and larger than the size of $I[1,p_k^2]$. That is, the degree of closeness of $\beta_4^{R_k}$ to 1 depends on the degree of closeness of the size of $I[p_k^2+1,m_k]$ to the size of $I[1,m_k]$.

Now, for a partial sum S_k where $k \geq 35$, the ratio of the size of $I[1,p_k^2]$ to the size of the interval $I[1,m_k]$ is less than 1.5×10^{-53}. Clearly, the size of the Left interval $I[1,p_k^2]$ is negligible compared to the size of the interval $I[1,m_k]$. Consequently, $\beta_4^{R_k}$ must be very close to 1 for a level $k \geq 35$.

Step 3. By definition,

$$\hat{\delta}_k^{L_k} \leq \delta_k^{L_k}. \tag{37}$$

On the other hand, by Lemma 7.8, we have the formula

$$\hat{\delta}_k^{L_k} = \delta_k - \beta_4^{R_k}\frac{\delta_k}{\delta_4}\left(\delta_4 - \hat{\delta}_4^{L_k}\right).$$

Therefore, substituting this into (37), we obtain

$$\delta_k - \beta_4^{R_k}\frac{\delta_k}{\delta_4}\left(\delta_4 - \hat{\delta}_4^{L_k}\right) \leq \delta_k^{L_k}. \tag{38}$$

From (36), we have $(\delta_4 - \hat{\delta}_4^{L_k}) < \epsilon$ if $k \geq 35$; then, substituting this into (38),

$$\delta_k - \beta_4^{R_k}\frac{\delta_k}{\delta_4}\epsilon \leq \delta_k^{L_k} \qquad (k \geq 35),$$

and assuming $\beta_4^{R_k} = 1$ we obtain

$$\delta_k - \frac{\delta_k}{\delta_4}\epsilon \leq \delta_k^{L_k} \qquad (k \geq 35).$$

Hence, it is easy to check that, with the preceding assumption, $\delta_k^{L_k} \geq \delta_k - \delta_k/\delta_4\epsilon > \delta_4$, for $k \geq 35$.

Step 4. We prove that indeed $\hat{\delta}_k^{L_k} > \delta_4$ for $k \geq 35$. Suppose that someone asked the following question: for a level $k \geq 35$, could $\hat{\delta}_k^{L_k} \leq \delta_4$, for a particular combination of selected remainders within the sequences s_h that form the partial sum S_k? In this case,

$$\hat{\delta}_k^{L_k} = \delta_k - \beta_4^{R_k} \frac{\delta_k}{\delta_4} \left(\delta_4 - \hat{\delta}_4^{L_k} \right) \leq \delta_4. \tag{39}$$

Since by Step 1 we have $(\delta_4 - \hat{\delta}_4^{L_k}) < \epsilon$, for $k \geq 35$, it is simple to check that (39) implies $\beta_4^{R_k} \geq 79.2$ ($k \geq 35$), and this contradicts Step 2. Therefore, the answer to this question is 'no', and we conclude that for every level $k \geq 35$ we have $\hat{\delta}_k^{L_k} > \delta_4$, no matter the combination of selected remainders in the sequences s_h that form the partial sum S_k.

∎

Definition 7.4. Let S_k be the partial sum associated to the Sieve II; recall that in Section 2 we have taken $\mathscr{B} = \{n : 1 \leq n \leq p_k^2\}$; let $T(\mathscr{B}, \mathscr{P}, p_k)$ be the sifting function of the Sieve II. We denote by $\{T(\mathscr{B}, \mathscr{P}, p_k)\}$ the set of the values of $T(\mathscr{B}, \mathscr{P}, p_k)$ for all the combinations of selected remainders in the sequences that form the partial sum S_k.

Now, we can obtain a lower bound for the sifting function of the Sieve II (that is, a lower bound for the number of permitted k-tuples within the Left interval $I[1, p_k^2]$ of S_k).

Lemma 7.11. *For level $k \geq 35$ ($p_k \geq 149$), we have* $\min\{T(\mathscr{B}, \mathscr{P}, p_k)\} > p_k/2$.

