Symmetries in Wigner 18-j and 21-j Symbols

Richard J. Mathar*
Sterrewacht Leiden, P.O. Box 9513, 2300 RA Leiden, The Netherlands
(Dated: February 29, 2012)

The symmetry group of the 18-j(H) Wigner symbol is restructured by splitting two symmetry equations (Yutsis et al. 1962) into three generators. The symmetry groups of two 21-j Wigner symbols (Ponzano 1965) are complemented to form groups of order 8. This summarizes systematic evaluation of the automorphisms of the associated simple cubic graphs with McKay's `nauty` program.

PACS numbers: 03.65.Fd, 31.15.xh, 02.10.Ox
Keywords: Angular Momentum Coupling, Wigner 18-j Symbol, Graphical Analysis

I. INTRODUCTION

A. Motivation

Following up on a tabulation of "irreducible" Wigner graphs complete up to the 21-j symbols [1], one of the symmetry operations of one 18-j symbol, labeled H in the reference [2], appeared to be missing.

Understanding that visual inspection of these trivalent graphs is tricky, this report shows the results of an industrial-scale investigation of the 18-j and 21-j symbols (graphs on 12 and 14 vertices) with Brendan McKay's `nauty` program [3, 4].

B. Notation

In the graphical representation of angular momentum coupling, the angular momenta are represented by $3n$ edges in a trivalent (commonly referred to as "cubic") graph [2, 5]. The number of edges is a multiple of three, and the number of vertices is $2n$.

In an effort of economizing of the notation, the vertices v_i will be enumerated from 0 up to $2n-1$. The angular momenta j carry two subscripts which show the two vertices they connect. In the standard literature, the symmetry operations describe which edge label is mapped onto which other edge label. Here, in the vertex-oriented notation, the operations are considered permutations of the vertex labels. Permutations are written in cycle notation $(v_1\ v_2\ \ldots v_i)(v_j\ v_k\ \ldots v_k)\ldots$, which specifies vertex replacements $v_1 \to v_2$, $v_2 \to v_3$,$\ldots v_i \to v_1$, $v_j \to v_k \ldots v_k \to v_j$ cyclically within each of the parentheses. Vertices that remain fixed are not written down.

Symmetry operations are transcribed back to the original notation of the edges by applying the operation separately to the two indices of each j-label. If the two indices emerge in the opposite order, a sign flip in the edge orientation is also induced, which generates a phase factor $(-1)^{j_{\cdot\cdot}}$ according to the rules of the graphical evaluation. We shall omit the $+$ and $-$ symbols of handedness on the vertices; phase factors that result from the restructuring of these will be ignored. In that sense, the symmetries reported below conserve absolute values, but not necessarily signs of the $3n$-j symbols.

II. SPLIT SYMMETRY FOR THE 18j(H) SYMBOL

The representation of one (out of 18) 18-j symbol is shown in Figure 1. (The labels in this manuscript generally differ from my table [1] and have been renumbered along a Hamiltonian circle in the planar view of the reference publications.) Labels of the reference work [2] emerge renamed in the economized vertex notation as follows: $j_{0,1} = k_1$, $j_{0,8} = k_1'$, $j_{0,11} = k_1''$, $j_{1,2} = j_2$, $j_{5,1} = j_1$, $j_{3,2} = k_2$, $j_{2,7} = j_3$, $j_{3,4} = k_2''$, $j_{3,9} = k_2'$, $j_{4,5} = j_3''$, $j_{11,4} = j_2''$, $j_{6,5} = k_3''$, $j_{6,7} = k_3$, $j_{6,10} = k_3'$, $j_{7,8} = j_1'$, $j_{8,9} = j_2'$, $j_{9,10} = j_3'$, $j_{10,11} = j_1''$.

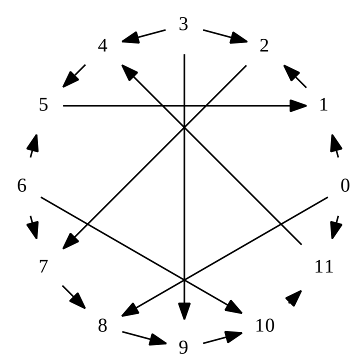

FIG. 1. 18-j class H from [2, Fig A. 3.8]. LCF notation [6,-5,-4,4,-5,4,6,-4,5,-4,4,5] [1, Fig 15].

