Calculation of the Area of a Sampled Boundary Using Fourier Components

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Abstract. A reformulation of the area of a planar two-dimensional object in the frequency domain allows for the computation of the true area of a band-limited boundary to be calculated.

Introduction

Represent the boundary of a planar object as an ordered set of points on the complex plane, \{u[j]\}, with j an integer in the range (0,N-1) as discussed in [1]. Applying the discrete Fourier transform to these points gives an equivalent set of points \(U[n]\), with n in the range (0,N-1). The area swept out by traversing such a boundary in an anti-clockwise direction, is the sum of all of the elementary trapezoids between adjacent boundary points and the x-axis, such as those identified by the four points \(x[j], u[j], u[j + 1], x[j + 1]\) in Fig. 1.

![Diagram](image)

Figure 1

The area of the object is the sum of these elements for all j:

\[
A = \sum_{j=0}^{N-1} \frac{1}{2} (x[j] - x[j + 1])(y[j] + y[j + 1])
\]

\[
\therefore A = \frac{1}{2} \sum_{j=0}^{N-1} (x[j]y[j + 1] - x[j + 1]y[j])
\]

Expressing this equation in terms of points on the complex plane gives:

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Transforming to the Frequency Domain

Transforming to the frequency domain using the general form of Parseval’s theorem, we get:

\[
A = \frac{1}{4i} \sum_{j=0}^{N-1} (u^*[j]u[j+1] - u[j]u^*[j+1])
\]

The sums involved remain true over any \(N\) consecutive points because of the cyclic continuation of the boundary and of its’ discrete Fourier transform; in particular, over the \(N\) points symmetrically disposed about zero.

If \(N\) is even, add an additional element with zero intensity at the midpoint so that the total number of elements becomes an odd number.

We now use this alternative form

This is the area defined by an \(N\) point polygonal. If the boundary is band limited in the frequency domain, or is made so by some pre-processing operation, we can increase the number of points represented by extending the spectrum with additional zero power elements as explained in an earlier paper [1]. In the limit, we can calculate the true area of a band limited boundary by increasing the number of points to infinity:

\[
A = \lim_{N \to \infty} \left( \frac{N}{2} \sum_{n=0}^{N-1} |U[n]|^2 \sin \left( \frac{2\pi n}{N} \right) \right)
\]

\[
A = \lim_{N \to \infty} \left( \frac{N}{2} \sum_{n} |U[n]|^2 \left( \frac{2\pi n}{N} + \frac{1}{3!} \left( \frac{2\pi n}{N} \right)^3 + \cdots \right) \right)
\]

\[
A = \pi \sum_{n} n|U[n]|^2
\]