Two Theories of Special Relativity ?

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Dedicated to Marie-Louise Nykamp

Abstract

Recently, [3], it was shown that Special Relativity is in fact based just about on one single physical axiom which is that of Reciprocity. Originally, Einstein, [1], established Special Relativity on two physical axioms, namely, the Galilean Relativity and the Constancy of the Speed of Light in inertial reference frames. Soon after, [2,4,5], it was shown that the Galilean Relativity alone, together with some implicit mathematical type conditions, is sufficient for Special Relativity. The references in [7,3] can give an idea about the persistence over the years, even if not the popularity, of the issue of minimal axiomatic foundation of Special Relativity. Here it is important to note that, implicitly, three more assumptions have been used on space-time coordinate transformations, namely, the homogeneity of space-time, the isotropy of space, and certain mathematical condition of smoothness type on the coordinate transformations. In [3], a weaker boundedness type condition on space-time coordinate transformations is used instead of the usual mathematical smoothness type conditions. In this paper it is shown that the respective boundedness condition is related to the Principle of Local Transformation Increment Ratio Limitation, or in short, PLTIRL, a principle introduced here, and one which has an obvious physical meaning. It is also shown that PLTIRL is *not* a stronger assumption than that of the mentioned boundedness in [3], and yet it can also deliver the Lorentz Transformations. Of interest is the fact that, by formulating PLTIRL as a physical axiom, the possibility is opened up for the acceptance, or on the contrary, rejection of this physical axiom PLTIRL, thus leading to *two* possible theories of Special Relativity. And to add further likelihood to such a possibility, the rejection of PLTIRL leads easily to effects which involve unlimited time and/or space intervals, thus are not accessible to usual experimentation for the verification of their validity, or otherwise. A conclusion is that a more careful consideration of the assumptions underlying Special Relativity is worth pursuing. In this regard, a corresponding trend has lately been observable in Quantum Mechanics and General Relativity. In the former, the respective analysis is more involved than has so far been the case for Special Relativity. As for the latter, the technical and conceptual difficulties are considerable. Regarding Quantum Field Theory, the situation is, so far, unique in Physics since, to start with, there is not even one single known rigorous and comprehensive enough mathematical model. This paper is a new version of [20].

> "Of all things, good sense is the most fairly distributed : everyone thinks he is so well supplied with it that even those who are the hardest to satisfy in every other respect never desire more of it than they already have."   :-)   :-)   :-)
>
> R Descartes
> Discourse de la Méthode

> "... creativity often consists of finding hidden assumptions. And removing those assumptions

can open up a new set of possibilities ...”

Henry R Sturman

“History is written with the feet ...”

Chinese Ex-Chairman Mao,
of the Long March fame ...

Science is not done scientifically, since it is mostly done by non-scientists ...

Anonymous

Physics is too important to be left to physicists ...

Anonymous

Is the claim about the validity of the so called “physical intuition” but a present day version of medieval claims about the sacro-sanct validity of theological revelations ?

Anonymous

0. Prologue

In recent times, there has been a remarkable amount of research regarding the foundations of Quantum Mechanics and General Relativity, with many such ventures, especially regarding quanta, searching for axioms of a manifestly physical nature. Often the attention is focused not only on possible reformulations of the usually accepted theories, but as well on their possible extensions, some of them quite daring as such. As for Quantum Field Theory, the foundational situation is still at its beginnings since the theory is not yet at the stage to have a rigorous and widely enough accepted formulation, let alone an axiomatic one.
Needless to say, the conceptual complexities involved in such foundational research are considerable in Quantum Mechanics, while in General Relativity tremendous technical difficulties come to be added in such ventures.
All such challenges seem, however, to pale in comparison to those one faces when it comes to Quantum Field Theory.

Special Relativity in this regard seems to be in a surprisingly convenient easy situation. Indeed, as first formulated in 1905 by Einstein, it is based on not more than two axioms, each of which has a clear physical meaning, namely, the Galilean Relativity and the Constancy of the Speed of Light in inertial reference frames. Soon after, see [2,4,5], and ever since, see for instance [7,3] and the references cited there, a number of attempts have been made in order to find a *minimal* axiomatic setup. The present paper has as one of its aims to further contribute to that venture.

A remarkable fact which emerges along that study of a possibly minimal axiomatic formulation of Special Relativity is what may be seen as a significant *disconnect* between physical type axioms, and on the other hand, the accepted mathematical models. Indeed, as seen below, and already pointed out earlier in the literature, see [2,4,5] and [7,3] and the references cited there, the Lorentz Transformations can be obtained *without* the axiom of the Constancy of the Speed of Light. And as far as the axiom of the Galilean Relativity is concerned, a clearly weaker form of it, namely the axiom of Reciprocity, [7,3], is sufficient, together with certain mathematical type conditions on the coordinate transformations involved. By the way, this axiom of Reciprocity merely asks that in inertial reference frames the laws of Physics remain the same when a velocity v is replaced with the velocity $-v$.