Three generators for permutations of labels that generate a group of 18 symmetries (isomorphic to the Dihedral Group D_{18}) for the graph are $g_1 \equiv$(1 11 8)(2 4 9)(5 10 7), $g_2 \equiv$(2 5)(3 6)(4 7)(8 11)(9 10) and $g_3 \equiv$(0 3)(1 2)(4 8)(5 7)(9 11).

This graph has been singled out from the set of 18 different 18-j symbols because only two symmetry relations have been published [2, A 4.8]:

---

* http://www.strw.leidenuniv.nl/~mathar

- The first part of equation [2, A 4.8] mixes edges in two long cycles $j_1 \to j_3'' \to j_2'' \to j_1'' \to j_3' \to j_2' \to j_1' \to j_3 \to j_2 \to j_1$ and $k_1 \to k_3'' \to k_2'' \to k_1'' \to k_3' \to k_2' \to k_1' \to k_3 \to k_2 \to k_1$. Translation to vertex labels with the list above says these equal $j_{5,1} \to j_{4,5} \to j_{11,4} \to j_{10,11} \to j_{9,10} \to j_{8,9} \to j_{7,8} \to j_{2,7} \to j_{1,2} \to j_{5,1}$ and $j_{0,1} \to j_{6,5} \to j_{3,4} \to j_{0,11} \to j_{6,10} \to j_{3,9} \to j_{0,8} \to j_{6,7} \to j_{3,2} \to j_{0,1}$. This is rewritten as the vertex permutation (1 5 4 11 10 9 8 7 2)(3 0 6), which is the element $g_2 g_3$ of the permutation group—application of g_3, then g_2—in terms of the three generators defined above.

- The second part of equation [2, A 4.8] swaps edges in pairs, $j_1 \leftrightarrow j_1'$, $j_2 \leftrightarrow j_3$, $j_2' \leftrightarrow j_3''$, $j_3' \leftrightarrow j_2''$, $k_1 \leftrightarrow k_3$, $k_1' \leftrightarrow k_3''$, $k_2' \leftrightarrow k_2''$, $k_3' \leftrightarrow k_1''$. In vertex-notation the maps are $j_{5,1} \leftrightarrow j_{7,8}$, $j_{1,2} \leftrightarrow j_{2,7}$, $j_{8,9} \leftrightarrow j_{4,5}$, $j_{9,10} \leftrightarrow j_{11,4}$, $j_{0,1} \leftrightarrow j_{6,7}$, $j_{0,8} \leftrightarrow j_{6,5}$, $k_{3,9} \leftrightarrow j_{3,4}$, $k_{6,10} \leftrightarrow k_{0,11}$, equivalent to the permutation (0 6)(1 7)(4 9)(5 8)(10 11), which is the group element $g_3 g_2 g_3$.

If we call the two permutations (symmetries) of the two parts of the equation $h_1 \equiv g_2 g_3$ and $h_2 \equiv g_3 g_2 g_3$, these are two generators of the full group of with 18 elements. In particular $g_1 = h_1^3$, $g_2 = h_2 h_1^{-2}$, and $g_3 = h_1 h_2$, where h_1^{-2} is the two-fold application of the inverse of h_1 and h_1^3 the three-fold application of h_1. The description of the group by the three generators g_1, g_2 and g_3 is nicer, because they have more fixed points and contain cycles of lower order.

The symmetry generated by iterating g_1 is illustrated by re-drawing the graph as in Figure 2: it sends vertex 1 to 11 and then to 8, and in parallel 2 to 4 and 9, with 5 to 10 and 7. The three vertices 3, 6 and 0 stay in the middle of the picture, fixed. They are essentially handing over their edges by rotating the 9 vertices in the "outer" cycle 1-2-7-8-9-10-11-4-5-1 counter-clockwise in a modulo-3 pattern.

Return to the original notation is easy with the list of associations noted above. Example: as g_1 maps 10 to 7 and 11 to 8, it maps $j_{10,11}$ to $j_{7,8}$, therefore j_1'' to j_1'.