Remark 7.6. Consider the partial sum S_k of the series $\sum s_k$. Recall the notation $\{\delta_k^{L_k}\}$ to denote the set of values of $\delta_k^{L_k}$, for all the combinations of selected remainders in the sequences that form the partial sum S_k; and recall the notation $\hat{\delta}_k^{L_k}$ to denote $\min\{\delta_k^{L_k}\}$. Note that within the Left interval $I[1, p_k^2]$ of the partial sum S_k we have p_k subintervals of size p_k. So, the minimum number of permitted k-tuples within the Left interval $I[1, p_k^2]$ of the partial sum S_k is $p_k \hat{\delta}_k^{L_k}$. Then, by definition, $\min\{T(\mathscr{B}, \mathscr{P}, p_k)\} = p_k \hat{\delta}_k^{L_k}$.

Proof. By the preceding remark, $\min\{T(\mathscr{B}, \mathscr{P}, p_k)\} = p_k \hat{\delta}_k^{L_k}$. Now, by Lemma 7.10, if $k \geq 35$ then $\hat{\delta}_k^{L_k} > \delta_4$. It follows that $\min\{T(\mathscr{B}, \mathscr{P}, p_k)\} > p_k \delta_4$, whenever $k \geq 35$. Using Lemma 3.1, it is easy to check that $\delta_4 = 1/2$, and so $\min\{T(\mathscr{B}, \mathscr{P}, p_k)\} > p_k/2$ if $k \geq 35$.

∎

The next result can be obtained as an easy consequence of the preceding lemma, or it can be obtained from Lemma 7.9. We shall take the second way.

Lemma 7.12. *As $k \to \infty$, we have* $\min\{T(\mathscr{B}, \mathscr{P}, p_k)\} \to \infty$.

Proof. By Lemma 7.9, as $k \to \infty$ we have $\hat{\delta}_k^{L_k} \sim \delta_k$, and by Theorem 3.4 we have $\delta_k \to \infty$; it follows that $\hat{\delta}_k^{L_k} \to \infty$ as $k \to \infty$. On the other hand, by Remark 7.6, we have $\min\{T(\mathscr{B}, \mathscr{P}, p_k)\} = p_k \hat{\delta}_k^{L_k}$. Since, as $k \to \infty$, both $p_k \to \infty$ and $\hat{\delta}_k^{L_k} \to \infty$, it follows that $\min\{T(\mathscr{B}, \mathscr{P}, p_k)\} \to \infty$ as well.

∎

8 Proof of the Main Theorem

In this section we prove the Main Theorem. We begin by defining the sequence of k-tuples of the Sieve associated with x (the Sieve I), where $x > 49$ is an even number.

Definition 8.1. Let $x > 49$ be an even number, and let k be the index of the greatest prime less than $\sqrt{x}$. Let $\{b_1, b_2, b_3, \ldots, b_k\}$ be the ordered set of the remainders of dividing x by $p_1, p_2, p_3, \ldots, p_k$. We define the sequence of k-tuples of remainders of level k, where in the sequences of remainders modulo p_h ($1 \leq h \leq k$) that form this sequence of k-tuples are applied the following rules for selecting remainders:

Rule 1. Within every period of size p_h of the sequence s_h ($1 \leq h \leq k$), the remainder 0 is selected.

Rule 2. Within every period of size p_h of the sequence s_h ($1 \leq h \leq k$), the remainder b_h is selected.

Now we can define formally the Sieve I, as follows.

Definition 8.2. Let $\mathscr{P}$ be the sequence of all primes; let $z = \sqrt{x}$, and let p_k be the greatest prime less than z. Let $\mathscr{A}$ be the set consisting of the indices of the sequence of k-tuples of the preceding definition, that lie in the interval $[1, x]$. For each $p = p_h \in \mathscr{P}$, $(1 \leq h \leq k)$, the subset $\mathscr{A}_p$ of $\mathscr{A}$ consists of the indices n of the sequence of k-tuples such that the remainder of dividing n by the modulus p_h is a selected remainder. Then, the indices of the prohibited k-tuples lying in $\mathscr{A}$ are sifted out; and the indices of the permitted k-tuples lying in $\mathscr{A}$ remain unsifted. See Remark 1.2. The sifting function

$$S(\mathscr{A}, \mathscr{P}, z) = \left| \mathscr{A} \setminus \bigcup_{\substack{p \in \mathscr{P} \\ p < z}} \mathscr{A}_p \right|,$$

is given by the number of permitted k-tuples whose indices lie in the interval $\mathscr{A}$.