It seems therefore that, at least as far as Special Relativity is concerned, its mathematical core - which is given by the Lorentz Transformations - is in fact *overstated* by its two usual axioms, the second of which is not even needed, while the first one is only used in an obviously weaker form.

It appears that such a *disconnect* should be of concern in Physics since it seems naturally more easy to have physical intuition come up with some axioms, than to guarantee that those axioms will as well be minimal. And in such a foundational situation the issue of minimality of physical type axioms is not reduced merely to the avoidance of redundancy. Indeed, when setting up physical type axioms the objective is clearly the achievement of a *precise* model of the part of physics intended for study. And then it may - and why not, also should - be considered as problematic the choice of a system of axioms which happens to overstate the case, and it does so without any other justification, except for being the product of some physical intuition ...

Last, and not least, regarding axiomatic approaches to Physics, various theories of Physics are also based on the *general mathematical assumptions* upon which corresponding mathematical models of such theories happen to be built during a given historical period in the development of science. And such assumptions are, of course, so deeply embedded in the mathematical culture of any such historical period as to be taken tacitely for granted, and thus not being subjected to any widely enough practiced questioning, let alone to the more systematic development of the consequences of alternative assumptions.

In this regard, a basic and general mathematical assumption for more than two millennia is about the *scalars* used in Physics. Namely, ever since Euclid set up Geometry in ancient Egypt, it is assumed that the geometric straight line, which since Descartes is identified with what in modern Mathematics is the field $\mathbb{R}$ of usual real numbers - and thus

gives the *scalars* upon which all else is constructed in Physics, including the complex numbers in $\mathbb{C}$ - does satisfy the Archimedean Axiom, [8-21]. This tacit and long accepted assumption on scalars, however, does not have any known motivation in modern Physics, albeit in ancient Egypt was of obvious practical importance in measuring the land after the yearly flood of the Nile.
And the remarkable fact is that, by setting aside the Archimedean Axiom, a large variety of considerably more rich and complex versions of the geometric straight line - and thus of scalars - can be constructed quite easily, [20,21]. In other words, one can obtain a large variety of scalars which extend $\mathbb{R}$, and thus correspondingly, the complex numbers in $\mathbb{C}$ as well. And these scalar extensions can give mathematical models of space-time, as well as of manifolds, Hilbert spaces, and other mathematical structures used in Physics, and do so with possible advantages due to their more rich and complex structure, advantages some of which are mentioned in [8-21].

As it happens, the Lorentz Transformations can also be deduced in such extensions, [18], along lines similar to those presented in this paper. Thus Special Relativity has a *second relativity* property, namely, it is independent also of a wide class of scalars which can be used in Physics, and not only of the inertial reference frames.
Regarding the possible validity of such a second relativity for other branches of Physics, the problem is still open. Further details in this regard can be found in [14].

1. A Most General Setup

For convenience, we briefly review the main result in [3]. Let S and S' be two reference frames with space-time coordinates (x, y, z, t), respectively, $(x\,', y\,', z\,', t\,')$. Further, let S' move with constant velocity v with respect to S, and do so parallel with the x-axis in S. Lastly, let assume the property

$$(1.1) \qquad x = y = z = t = 0 \implies x\,' = y\,' = z\,' = t\,' = 0$$

which means that at $t = t\,' = 0$, the origins of coordinates in S and

S' coincide.

Let us now consider the most general possible space-time coordinate transformation, [3], namely, of the form

$$(1.2) \qquad \begin{aligned} x' &= X(x,y,z,t,v) \\ y' &= Y(x,y,z,t,v) \\ z' &= Z(x,y,z,t,v) \\ t' &= T(x,y,z,t,v) \end{aligned}$$

where $(x,y,z,t), (x',y',z',t') \in \mathbb{R}^4$ and $v \in \mathbb{R}$.

For ease of notation, and following [3], for each $v \in \mathbb{R}$, let us define a mapping

$$(1.3) \qquad f_v : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$$

where for $(x,y,z,t) \in \mathbb{R}^4$, we have

$$(1.4) \qquad \begin{aligned} & f_v(x,y,z,t) = \\ & \quad = (X(x,y,z,t,v), Y(x,y,z,t,v), Z(x,y,z,t,v), T(x,y,z,t,v)) \end{aligned}$$

Then (1.2) takes the form

$$(1.5) \qquad u' = f_v(u), \qquad u, u' \in \mathbb{R}^4, \quad v \in \mathbb{R}$$

Let us now take any $u_0 = (x_0, y_0, z_0, t_0), \Delta u = (\Delta x, \Delta y, \Delta z, \Delta t) \in \mathbb{R}^4$, and then define $u_0{}' = (x_0{}', y_0{}', z_0{}', t_0{}'), \Delta u' = (\Delta x', \Delta y', \Delta z', \Delta t') \in \mathbb{R}^4$ by

$$(1.6) \qquad u_0{}' = f_v(u_0)$$

$$(1.7) \qquad \Delta u' = f_v(u_0 + \Delta u) - f_v(u_0)$$

We note that (1.6), (1.7) can be interpreted as follows. We have a reference frame S_0 which is identical with S, except that it has its origin at the point u_0 in S. Similarly, we have a reference frame S'_0

which is identical with S', except that it has its origin at the point ${u_0}'$ in S'.