One might ask: is the mirror operation which swaps the right and left vertices in Figure 2 as $8 \leftrightarrow 9$, $7 \leftrightarrow 10$, $0 \leftrightarrow 3$, $2 \leftrightarrow 11$, $1 \leftrightarrow 4$, not missing? This symmetry is actually represented by the group member $g_1 g_3$, i. e., application of (0 3)(1 2)(4 8)(5 7)(9 11) followed by (1 11 8)(2 4 9)(5 10 7).

Similarly, the symmetry operation which mirrors the elements in Figure 1 from the left to the right is $g_3 g_1^{-1} g_2 g_3$, where g_1^{-1} denotes the inverse operation of g_1 (shifting elements cyclically left in each parenthesis).

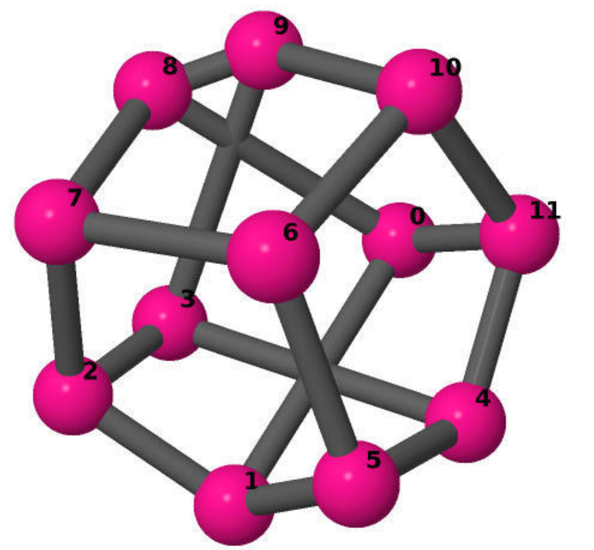

FIG. 2. 18-j symbol H, illustrating the symmetry (1 11 8)(2 4 9)(5 10 7) where 0 acts as the symmetry center for the $1 \to 11 \to 8$ cycle, 3 as the symmetry center for the $2 \to 4 \to 9$ cycle, and 6 for the $5 \to 10 \to 7$ cycle.

TABLE I. Orders of the Automorphism groups of all 18-j symbols.

$\vert\mathcal{A}\vert$	18-j symbol [2]
48	C
24	A, B
18	H
16	F, G
8	D, E, I, K
4	L, M, N, P
2	R, S, T, V

III. PONZANO'S 21-j SYMBOLS

A. 21-j(1)

We tersely review the eight 21-j symbols that have been mentioned in a previous publication [6].

Ponzano's 21-j(1) symbol has a symmetry group of order 2 with generator (3 5)(6 8)(9 11), as published, if referring to the labeling of Figure 3.

B. 21-j(2)

Ponzano's 21-j(2) symbol has a symmetry group of order 8, isomorphic to the direct product $C_2 \times C_2 \times C_2$. The three generators from `nauty` are (1 13)(2 3)(4 5)(6 8)(9 10)(11 12), (2 11)(3 12)(4 9)(5 10) and (0 7)(1 8)(2 4)(3 5)(6 13)(9 11)(10 12), referring to the labeling convention of Figures 4 and 5. The generators are two mirror

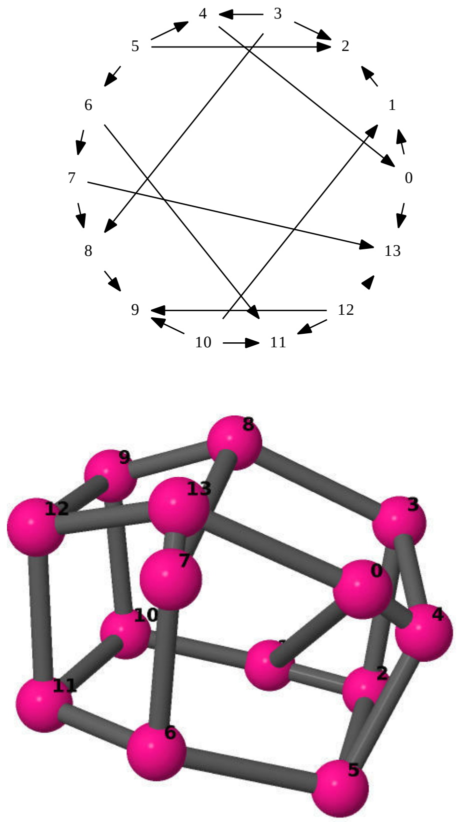

FIG. 3. 21-j(1) symbol $[-5, 5, -5, 7, -6, 3, -5, 6, -3, 5, 7, 5, 6, -6]$ [6].

