The finite Yang-Laplace Transform in fractal space

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Abstract: In this paper, we establish finite Yang-Laplace Transform on fractal space, considered some properties of finite Yang-Laplace Transform.

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1 Introduction

Local fractional calculus has played an important role in areas ranging from fundamental science to engineering in the past ten years [1-18]. It is significant to deal with the continuous functions (fractal functions), which are irregular in the real world. Recently, Yang-Laplace transform based on the local fractional calculus was introduced [9] and Yang continued to study this subject [10]. The Yang-Laplace transform of \( f(x) \) is given by [9,10]

\[
L_{\alpha} \{ f(x) \} = f_{s}^{L,\alpha}(s) := \frac{1}{\Gamma(1+\alpha)} \int_{0}^{\infty} E_{\alpha}(-s^{\alpha}x^{\alpha}) f(x)(dx)^{\alpha}, \quad 0 < \alpha \leq 1,
\]

(1.1)

And its Inverse formula of Yang-Laplace’s transforms as follows

\[
f(x) = L_{\alpha}^{-1} \{ f_{s}^{L,\alpha}(s) \} := \frac{1}{(2\pi)^{\alpha}} \int_{\beta-i\infty}^{\beta+i\infty} E_{\alpha}(s^{\alpha}x^{\alpha}) f_{s}^{L,\alpha}(s)(ds)^{\alpha}
\]

(1.2)

The purpose of this paper is to establish the finite Yang-Laplace Transforms based on the Yang-Laplace transforms and consider its some properties.

2 The Finite Yang-Laplace Transform and its properties

In the section, both finite Yang-Laplace transform and its inverse are defined from the corresponding Yang-Laplace transform and its inverse.

Definition 2.1 (The Finite Yang-Laplace Transform). If \( f(x) \) is a continuous or piecewise continuous function on a finite interval \( 0 < x < T \), the finite local fractional Laplace transform of \( f(x) \) is defined by

\[
L_{\alpha,T} \{ f(x) \} = f_{s}^{L,\alpha}(s,T) := \frac{1}{\Gamma(1+\alpha)} \int_{0}^{T} E_{\alpha}(-s^{\alpha}x^{\alpha}) f(x)(dx)^{\alpha}
\]

(2.1)

where \( s \) is a real or complex number and \( T \) is a finite number that may be positive or negative so that (2.1) can be defined in any interval \((-T, T)\). Clearly, \( L_{\alpha,T} \) is a linear integral transformation.

The inverse finite Yang-Laplace transform is defined by the complex integral

\[
f(x) = L_{\alpha,T}^{-1} \{ f_{s}^{L,\alpha}(s,T) \} := \frac{1}{(2\pi)^{\alpha}} \int_{\beta-i\infty}^{\beta+i\infty} E_{\alpha}(s^{\alpha}x^{\alpha}) f_{s}^{L,\alpha}(s,T)(ds)^{\alpha}
\]

(2.2)
where the integral is taken over any open contour $\Gamma$ joining any two points $\beta - iR$ and $\beta + iR$ in the finite complex $s$ plane as $R \to \infty$.

If $f(x)$ is almost piecewise continuous, that is, it has at most a finite number of simple discontinuities in $0 \leq x \leq T$. Moreover, in the intervals where $f(x)$ is continuous, it satisfies a Lipschitz condition of order $\gamma > 0$. Under these conditions, it can be shown that the inversion integral (2.2) is equal to

$$
\frac{1}{(2\pi i)^{\alpha}} \int_{\Gamma} E_{\alpha}(s^a x^a) f_s^{L,\alpha}(s, T)(ds)^\alpha = \frac{1}{2} [f(x - 0) + f(x + 0)]
$$

(2.3)

where $\Gamma$ is an arbitrary open contour that terminates with finite constant $\beta$ as $R \to \infty$. This is due to the fact that $f_s^{L,\alpha}(s, T)$ is an entire function of $s$.

**Example 2.1** if $f(x) = 1$, then

$$
L_{\alpha, T} \{1\} = -\frac{1}{s^\alpha} E_{\alpha}(-s^a x^a) \Big|_{s=0} = -[E_{\alpha}(-s^a T^a) - 1] = [1 - E_{\alpha}(-s^a T^a)]
$$

(2.4)

**Example 2.2** if $f(x) = E_{\alpha}(a^a x^a)$,

$$
L_{\alpha, T} \{E_{\alpha}(a^a x^a)\} = -\frac{1}{s^\alpha - a^\alpha} [E_{\alpha}(-s^a(a^a T^a) - 1]

= \frac{1}{s^\alpha - a^\alpha} [1 - E_{\alpha}(-(s^a - a^a) T^a)]
$$

(2.5)

**Theorem 2.1** if $L_{\alpha, T} \{f(x)\} = f_s^{L,\alpha}(s, T)$, then

$$
L_{\alpha, T} \{E_{\alpha}(-s^a x^a) f(x)\} = \tilde{f}_s^{L,\alpha}(s + a, T)
$$

(Shifting) (2.6)

$$
L_{\alpha, T} \{f(ax)\} = \frac{1}{a^\alpha} \tilde{f}_s^{L,\alpha} \left(\frac{s}{a}, aT\right)
$$

(Scaling) (2.7)