Then clearly, S_0 and S'_0 relate to one another in the same way as S and S' do. Namely, S'_0 moves with velocity v with respect to S_0, and, see (1.1)

$$(1.8) \qquad \Delta u = 0 \implies \Delta u\,' = 0$$

Hence due to the homogeneity of space-time, we have, see (1.5)

$$(1.9) \qquad \Delta u\,' = f_v(\Delta u)$$

And then, (1.7), (1.9) give

$$(1.10) \qquad f_v(u_0 + \Delta u) = f_v(u_0) + f_v(\Delta u)$$

Thus the functions f_v, with $v \in \mathbb{R}$, are *additive*, namely

$$(1.11) \qquad f_v(u + w) = f_v(u) + f_v(w), \qquad u, w \in \mathbb{R}^4$$

On the other hand, (1.1) implies

$$(1.12) \qquad f_v(0) = 0$$

hence

$$(1.13) \qquad f_v(-u) = -f_v(u), \qquad u \in \mathbb{R}^4$$

Consequently, it is easy to show that, see (18)-(20) in [3]

$$(1.14) \qquad f_v(ru) = r f_v(u), \qquad r \in \mathbb{Q}, u \in \mathbb{R}^4$$

Now it is precisely here that one needs the assumption of certain *mathematical* properties on the functions f_v, that is, on the coordinate transformations (1.2) - (1.5), in order to be able to make the transition from property (1.14) of the homogeneity of f_v with respect to arbitrary *rational* scalars, to the *stronger* property

$$(1.15) \qquad f_v(cu) = cf_v(u), \qquad c \in \mathbb{R}, u \in \mathbb{R}^4$$

which is that of the homogeneity of f_v with respect to arbitrary *real* scalars.

Obviously, if one assumes that f_v is *continuous* on $\mathbb{R}^4$, then (1.14) implies (1.15), since $\mathbb{Q}$ is a dense subset of $\mathbb{R}$. In [7], a stronger property of f_v is assumed, namely, its differentiability on $\mathbb{R}^4$.

On the other hand, in [3] it is shown that the following *boundedness* property of f_v - which obviously is a *weaker* condition than the continuity of f_v - is sufficient in order to obtain (1.15) from (1.14). Namely, we only need that

$$(1.16) \qquad \begin{array}{l} \forall \;\; v \in \mathbb{R} \;\;\; : \\ \\ \exists \;\; M > 0 \;\;\; : \\ \\ \forall \;\; u \in \mathbb{R}^4 \;\;\; : \\ \\ \quad ||\, u \,|| \leq 1 \;\;\Longrightarrow\;\; ||\, f_v(u) \,|| \leq M \end{array}$$

where for $a = (a_1, a_2, a_3, a_4) \in \mathbb{R}^4$, we define the norm $||\, a \,|| = |\, a_1 \,| + |\, a_2 \,| + |\, a_3 \,| + |\, a_4 \,|$.

And as is shown in [3], this boundedness condition (1.16) implies the following *continuity* property

$$(1.17) \qquad \begin{array}{l} \forall \;\; v \in \mathbb{R}, \;\;\; \epsilon > 0 \;\;\; : \\ \\ \exists \;\; \delta > 0 \;\;\; : \\ \\ \forall \;\; u \in \mathbb{R}^4 \;\;\; : \\ \\ \quad ||\, u \,|| \leq \delta \;\;\Longrightarrow\;\; ||\, f_v(u) \,|| \leq \epsilon \end{array}$$

Indeed, if $||\, u \,|| \leq \frac{1}{n}$, for some $n \in \mathbb{N}$, then $||\, nu \,|| \leq 1$, hence (1.16) implies $||\, f_v(nu)\,|| \leq M$, which in view of (1.15), yields $||\, f_v(u)\,|| \leq \frac{M}{n}$. But for a given $\epsilon > 0$, we have $\frac{M}{n} \leq \epsilon$, as soon as $\frac{1}{n} \leq \frac{\epsilon}{M}$. Hence in (1.17), we can take $\delta \leq \frac{1}{n}$.