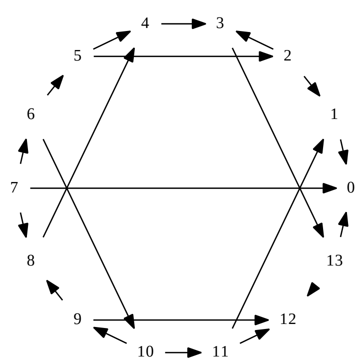

FIG. 4. 21-j(2) symbol [6].

plane and one inversion operation on the graph— obvious symmetries in the two figures.

The number of three equations by Ponzano [6, (2)] matches the number of generators, so they fully catch the symmetry.

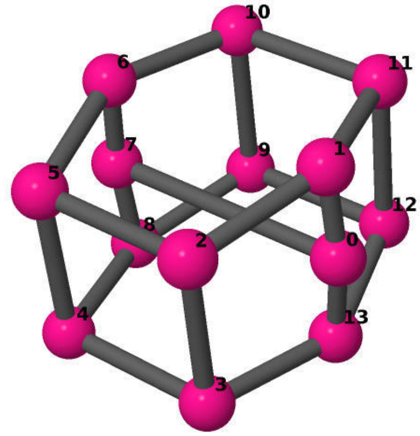

FIG. 5. 21-j(2) symbol $[7, -4, 3, -4, 4, -3, 4]^2$, 3D view of Figure 4.

C. 21-j(3)

Ponzano's 21-j(3) symbol has a symmetry group of order 8, isomorphic to the Dihedral Group D_8. Three generators proposed by `nauty` are $g_1 \equiv$(0 2)(3 13)(4 7)(5 8)(6 9), $g_2 \equiv$(0 3)(1 12)(2 13)(4 8)(5 7), and $g_3 \equiv$(0 4 13 5)(1 9 12 6)(2 8 3 7)(10 11). The translation of the vertex-labeled nomenclature proposed in Figure 6 to the edge-labeled nomenclature is: $j_{1,0} = k_1'$, $j_{1,11} = k_2'$, $j_{2,1} = k_3'$, $j_{2,3} = p'$, $j_{3,4} = j$, $j_{3,12} = j_3'$, $j_{4,9} = j_3$, $j_{5,2} = k$, $j_{5,4} = p$, $j_{5,6} = k_3$, $j_{6,7} = k_1$, $j_{6,10} = k_2$, $j_{8,0} = g$, $j_{8,7} = q$, $j_{9,8} = j_1$, $j_{10,9} = j_2$, $j_{10,11} = n$, $j_{12,11} = j_2'$, $j_{12,13} = j_1'$, $j_{13,0} = q'$, $j_{13,7} = h$.

Only two equations are found in the reference article [6].

- The first implies the right-left mirror operation (0 7)(1 6)(2 5)(3 4)(8 13)(9 12)(10 11), which is $g_3^{-1}g_1$ in terms of the three generators.
- The second equation is the map $k_3 \leftrightarrow j_3$, $k_1 \leftrightarrow j_1$, $k_2 \leftrightarrow j_2$, $k_2' \leftrightarrow j_2'$, $k \leftrightarrow j$, $k_1' \leftrightarrow j_1'$, $k_3' \leftrightarrow j_3'$, $h \leftrightarrow g$, which translates to the vertex-oriented labeling $j_{5,6} \leftrightarrow j_{4,9}$, $j_{6,7} \leftrightarrow j_{9,8}$ $j_{6,10} \leftrightarrow j_{10,9}$, $j_{1,11} \leftrightarrow j_{12,11}$, $j_{5,2} \leftrightarrow j_{3,4}$, $j_{1,0} \leftrightarrow j_{12,13}$, $j_{2,1} \leftrightarrow j_{3,12}$, $j_{13,7} \leftrightarrow j_{8,0}$. This operation is found to be the permutation (1 12)(2 3)(6 9)(4 5)(7 8)(0 13), which is $g_2 g_1$.

These two equations combined generate only a subgroup of order 4, isomorphic to $C_2 \times C_2$, so by using g_1, g_2 and g_3 the symmetry is indeed extended.