Remark 8.1. Every sequence s_h $(1 \leq h \leq k)$ that form the sequence of k-tuples associated to the Sieve I consists of the remainders of dividing n by p_h. If a remainder is equal to 0, it is always a selected remainder. If a remainder is equal to b_h, it is also a selected remainder. If x is divisible by p_h, then $b_h = 0$ and therefore, in every period p_h of s_h there is only one selected remainder.

The following theorem shows that if n is the index of a permitted k-tuple belonging to the set $\mathscr{A}$ and $1 < n < x$, then n is a prime such that either $x - n = 1$ or $x - n$ is also a prime.

Theorem 8.1. *Let us consider the Sieve I, and its associated sequence of k-tuples. If n $(1 < n < x)$ is an unsifted element of the set $\mathscr{A}$, then n is a prime such that either $x - n = 1$ or $x - n$ is also a prime.*

Proof. Step 1. By definition, the set $\mathscr{A}$ consists of the indices of the sequence of k-tuples associated to the Sieve I, which lie in the interval $[1, x]$. Since n is an unsifted element of the set $\mathscr{A}$, by definition, n is the index of a permitted k-tuple. In the sequences of remainders modulo p_h $(1 \leq h \leq k)$ that form the sequence of k-tuples associated to the Sieve I, if a remainder is equal to 0 then it is a selected remainder. Then, by definition, a permitted k-tuple in this sequence has no element equal to 0 (see Remark 1.1). This means that n is not divisible by any of the primes $p_1, p_2, p_3, \ldots, p_k$ less than $z = \sqrt{x}$. Since $1 < n < x$, it follows at once that n is a prime.

Step 2. Let $\{b_1, b_2, b_3, \ldots, b_k\}$ be the ordered set of the remainders of dividing x by $p_1, p_2, p_3, \ldots, p_k$. Let r_h $(1 \leq h \leq k)$ be the elements of the permitted k-tuple whose index is n. In the sequences of remainders modulo p_h $(1 \leq h \leq k)$ that form the sequence of k-tuples associated to the Sieve I, by definition, if a given remainder of the sequence is equal to $b_h \in \{b_1, b_2, b_3, \ldots, b_k\}$, then it is a selected remainder. Consequently, by definition, for the permitted k-tuple whose index is n we have $r_h \neq b_h$ $(1 \leq h \leq k)$; this implies $n \not\equiv x \pmod{p_h}$, for every prime $p_h < \sqrt{x}$ (see Remark 1.1).

Step 3. By Step 1, n is a prime; furthermore $n \not\equiv x \pmod{p_h}$, where $p_h < \sqrt{x}$, by Step 2. This last implies that $x - n$ is not divisible by any prime $p_h < \sqrt{x}$. Since $\sqrt{x - n} < \sqrt{x}$, it follows that either $x - n = 1$ or $x - n$ is also a prime.

∎

Note that, given the level k, and given an even integer x $(p_k^2 < x < p_{k+1}^2)$, there is a sequence of k-tuples associated to the Sieve I, which has specific selected remainders for this particular x. On the other hand, given k, there is a partial sum S_k associated to the Sieve II, where there are multiple choices for selecting remainders, allowed by the rules defined in Section 2. Both are sequences of k-tuples of remainders, but they differ in the rules for selecting remainders in each one of them. The following lemma gives the relation between the number of permitted k-tuples within the interval $I[1, p_k^2]$ of the partial sum S_k (the sifting function of the Sieve II), and the number of permitted k-tuples within the interval $I[1, x]$ of the sequence of k-tuples associated to the Sieve I (the sifting function of the Sieve I).