Now we recall (1.11), and then the continuity property (1.17) implies the following stronger *uniform continuity*

$$
\begin{array}{ll}
 & \forall \;\; v \in \mathbb{R}, \quad \epsilon > 0 \quad : \\
 & \exists \;\; \delta > 0 \quad : \\
(1.18) & \forall \;\; u, w \in \mathbb{R}^4 \quad : \\
 & \quad ||\, w - u \,|| \leq \delta \quad \Longrightarrow \quad ||\, f_v(w) - f_v(u) \,|| \leq \epsilon
\end{array}
$$

2. A Class of Lorentz Type Transformations

For convenience, let us recall that based on (1.15), one can obtain the Lorentz Transformations, as shown in [7]. For simplicity, we only consider the case of the one-dimensional space, when it follows that (1.2) takes the form

$$(2.1) \qquad x\,' = \frac{x - vt}{\sqrt{1 - Kv^2}}, \qquad t\,' = \frac{t - Kv\,x}{\sqrt{1 - Kv^2}}$$

for an arbitrary $K \in \mathbb{R}$, such that $1 - Kv^2 > 0$, thus

$$(2.2) \qquad K < \frac{1}{v^2}$$

Clearly, for $K = 0$, the transformations (2.1) become

$$(2.3) \qquad x\,' = x - vt, \qquad t\,' = t$$

which corresponds to the usual Galilean case.

As for other possible values of K one can proceed as follows, [7]. A simple argument based on (2.1) gives the law of addition of velocities

$$(2.4) \qquad w = \frac{u+v}{1+Kuv}$$

therefore, assuming

$$(2.5) \qquad K > 0$$

and taking $u = v = \frac{1}{\sqrt{K}}$, one obtains $w = \frac{1}{\sqrt{K}} = u = v$. Hence the speed

$$(2.6) \qquad \frac{1}{\sqrt{K}}$$

is independent of the reference frame.

Here then we can bring in the Axiom of the Constancy of the Speed of Light in inertial reference frames, and conclude that $\frac{1}{\sqrt{K}} = c$, thus

$$(2.7) \qquad K = \frac{1}{c^2} > 0$$

Finally, as shown in [7], the assumption $K < 0$ leads to an impossibility.

Remark 2.1.

As seen above, in order to obtain the Lorentz Transformations, one does *not* need the full power of the two usual axioms of Special Relativity, namely, the axioms of Galilean Relativity and Constancy of the Speed of Light in inertial reference frames.

Indeed, instead of the axiom of Galilean Relativity one only needs the considerably weaker axiom of Reciprocity.

As for the axiom of Constancy of the Speed of Light in inertial reference frames, one can obtain (2.6) without any use of it. Therefore, the content of this axiom is only used in part, namely, when obtaining (2.7), and thus in order to distinguish a special physical value for $\frac{1}{\sqrt{K}}$, which is that of the speed of light c.

This is precisely the substance of what may be called the *disconnect* between Physics and its Mathematical Model in the case of Special Relativity.

3. The Principle of Local Transformation Increment Ratio Limitation, or PLTIRL

The idea in [3] to use the *weaker boundedness* condition (1.16) on the coordinate transformations (1.2) - (1.5), instead of usual stronger continuity or smoothness conditions, such as for instance in [7], brings to attention the possible role certain other conditions may play in the axiomatic foundation of Special Relativity, conditions which may also lend themselves in an easier manner to a physical interpretation. And once such physical interpretations are available, their acceptance or otherwise may open *branchings* in Special Relativity, see section 5 in the sequel, which may be physically meaningful.

In this regard, here we introduce the following *Principle of Local Transformation Increment Ratio Limitation*, or in short, PLTIRL, formulated in terms of the coordinate transformations (1.2) - (1.5). And as seen in this section, PLTIRL is - at least formally and in the given context of (1.11) - (1.14) - *not* a stronger condition than (1.16), and yet, it can still lead to the Lorentz Transformations.
The interest in PLTIRL is in the fact that it lends itself to a *physical interpretation*, as seen in the sequel, in section 4.

Here it should be mentioned that the main issue is the following :

- The properties (1.11) - (1.14), as seen in the literature, and recently in [7,3], can be obtained from weaker assumptions than the two usual axioms of Special Relativity.

- Therefore, in order to obtain the Lorentz Transformations, all that is needed is to find appropriate weakest possible assumptions, say $(\mathcal{C})$, which can lead from (1.11) - (1.14) to (1.15).

On the other hand, any such an assumption $(\mathcal{C})$ will operate not alone and all on its own, but in the context of the already proved properties (1.11) - (1.14). Therefore, in comparing different possible such assumptions $(\mathcal{C})$, one may proceed in two ways : compare those assumptions strictly on their own terms and without any reference to (1.11) - (1.14), or compare them in the context of (1.11) - (1.14).

However, we should not forget the overall aim, namely, to find out which may indeed be the *minimal* assumptions needed in order to obtain the Lorentz Transformations. And then the second of the above two kind of comparisons between various possible additional assumptions $(\mathcal{C})$ clearly has a greater importance. Specifically, this is precisely why condition (1.16), see [3], is welcome, since in the context of (1.11) - (1.14), it can still deliver the Lorentz Transformations, although, all on its own, it is weaker than the usual continuity or smoothness assumptions in the literature, see for instance [7].