It may be interesting to note that the symmetry $g_3 \equiv$(0 4 13 5)(1 9 12 6)(2 8 3 7)(10 11) maps vertex 0 to 4 and vertex 1 to 9, for example, therefore $j_{0,1} \to j_{4,9}$, which is $k_1' \to j_3$ in the original publication. This rearrangement induces a cross-mix of entries in the reduction to a sum over products of 9-j and 18-j symbols [6, (3)]. It cannot be stated to be a symmetry within the 9-j symbol or the 18-j symbol of the reduction, and this is one of the reasons why it might have been missed.

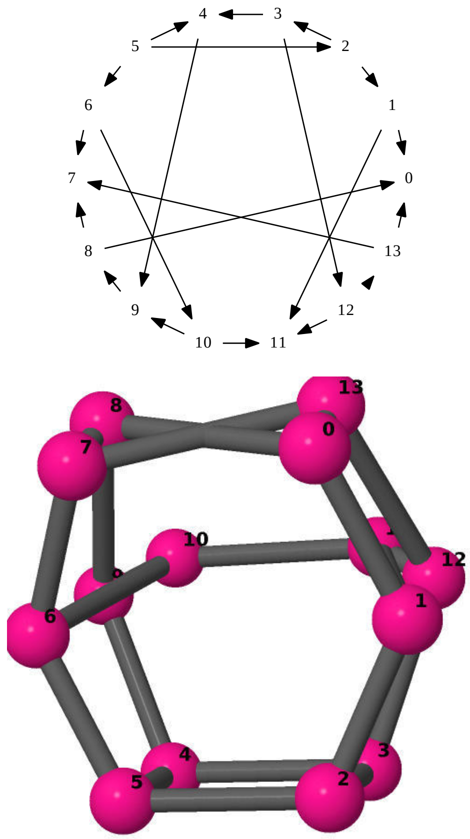

FIG. 6. 21-$j(3)$ symbol $[5, -3, 4, 6, 6, -5, -4; -]$ [6].

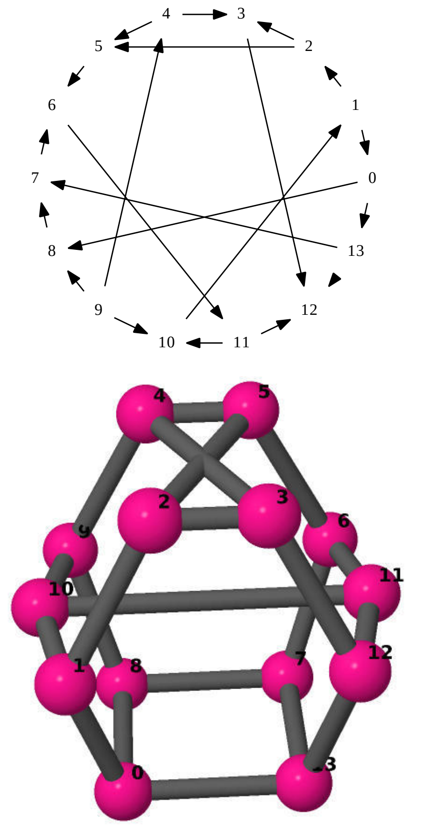

FIG. 7. 21-$j(4)$ symbol $[5, -3, 5, 6, 6, -5, 5; -]$ [6].

D. 21-$j(4)$

Ponzano's 21-$j(4)$ symbol has a symmetry group of order 4, isomorphic to the direct product $C_2 \times C_2$. `nauty` proposes the two generators (0 7)(1 6)(2 5)(3 4)(8 13)(9 12)(10 11), and (0 8)(1 9)(2 4)(3 5)(6 12)(7 13), which matches the number of equations given by Ponzano. The first generator represents the left-right mirror symmetry in Figure 7 (planar diagram), and the second a front-back mirror plane in Figure 7 (3D view).

E. 21-j(5)

Ponzano's 21-j(5) symbol has a symmetry group of order 4, isomorphic to the direct product $C_2 \times C_2$. The two generators can be chosen as $g_1 \equiv$(1 4)(2 5)(3 6)(9 12)(10 11) and $g_2 \equiv$(0 7)(1 3)(4 6)(8 13) using the labels in Figure 8. The number of generators matches the number of equations given by Ponzano. The left-right mirror symmetry in the planar view of Figure 8 is $g_2 g_1 =$(0 7)(1 6)(2 5)(3 4)(8 13)(9 12)(10 11).