Recall that we denote by $\{T(\mathscr{B}, \mathscr{P}, p_k)\}$ the set of the values of $T(\mathscr{B}, \mathscr{P}, p_k)$ for all the combinations of selected remainders in the sequences that form the partial sum S_k associated to the Sieve II.

Lemma 8.2. *Let $\mathscr{P}$ be the sequence of all primes. Let $x > 49$ be an even number, and let k be the index of the greatest prime less than $z = \sqrt{x}$; that is, $p_k^2 < x < p_{k+1}^2$. Consider the Sieve I, the Sieve II, and their associated sequences of k-tuples. We have $S(\mathscr{A}, \mathscr{P}, z) \geq \min\{T(\mathscr{B}, \mathscr{P}, p_k)\}$.*

Proof. By definition, the sequences of remainders modulo p_h $(1 < h \leq k)$ that form the sequence of k-tuples associated to the Sieve I can have one or two selected remainders in every period (see Remark 8.1). However, the sequences s_h $(1 < h \leq k)$ that form the partial sum S_k associated to the Sieve II, by definition, have always two selected remainders in every period. Suppose that we perform on the sequence of k-tuples associated to the Sieve I the following operation: in each sequence of remainders modulo p_h $(1 < h \leq k)$ that have only one selected remainder in every period, we choose an arbitrary second selected remainder. We obtain a partial sum S_k with a particular combination of selected remainders, where the number of permitted k-tuples within the interval $I[1, p_k^2]$ is greater than or equal to $\min\{T(\mathscr{B}, \mathscr{P}, p_k)\}$. It is obvious that in the interval $I[1, p_k^2]$ of the sequence of k-tuples associated to the Sieve I before performing the operation, the number of permitted k-tuples is also greater than or equal to $\min\{T(\mathscr{B}, \mathscr{P}, p_k)\}$. Since $I[1, p_k^2] \subset I[1, x]$, it follows that $S(\mathscr{A}, \mathscr{P}, z) \geq \min\{T(\mathscr{B}, \mathscr{P}, p_k)\}$.

∎

We need one more lemma before proving the Main theorem.

Lemma 8.3. *In the sequence of k-tuples associated to the Sieve I, if n $(1 < n < x)$ is the index of a permitted k-tuple, then $n' = x - n$ is the index of another permitted k-tuple.*

Proof. Step 1. Let $\{p_1, p_2, p_3, \ldots, p_k\}$ be the ordered set of the primes less than $z = \sqrt{x}$; and let $\{b_1, b_2, b_3, \ldots, b_k\}$ be the ordered set of the remainders of dividing x by $p_1, p_2, p_3, \ldots, p_k$. Recall that in the sequences of remainders modulo p_h $(1 \leq h \leq k)$ that form the sequence of k-tuples associated to the Sieve I, we have that 0 is a selected remainder, and b_h is also a selected remainder. Therefore, a given k-tuple whose elements are neither 0 nor b_h $(1 \leq h \leq k)$, by definition, is a permitted k-tuple.

Step 2. Let r_h $(1 \leq h \leq k)$ be the elements of the permitted k-tuple whose index is n. By definition, for the permitted k-tuple whose index is n we have $r_h \neq b_h$ $(1 \leq h \leq k)$, since every b_h is a selected remainder; this implies $n \not\equiv x \pmod{p_h}$, for every prime $p_h < \sqrt{x}$. Hence $n' = x - n \not\equiv 0 \pmod{p_h}$, for every prime $p_h < \sqrt{x}$. It follows that the k-tuple whose index is n' has no element equal to 0.

Step 3. Let r'_h $(1 \leq h \leq k)$ be the elements of the k-tuple whose index is $n' = x - n$. By definition, the permitted k-tuple whose index is n has no element equal to 0, since it is a selected remainder. This means that n is not divisible by any of the primes $p_1, p_2, p_3, \ldots, p_k$ less than $\sqrt{x}$. It follows that $n \not\equiv 0 \pmod{p_h} \implies n + x \not\equiv x \pmod{p_h} \implies n' = x - n \not\equiv x \pmod{p_h}$, for every prime $p_h \in \{p_1, p_2, p_3, \ldots, p_k\}$. So, for the k-tuple whose index is $n' = x - n$ we have $r'_h \neq b_h$ $(1 \leq h \leq k)$. From Step 1, Step 2 and this step, the k-tuple whose index is n' is a permitted k-tuple.