Consequently, under the inspiration given by the boundedness type condition (1.16) in [3], a main point in this paper is to replace that condition with PLTIRL in the axiomatic foundation of Special Relativity. The advantage obtained in this way is that PLTIRL is a condition which lends itself to a physical interpretation. And as seen in section 4, this physical interpretation opens up the possibility of *two* versions of Special Relativity, namely, according to having the physical PLTIRL accepted, or on the contrary, rejected.

And now, to the formulation of PLTIRL. Given a velocity $v \in \mathbb{R}$ and two sets of space-time coordinates $(x_0, y_0, z_0, t_0), (x, y, z, t) \in \mathbb{R}^4$, with their corresponding transformed coordinates $(x'_0, y'_0, z'_0, t'_0), (x', y', z', t') \in \mathbb{R}^4$ through (1.2) - (1.5), then the boundedness condition PLTIRL asks that there exist $K, \rho > 0$, such that

$$|| P - P_0 || \leq \rho \implies || P' - P'_0 || \leq K || P - P_0 ||$$

where $P_0 = (x_0, y_0, z_0, t_0), P = (x, y, z, t), P'_0 = (x'_0, y'_0, z'_0, t'_0), P' = (x', y', z', t')$.

We note that K and ρ may depend on v and $P_0 = (x_0, y_0, z_0, t_0)$, which are supposed to be given. As for P'_0, it results from v and P_0 through the coordinate transformations (1.2) - (1.5). Further, P is arbitrary, and then $P\,'$ is given by (1.2) - (1.5) applied to v and P.

Consequently, a more detailed formulation of PLTIRL is as follows :

$$\text{(PLTIRL)} \quad \begin{array}{l} \forall \;\; v \in \mathbb{R}, \;\; P_0 \in \mathbb{R}^4 \;\; : \\ \\ \exists \;\; K, \rho > 0 \;\; : \\ \\ \forall \;\; P \in \mathbb{R}^4 \;\; : \\ \\ \quad || \, P - P_0 \, || \leq \rho \quad \Longrightarrow \quad || \, P\,' - P\,'_0 \, || \leq K || \, P - P_0 \, || \end{array}$$

Obviously, the point in the above condition is the implication

$$(3.1) \qquad || \, P - P_0 \, || \leq \rho, \quad P \neq P_0 \qquad \Longrightarrow \qquad \frac{|| \, P\,' - P\,'_0 \, ||}{|| \, P - P_0 \, ||} \leq K$$

which is precisely about a *local transformation increment ratio limitation*, where the transformation increment is $|| \, P\,' - P\,'_0 \, ||$, the increment is $|| \, P - P_0 \, ||$, and then their ratio is limited by K, while the local aspect is due to the fact that one can only deal with increments $|| \, P - P_0 \, ||$ which are limited by an appropriate ρ.

Here we can note that the *transformation increment ratio* $\dfrac{|| \, P\,' - P\,'_0 \, ||}{|| \, P - P_0 \, ||}$ can in fact be seen as a certain kind of *local velocity* around the space-time event P_0. More precisely, it can be seen as the local velocity of the reference frame S' with respect to the reference frame S. Indeed, when for simplicity, one considers that transformation increment in terms of the one dimensional Lorentz Transformations

$$x\,' = \frac{x - vt}{\sqrt{1 - (\dfrac{v}{c})^2}}, \qquad t\,' = \frac{t - \dfrac{v}{c^2}\, x}{\sqrt{1 - (\dfrac{v}{c})^2}}$$

and assumes, see (1.1), (1.2), that $P_0 = (x_0, t_0) = (0, 0), \;\; P = (x, t), \;\; P_0\,' = (x_0\,', t_0\,') = (0, 0), \;\; P\,' = (x\,', t\,')$, then

$$|| \, P - P_0 \, || = | \, x - x_0 \, | + | \, t - t_0 \, | = | \, x \, | + | \, t \, |$$

$$|| \, P' - P'_0 \, || = | \, x' - x_0{}' | + | \, t' - t_0{}' | = | \, x' \, | + | \, t' \, |$$

hence

$$\frac{|| \, P' - P'_0 \, ||}{|| \, P - P_0 \, ||} = \frac{| \, x' \, | + | \, t' \, |}{| \, x \, | + | \, t \, |} = \frac{| \, x - vt \, | + | \, t - \frac{v}{c^2} x \, |}{| \, x \, | + | \, t \, |} \frac{1}{\sqrt{1 - (\frac{v}{c})^2}}$$

thus for $x = 0, v, t > 0$, one obtains an expression independent of t, namely

$$\frac{|| \, P' - P'_0 \, ||}{|| \, P - P_0 \, ||} = \frac{v + 1}{\sqrt{1 - (\frac{v}{c})^2}} \tag{3.2}$$

while for $t = 0, x, v > 0$, results an expression independent of x, namely

$$\frac{|| \, P' - P'_0 \, ||}{|| \, P - P_0 \, ||} = \frac{\frac{v}{c^2} + 1}{\sqrt{1 - (\frac{v}{c})^2}} \tag{3.3}$$

and clearly, (3.2), (3.3) are closely related to the velocity v of the reference frame S' with respect to the reference frame S.