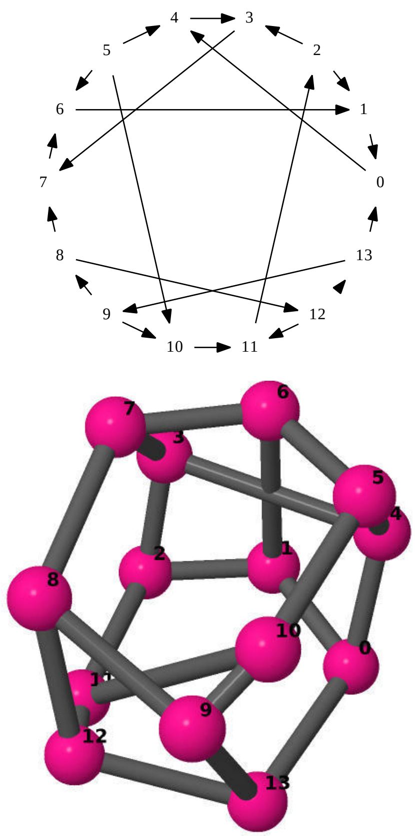

FIG. 8. 21-j(5) symbol $[5, 6, -5, 6, 6, -5, 6; -]$, planar and 3D view.

F. 21-j(6)

Ponzano's 21-j(6) symbol has a symmetry group of order 16, isomorphic to the direct product $C_2 \times D_8$. The four generators proposed by `nauty` are $g_1 \equiv$(3 5)(6 8)(9 11), $g_2 \equiv$(1 13)(2 4)(10 12), $g_3 \equiv$(2 10)(3 9)(4 12)(5 11), and $g_4 \equiv$(0 7)(1 8 13 6)(2 3 4 5)(9 12 11 10), assuming the labels of Figure 9.

Ponzano provided matching four equations. The first equation [6, (6)], for example, is characterized by the following substitutions of cycle order 2: $j_1 \leftrightarrow j_1'$, $j_2 \leftrightarrow j_2'$, $j_3 \leftrightarrow j_3'$, $j_4 \leftrightarrow j_4'$, $l_1 \leftrightarrow l_3$, $k_1 \leftrightarrow k_1'$, $k_2 \leftrightarrow k_2'$, $l_1' \leftrightarrow l_3'$, which represents the right-left mirror operation on the

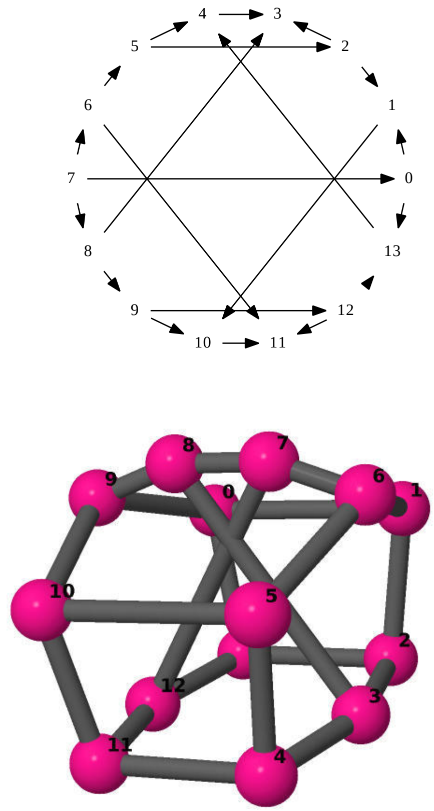

FIG. 9. 21-j(6) symbol $[-3, 5, 7, -5, 3, 5, -5]^2$, planar and 3D view.

associated planar view. The equivalent vertex-labeled permutation is (0 7)(1 6)(2 5)(3 4)(13 8)(12 9)(11 10) $= g_4^{-1} g_1$, applying g_1 followed by the inverse of g_4.

G. 21-j(7)

Ponzano's 21-j(7) symbol, Figure 10, has a symmetry group of order 1, containing only the identity. This matches the original article which gave no equation.