∎

Finally, we prove the Main Theorem. First, a definition:

Definition 8.3. Let $x > 49$ be an even number. We define the *partition function* $g(x)$ as the number of representations of the even number x as the sum $p + q$ of two primes $(p \leq q)$, that is, the number of Goldbach partitions [11] of the even number x.

Remark 8.2. Let $x > 49$ be an even number. Assume that in the sequence of k-tuples associated to the Sieve I there is a permitted k-tuple at position $n = x - 1$. Then, by Lemma 8.3, there is another permitted k-tuple at position 1; and furthermore, $n = x - 1$ is a prime, by Lemma 8.1, Step 1. So, 1 and $x - 1$ will appear among the unsifted members of the set $\mathscr{A}$. Note that in this case x is an even number of the form $p + 1$, where p is a prime.

Remark 8.3. Let $x > 49$ be an even number. Suppose that in the sequence of k-tuples associated to the Sieve I there is a permitted k-tuple at position $n = x/2$. Then, by Lemma 8.1, Step 1, the even number x is of the form $2p$, where p is a prime. In this case, there is a Goldbach partition $x = p + p$, but for this partition we have only one permitted k-tuple at position $n = p$ in the sequence of k-tuples.

Theorem 8.4. *The Main Theorem*
Let $x > 49$ be an even number, and let k be the index of the greatest prime less than $z = \sqrt{x}$.

(a) Every even number $x > p_{35}^2$ $(p_{35}^2 = 149^2 = 22201)$ is the sum of two odd primes.

(b) As $x \to \infty$, we have $g(x) \to \infty$.

Proof. Step 1. Recall that $S(\mathscr{A}, \mathscr{P}, z)$ denotes the sifting function of the Sieve I; assume that $S(\mathscr{A}, \mathscr{P}, z) \geq 3$. By Remark 8.2, among the unsifted members of the set $\mathscr{A}$ might appear 1 and $x - 1$. So, we can see that there are at least $S(\mathscr{A}, \mathscr{P}, z) - 2$ integers n in $\mathscr{A}$ such that n is a prime and $x - n$ is also a prime, by Theorem 8.1.

Step 2. By Lemma 8.1, the unsifted members of the set $\mathscr{A}$ could be primes p such that $x = p + q$, where q is also a prime that belongs to $\mathscr{A}$. If $p = q$ then x is of the form $2p$ (see Remark 8.3); in this case we have a Goldbach partition $x = p + p$, and only one member p of the set $\mathscr{A}$; otherwise, we have a Goldbach partition $x = p + q$, and two members p, q of the set $\mathscr{A}$. By Step 1, it is easy to check that in either case, the number $g(x)$ of Goldbach partitions of the even number x must be at least $\lceil (S(\mathscr{A}, \mathscr{P}, z) - 2)/2 \rceil$.

Step 3. We prove part *(a)* of the theorem. By Lemma 7.11, for every level $k \geq 35$, we have $\min\{T(\mathscr{B}, \mathscr{P}, p_k)\} > p_k/2 \geq p_{35}/2$. On the other hand, $S(\mathscr{A}, \mathscr{P}, z) \geq \min\{T(\mathscr{B}, \mathscr{P}, p_k)\}$ for every even number x such that $p_k^2 < x < p_{k+1}^2$ ($k \geq 35$), by Lemma 8.2. It follows that $S(\mathscr{A}, \mathscr{P}, z) > p_{35}/2$ for every even number $x > p_{35}^2$. Then, by Step 1, if $x > p_{35}^2$ there must be at least one unsifted member $n < x$ of $\mathscr{A}$, which is a prime such that $x - n$ is also a prime.