**Proof**

$$
L_{\alpha, T} \{E_{\alpha}(-s^a x^a) f(x)\} = \frac{1}{\Gamma(1 + \alpha)} \int_0^\gamma E_{\alpha}(-s^a x^a) E_{\alpha}(-s^a x^a) f(x)(dx)^\alpha

= \frac{1}{\Gamma(1 + \alpha)} \int_0^\gamma E_{\alpha}(-(s + a)^a x^a) f(x)(dx)^\alpha = \tilde{f}_s^{L,\alpha}(s + a, T)
$$

Let $y = ax$, we have

$$
L_{\alpha, T} \{f(ax)\} = \frac{1}{\Gamma(1 + \alpha)} \int_0^\gamma E_{\alpha}(-s^a x^a) f(ax)(dx)^\alpha

= \frac{1}{a^\alpha \Gamma(1 + \alpha)} \int_0^{a\gamma} E_{\alpha}(-\frac{s^a}{a^a} x^a) f\left(\frac{s}{a}\right)(dy)^\alpha = \tilde{f}_s^{L,\alpha} \left(\frac{s}{a}, aT\right)
$$

**Theorem 2.2** (Finite local fractional Laplace Transforms of Derivatives).

if $L_{\alpha, T} \{f(x)\} = f_s^{L,\alpha}(s, T)$, then

$$
L_{\alpha, T} \{f^{(\alpha)}(x)\} = s^\alpha \tilde{f}_s^{L,\alpha}(s, T) - f(0) + E_{\alpha}(-s^a T^a) f(T)
$$

(2.8)
More generally,

\[ L_{\alpha,T} \{ f^{(\alpha)}(x) \} = s^{\alpha} \tilde{f}_s^{L,\alpha}(s,T) - s^{\alpha} f(0) - f^{(\alpha)}(0) + s^{\alpha} f(T) E_{\alpha}(-s^{\alpha}T^{\alpha}) + f^{(\alpha)}(T) E_{\alpha}(-s^{\alpha}T^{\alpha}) \]  \hspace{1cm} (2.9) \]

**Proof.** Integrating by parts, we have

\[ L_{\alpha,T} \{ f^{(\alpha)}(x) \} = s^{\alpha} \tilde{f}_s^{L,\alpha}(s,T) - \sum_{k=1}^{n} s^{(n-k)\alpha} f^{((n-k)\alpha)}(0) + E_{\alpha}(-s^{\alpha}T^{\alpha}) \sum_{k=1}^{n} s^{(n-k)\alpha} f^{((n-k)\alpha)}(T) \]  \hspace{1cm} (2.10) \]

**Theorem 2.3** (Finite local fractional Laplace Transform of Integrals). If

\[ F(x) = \frac{1}{\Gamma(1+\alpha)} \int_0^x f(t)(dt)^\alpha \]  \hspace{1cm} (2.11) \]

so that \( F^{(\alpha)}(x) = f(x) \) for all \( x \), then

\[ L_{\alpha,T} \{ F(x) \} = \frac{1}{\Gamma(1+\alpha)} \int_0^\tau E_{\alpha}(-s^{\alpha}x^{\alpha})F(x)(dx)^\alpha \]

**Proof.** We have from (2.10)

\[ L_{\alpha,T} \{ F^{(\alpha)}(x) \} = s^{\alpha} L_{\alpha,T} \{ F(x) \} - F(0) + E_{\alpha}(-s^{\alpha}T^{\alpha}) F(T) \]

Or

\[ L_{\alpha,T} \{ f(x) \} = s^{\alpha} L_{\alpha,T} \{ \frac{1}{\Gamma(1+\alpha)} \int_0^\tau f(t)(dt)^\alpha \} + E_{\alpha}(-s^{\alpha}T^{\alpha}) F(T) \]

Hence

\[ L_{\alpha,T} \left\{ \frac{1}{\Gamma(1+\alpha)} \int_0^\tau f(t)(dt)^\alpha \right\} = \frac{1}{s^{\alpha}} [L_{\alpha,T} \{ f(x) \} - E_{\alpha}(-s^{\alpha}T^{\alpha}) F(T)] \]

\[ L_{\alpha,T} \{ F(x) \} = \frac{1}{\Gamma(1+\alpha)} \int_0^\tau E_{\alpha}(-s^{\alpha}x^{\alpha})F(x)(dx)^\alpha \]

**Theorem 2.4** if \( L_{\alpha,T} \{ f(x) \} = \tilde{f}_s^{L,\alpha}(s,T) \), then

\[ \frac{d^\alpha}{ds^\alpha} \tilde{f}_s^{L,\alpha}(s,T) = L_{\alpha,T} \{ (-x)^{\alpha} f(x) \} \]  \hspace{1cm} (2.12) \]

\[ \frac{d^{2\alpha}}{ds^{2\alpha}} \tilde{f}_s^{L,\alpha}(s,T) = L_{\alpha,T} \{ (-x)^{2\alpha} f(x) \} \]  \hspace{1cm} (2.13) \]

More generally,

\[ \frac{d^{n\alpha}}{ds^{n\alpha}} \tilde{f}_s^{L,\alpha}(s,T) = L_{\alpha,T} \{ (-x)^{n\alpha} f(x) \} \]  \hspace{1cm} (2.14) \]

**Proof.**
Similarly, we obtain (2.13) and (2.14).

References