H. 21-j(8)

Ponzano's 21-j(8) symbol has a symmetry group of order 8, isomorphic to D_8. Three generators can be chosen as follows: $g_1 \equiv$(0 2)(3 13)(4 8)(5 7), $g_2 \equiv$(0 3)(1 12)(2 13)(4 5)(6 9)(7 8) and $g_3 \equiv$(0 4 2 8)(1 9)(3 7 13 5)(6 12)(10 11) if labels follow the convention of Figure 11.

Given only two equalities in the reference [6, (8)], at least one generator is missing:

- The first two terms in [6, (8)] represent the left-right symmetry in the planar view in Figure 11, realized by (0 7)(1 6)(2 5)(3 4)(8 13)(9 12)(10 11), which is $g_2 g_3^{-1}$ in terms of these three generators.

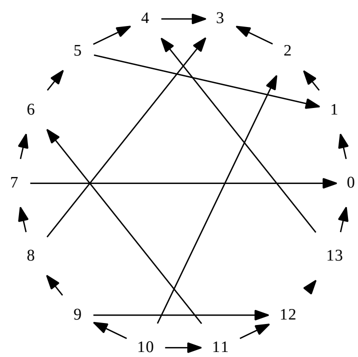

FIG. 10. 21-$j(7)$ symbol $[-5,-4,4,7,-5,3,-4,5,-3,5,7,4,-5,5]$ [6].

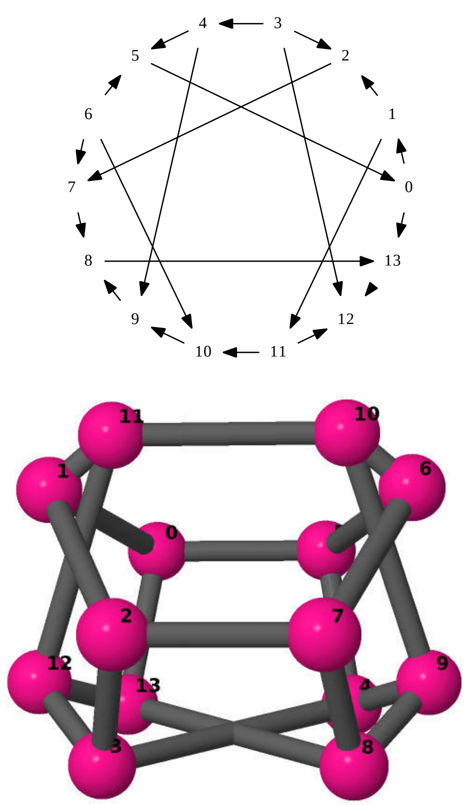

FIG. 11. $21j(8)$ symbol $[4,6,-5,5,-4,5,6;-]$ [6].

- The first and third term describe the operation $l_3 \leftrightarrow l'_3$, $l_1 \leftrightarrow l'_1$, $l_2 \leftrightarrow l'_2$, $h \leftrightarrow k$, $s \leftrightarrow t$, $j_2 \leftrightarrow j'_2$, $j_1 \leftrightarrow j'_1$, $j_3 \leftrightarrow j'_3$. In vertex-label notation, this is $j_{6,5} \leftrightarrow j_{0,1}$, $j_{4,9} \leftrightarrow j_{13,12}$, $j_{4,5} \leftrightarrow j_{0,13}$, $j_{10,9} \leftrightarrow j_{11,12}$, $j_{6,10} \leftrightarrow j_{1,11}$, $j_{3,2} \leftrightarrow j_{7,8}$, $j_{3,12} \leftrightarrow j_{9,8}$, $j_{1,2} \leftrightarrow j_{6,7}$, which represents the permutation (5 0)(6 1)(4 13)(9 12)(10 11)(3 8)(2 7), equivalent to the right-left symmetry in the 3D view in Figure 11. This permutation is the group element $g_3^{-1}g_2$ in terms of the three generators.

Altogether, these two symmetry elements construct a subgroup of only order 4, isomorphic to $C_2 \times C_2$.

IV. 21-j SYMBOLS OF HIGH SYMMETRY

A. Order 336

The symbol in Figure 12 has an automorphism group of order 336, isomorphic to the semi-direct product $PSL(3,2) \wedge C_2$, generated by (4 8)(5 7)(9 13)(10 12), (3 11)(4 10)(8 12)(9 13), (1 13)(2 4)(5 11)(6 12), and (0 1 2 3 4 5 10 9)(6 11 8 13)(7 12).