Step 4. We prove part *(b)* of the theorem. By Lemma 8.2, we have $S(\mathscr{A}, \mathscr{P}, z) \geq \min\{T(\mathscr{B}, \mathscr{P}, p_k)\}$ for every even number x such that $p_k^2 < x < p_{k+1}^2$ ($k \geq 4$). On the other hand, $\min\{T(\mathscr{B}, \mathscr{P}, p_k)\} \to \infty$ as $k \to \infty$, by Lemma 7.12. It follows that $S(\mathscr{A}, \mathscr{P}, z) \to \infty$ as $x \to \infty$. Hence $g(x) \to \infty$ as $x \to \infty$, by Step 2. The Main Theorem is proved.

∎

Now, it is a known fact that the strong Goldbach conjecture has been checked for even numbers larger than $p_{35}^2 = 22,201$. From this and the Main theorem, we conclude that every even number $x > 4$ can be expressed as the sum of two odd primes; therefore, the binary Goldbach conjecture is proved.

Acknowledgements

The author wants to thank Olga Vinogradova (Mathematics student, FCEyN, Universidad de Buenos Aires), Dra. Patricia Quattrini (Departamento de Matematicas, FCEyN, Universidad de Buenos Aires), Dr. Ricardo Duran (Departamento de Matematicas, FCEyN, Universidad de Buenos Aires) for helpful conversations and Dr. Hendrik W. Lenstra (Universiteit Leiden, The Netherlands), for his extremely useful suggestions.

References

[1] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer-Verlag, Berlin, 1976. MR0434929 (55 #7892).

[2] H. Halberstam and H.-E. Richert, *Sieve Methods*, Academic Press, London, 1974.

[3] A. C. Cojocaru and M. Ram Murty, *An Introduction to Sieve Methods an Their Applications*, Cambridge University Press, London, 2005.

[4] V. Brun, La série $1/5+1/7+1/11+1/13+1/17+1/19+1/29+1/31+1/41+1/43+1/59+1/61+\ldots$, où les dénominateurs sont nombres premieres jumeaux est convergente où finie, *Bull. Sci. Math.*, **43**, (1919), 124–128.

[5] V. Brun, Le crible d'Eratosthène et le théorème de Goldbach, *C. R. Acad. Sci. Paris*, **168**, (1919), 544–546.

[6] J. R. Chen, On the representation of a large even integer as the sum of a prime and the product of at most two primes. II, *Sci. Sinica* **21** (1978), no. 4, 421–430. MR0511293 (80e:10037)

[7] R. K. Guy, *Unsolved Problems in Number Theory*, third edition, Springer-Verlag, Berlin, 2004. MR2076335 (2005h:11003).

[8] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, sixth edition, Oxford Univ. Press, Oxford, 2008. MR2445243 (2009i:11001).

[9] E. Landau, *Elementary Number Theory*, translated by J. E. Goodman, Chelsea, New York, 1958. MR0092794 (19,1159d).

[10] H. L. Montgomery and R. C. Vaughan, The exceptional set in Goldbach's problem, *Acta Arith.* **27** (1975), 353–370. MR0374063 (51 #10263).

[11] T. Oliveira e Silva, Goldbach conjecture verification, `http://www.ieeta.pt/~tos/goldbach.html`.

[12] J. Rey Pastor, P. Pi Calleja and C. A. Trejo, *Mathematical Analysis. Vol. I* (in Spanish), Fifth edition], Editorial Kapelusz, Buenos Aires, 1960. MR0145013 (26 #2552a).

[13] I.M. Vinogradov, The Representation of an Odd Number as the Sum of Three Primes, *Dokl. Akad. Nauk SSSR* **16** (1937), 139–142.

[14] H. Helfgott, The ternary Goldbach conjecture is true, `http://www.arXiv.org/abs/1312.7748`.

[15] Terence Tao, Open question: The parity problem in sieve theory, `https://terrytao.wordpress.com/2007/06/05/open-question-the-parity-problem-in-sieve-theory/`.

[16] A.Selberg, The general sieve method and its place in prime number theory, *Proc. Int. Congress of Mathematicians Cambridge*, MA (Providence, RI. American Mathematical Society), **1** (1950), 262–289.

Ricardo G. Barca
Universidad Tecnologica Nacional
Buenos Aires (Argentina)
E-mail address: rbarca@frba.utn.edu.ar