And now let us show that, in the context of (1.11) - (1.14), the above condition PLTIRL is *not* stronger than (1.16).

In this respect, first we note that PLTIRL can also be seen as a *local Lipschitz* type continuity property, while (1.18) is a *uniform continuity* property. Thus as they stand, none of them is in general stronger, or for that matter, weaker than the other one.

However, the assumption in [3] is not (1.18), and instead of it, it is (1.16). Yet clearly, (1.16) leads to the following *global uniform Lipschitz* property

$$
(3.4) \quad
\begin{array}{l}
\forall \;\; v \in \mathbb{R} \;\; : \\
\\
\exists \;\; L > 0 \;\; : \\
\\
\forall \;\; u, w \in \mathbb{R}^4 \;\; : \\
\\
\quad || \, f_v(w) - f_v(u) \, || \leq L || \, w - u \, ||
\end{array}
$$

Indeed, in view of (1.14), we can proceed as follows. Let $s \in \mathbb{R},\; s > 0$, be such that $r = || \, u \, || + s \in \mathbb{Q}$, then for $w = \frac{1}{r} \, u$, we have $|| \, w \, || < 1$, thus (1.16) gives $|| \, f_v(w) \, || \leq M$. However $f_v(w) = \frac{1}{r} \, f_v(u)$, hence $|| \, f_v(u) \, || \leq M(|| \, u \, || + s)$, and since s can be arbitrary small, we obtain

$$
(3.5) \qquad || \, f_v(u) \, || \leq M || \, u \, ||, \qquad v \in \mathbb{R}, u \in \mathbb{R}^4
$$

from which (3.4) follows obviously, with $L = M$.

We can now conclude that PLTIRL, when seen in the context of (1.11) - (1.14), is *not* a stronger assumption than the boundedness assumption (1.16) in [3], since the latter implies (3.4) which is obviously at least as strong as PLTIRL.

Lastly, since PLTIRL is also a local Lipschitz continuity property, it follows that, together with (1.14), it still implies (1.15), thus again gives the Lorentz Transformations.

4. Accepting, or Rejecting, the Physical Axiom PLTIRL

Let us show now that PLTIRL can indeed be associated with a physical interpretation in a more obvious manner than condition (1.16) in [3].

For that purpose, let us further clarify the meaning of PLTIRL by assuming that it does not hold. This means that we have

$$
\text{(Non-PLTIRL)} \quad
\begin{array}{l}
\exists \;\; v \in \mathbb{R}, \quad P_0 \in \mathbb{R}^4 \;\; : \\[2ex]
\forall \;\; K, \rho > 0 \;\; : \\[2ex]
\exists \;\; P \in \mathbb{R}^4 \;\; : \\[2ex]
\quad ||\, P - P_0 \,|| \leq \rho \\[2ex]
\quad ||\, P\,' - P\,'_0 \,|| > K ||\, P - P_0 \,||
\end{array}
$$

These two inequalities above mean, respectively, that we have

$$
(4.1) \qquad |\, x - x_0 \,| + |\, y - y_0 \,| + |\, z - z_0 \,| + |\, t - t_0 \,| \leq \rho
$$

as well as

$$
(4.2) \qquad
\begin{array}{l}
|\, x\,' - x\,'_0 \,| + |\, y\,' - y\,'_0 \,| + |\, z\,' - z\,'_0 \,| + |\, t\,' - t\,'_0 \,| > \\[1ex]
\qquad\qquad > K(\, |\, x - x_0 \,| + |\, y - y_0 \,| + |\, z - z_0 \,| + |\, t - t_0 \,| \,)
\end{array}
$$

Clearly, (4.2) means that at least one of the following four relations holds

$$
(4.3) \qquad
\begin{array}{l}
|\, x\,' - x\,'_0 \,| > K(\, |\, x - x_0 \,| + |\, y - y_0 \,| + |\, z - z_0 \,| + |\, t - t_0 \,| \,)/4 \\[2ex]
|\, y\,' - y\,'_0 \,| > K(\, |\, x - x_0 \,| + |\, y - y_0 \,| + |\, z - z_0 \,| + |\, t - t_0 \,| \,)/4 \\[2ex]
|\, z\,' - z\,'_0 \,| > K(\, |\, x - x_0 \,| + |\, y - y_0 \,| + |\, z - z_0 \,| + |\, t - t_0 \,| \,)/4 \\[2ex]
|\, t\,' - t\,'_0 \,| > K(\, |\, x - x_0 \,| + |\, y - y_0 \,| + |\, z - z_0 \,| + |\, t - t_0 \,| \,)/4
\end{array}
$$

Consequently, the negation of PLTIRL means the existence of a *finite velocity* $v \in \mathbb{R}$ and of a *space-time event* $P_0 \in \mathbb{R}^4$, such that, no matter how near to P_0, there exist space-time events P for which at least one of the ratios between, on one hand, the space-time coordinates of the increments between the transformations of P and P_0, and on the other hand, the increment between P and P_0, can become *arbitrarily large.*

Indeed, in view of (4.3), the negation of PLTIRL takes the form

$$\exists \;\; v \in \mathbb{R}, \quad P_0 \in \mathbb{R}^4 \;\; :$$

$$\forall \;\; K, \rho > 0 \;\; :$$

$$\exists \;\; P \in \mathbb{R}^4, \quad P \neq P_0 \;\; :$$

$$|\, x - x_0 \,| + |\, y - y_0 \,| + |\, z - z_0 \,| + |\, t - t_0 \,| \leq \rho$$

and at least one of the following four relations holds

(Non-PLTIRL)

$$\frac{|\, x\,' - x\,'_0 \,|}{|\, x - x_0 \,| + |\, y - y_0 \,| + |\, z - z_0 \,| + |\, t - t_0 \,|} > K$$

$$\frac{|\, y\,' - y\,'_0 \,|}{|\, x - x_0 \,| + |\, y - y_0 \,| + |\, z - z_0 \,| + |\, t - t_0 \,|} > K$$

$$\frac{|\, z\,' - z\,'_0 \,|}{|\, x - x_0 \,| + |\, y - y_0 \,| + |\, z - z_0 \,| + |\, t - t_0 \,|} > K$$

$$\frac{|\, t\,' - t\,'_0 \,|}{|\, x - x_0 \,| + |\, y - y_0 \,| + |\, z - z_0 \,| + |\, t - t_0 \,|} > K$$

5. Two Alternative Theories of Special Relativity ?

The above considerations open up *two alternatives* in Special Relativity. Namely, one can - as a *physical axiom* - accept PLTIRL, that is, the Principle of Transformation Increment Ratio Limitation, or on the contrary, based on certain physical arguments, one can reject it.

Now, since one of the essential features of Special Relativity is the *limitation* on the velocity of propagation of any physical phenomenon, it appears to be more natural *not* to reject PLTIRL.

However, it is well known, see [3,7] and the literature cited there, that the mentioned velocity limitation is not a perfectly independent axiom of Special Relativity, since it follows from the physical axiom of Galilean Relativity, and in fact, from the physical axiom of Reciprocity, under rather general conditions, as shown in [3].

And here, one can note an interesting fact.

Namely, what one adds in [3] to Galilean Relativity, more precisely, to Reciprocity, in order to obtain the Lorentz Transformations, and thus the relativistic rule of velocity addition, as well as the mentioned velocity limitation, is a *boundedness* condition from which a continuity property results, a property closely related to PLTIRL, as seen above.

Thus in a way, velocity limitation is assumed, for instance, in the sense of (3.2), (3.3), in order to obtain velocity limitation in the sense of the Axiom of Constancy of the Speed of Light ...

This remark is *not* a criticism of the approach in [3], and instead, points out the fact that, to a certain extent the Galilean Relativity, or for that matter, Reciprocity - taken all alone and in itself - is *not* sufficient in order to obtain Special Relativity.

In this regard it may be of interest to recall that Einstein himself kept on numerous occasions presenting Special Relativity as being based on *two* physical axioms, namely, the Galilean Relativity *and* the velocity limitation, in which the highest possible one is of light in void.

We can, therefore, conclude that to the extent one is not rejecting highly discontinuous physical processes, there can be two rather distinct theories of Special Relativity, namely, the usual one, obtainable under PLTIRL, for instance, and on the other hand, one that obeys Galilean Relativity, and in fact, merely Reciprocity, and also assumes the homogeneity of space-time and isotropy of space, yet it is described by space-time coordinate transformations more general than the Lorentz ones.

In case PLTIRL is, however, rejected, then as seen above in (Non-PLTIRL), at least one of the four ratios

$$\frac{|\,x' - x'_0\,|}{|\,x - x_0\,| + |\,y - y_0\,| + |\,z - z_0\,| + |\,t - t_0\,|}$$

$$\frac{|\,y' - y'_0\,|}{|\,x - x_0\,| + |\,y - y_0\,| + |\,z - z_0\,| + |\,t - t_0\,|}$$

$$\frac{|\,z' - z'_0\,|}{|\,x - x_0\,| + |\,y - y_0\,| + |\,z - z_0\,| + |\,t - t_0\,|}$$

$$\frac{|\,t' - t'_0\,|}{|\,x - x_0\,| + |\,y - y_0\,| + |\,z - z_0\,| + |\,t - t_0\,|}$$

becomes arbitrarily large in *any* neighbourhood of at least *one* space-time event P_0.

Here however, it is worth noting that the last of the above ratios involves the *unboundedness of time*, and not of space, as is the case with the first three ratios. And such time unboundedness is a phenomenon *not* so easy to detect under usual experiments.

Consequently, an unorthodox kind of Special Relativity, one whose possibility was mentioned above in case PLTIRL is rejected, may escape the means of detection by usual experiments which, as a rule, do not involve arbitrarily large time intervals.

Needless to say, the detection by usual experiments of the space unboundedness which may result from the first three ratios above is, similarly, not an easy task, in case Non-PLTIRL is accepted.

6. Further Possibilities in Non-Archimedean Space-Times

As seen in [18], the Lorentz Transformations of Special Relativity can be obtained within far *larger* space-times than the usual four dimensional Minkowski one. Consequently, the above arguments in which the formulation of PLTIRL opens up the possibility of two rather different theories of Special Relativity may lead to a yet richer such possibility which will be dealt with elsewhere, since it goes considerably beyond the usual Euclidean framework, thus of that of Minkowski as well, and as such, it requires a suitable preliminary mathematical setup.

7. Comment on the Disconnect between Physics and the Mathematical Models

As it may often happen, physicists would expect that, in the study of various disciplines of Physics, a good physical intuition leads to physically meaningful axioms which can then be formulated in suitable mathematical models. And needless to say, a close connection is supposed to be exhibited between the physically meaningful axioms and the respective discipline of Physics, close in the sense that the axioms are not supposed to model only a part of the discipline under study, as much as they are not supposed to contain redundancy by overlapping with one another. Therefore, in a certain sense - and as it should be with every better axiomatic system - the axioms should satisfy two rather conflicting criteria, namely, to be sufficiently inclusive, and at the same time, to be minimal.
Here however, one should realize that the possible failure of a certain system of axioms in Physics to satisfy these two conflicting demands may happen to show up more clearly in the resulting mathematical model, and less so in other ways.
Therefore, this *failure*, although one between a certain discipline of Physics as such, and on the other hand, a given system of physically meaningful axioms which is supposed to describe it, rather ends up manifested as a *disconnect* between Physics, and on the other hand, one or another of its mathematical models.

As seen above, and in the references cited, the two usual axioms of Special Relativity happen not to satisfy such a requirement of being at the same time inclusive and minimal. Thus in the sense just mentioned, we witness here a disconnect between Special Relativity and its mathematical models. And this situation is of interest due to the following :

- In order to obtain the Lorentz Transformations, which are considered to contain the essence of Special Relativity, one needs less than both usual axioms of Special Relativity, namely, the Galilean Relativity and the Constancy of the Speed of Light in inertial reference frames. Indeed, instead of the full Galilean Relativity, only the axiom of Reciprocity, plus condition (1.16), or alternatively, (PLTIRL) are needed.

- The second above axiom is thus nearly a consequence of the axiom of Reciprocity, as noted in Remark 2.1., and pointed out in [3].

Thus the questions arise :

- Are the mentioned two usual axioms of Special Relativity too much for that theory of Physics ?
- And if not, then what are those parts of Special Relativity which have not yet been found, since the Lorentz Transformations do not need the full use of both those usual axiom ?

In short :

- Which is a system of axioms for Special Relativity which is both sufficient and minimal ?

Also as seen above, the crux of the issue is the following. Property (1.14), which is the homogeneity property of the coordinate transformations (1.2) - (1.5) with respect to *rational* scalars, can be obtained from considerably less than the two usual axioms of Special Relativity, namely, Galilean Relativity and the Constancy of the Speed of Light. And then, in order to obtain property (1.15), which is the homogeneity property of the coordinate transformations (1.2) - (1.5) with respect to *real* scalars, one needs certain additional *mathematical* assumptions.

Now the obvious such assumption is that, for each velocity $v \in \mathbb{R}$, the coordinate transformations (1.2) - (1.5) are continuous with respect to the space-time events $(x, y, z, t) \in \mathbb{R}^4$.

However, on the way of obtaining (1.14), one also obtained (1.11) which is the *additivity* of the coordinate transformations (1.2) - (1.5).

And then, we are already in a significantly *particular* situation regarding the coordinate transformations (1.2) - (1.5). Indeed, in the case of additive mappings there is a close connection between continuity and boundedness.
This is precisely why in [3] it was possible to ask for the boundedness condition (1.16), in order to obtain (1.15) from (1.14).

It could therefore at first sight appear that, given the additivity of the coordinate transformations (1.2) - (1.5), there is not much interest in distinguishing between continuity and boundedness conditions on such transformations.

Nevertheless, at a more careful consideration, it appears that there may indeed be such an interest. Namely, and as seen above, by choosing various conditions, say $(\mathcal{C})$, which may have not only a mathematical, but also a physical meaning, and choosing them such that

$$(7.1) \qquad (1.14) + (\mathcal{C}) \quad \Longrightarrow \quad (1.15)$$

one can open up the possibility for a better insight into Special Relativity, and insight which may help in dealing with the mentioned disconnect.

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