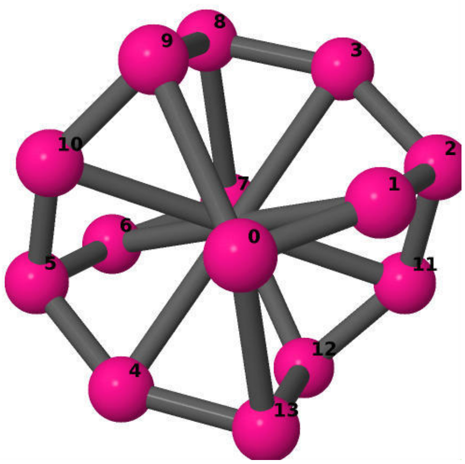

FIG. 12. 21-j symbol $[-5,5]^7$, the Heawood graph.

B. Order 28

The Möbius ladder graph in Figure 13 has an automorphism group of order 28, with generators (1 13)(2 12)(3 5)(4 6)(7 11)(8 10) and (0 1 2 3 4 11 10 9 8 7 6 5 12 13).

The graph $G_{7,2}$ in Figure 14 has also an automorphism group of order 28, with generators (2 7)(3 6)(4 5)(8 13)(9 12)(10 11), (0 1)(2 8)(3 9)(4 10)(5 11)(6 12)(7 13), and (0 2)(1 13)(3 8)(4 9)(5 10)(6 11)(7 12).

C. Order 16

The graph in Figure 15 has an automorphism group of order 16, with generators (7 9)(10 12), (3 5)(6 8), (2 11)(3 10)(4 13)(5 12)(6 7)(8 9), and (0 1)(2 4)(11 13). The first, second and last of the generators are easily understood as unwinding one of the crossings in the picture and recreating it at the adjacent position along the ring.

The graph in Figure 16 has an automorphism group also of order 16, with generators $g_1 \equiv$(1 13)(5 11)(6 12),

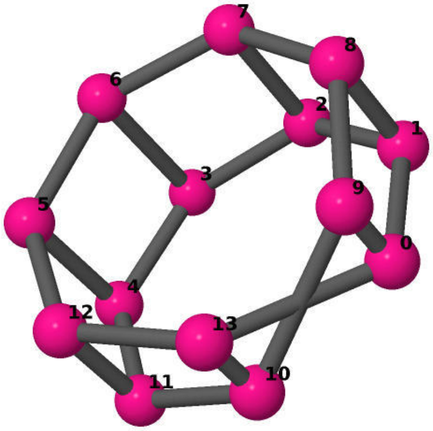

FIG. 13. 21-j symbol $[7]^{14}$, the Wigner 21-j symbol of the first kind.

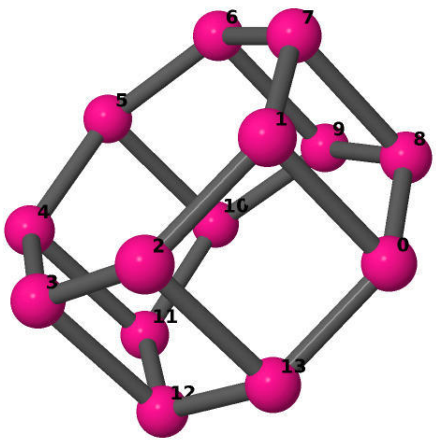

FIG. 14. 21-j symbol $[5, 3, -6, 6, -3, -5, 7]^2$, the Wigner 21-$j$ symbol of the second kind.

$g_2 \equiv$(0 1 2 13)(3 12 9 6)(4 11 10 5)(7 8), and $g_3 \equiv$(0 4)(1 5)(2 10)(3 9)(11 13). Two easily recognized mirror plane symmetries are (0 10)(1 5)(11 13)(2 4)$= g_3 g_2^{-2} g_1$ and (0 1)(2 13)(7 8)(6 9)(3 12)(5 10)(4 11)$= g_2 g_1$.

D. Order 14

Figure 17 is the sole Wigner 21-j graph with automorphism group of order 14, with generators $g_1 \equiv$(1 5)(2 6)(3 7)(4 8)(9 12)(10 11) and $g_2 \equiv$(0 1)(2 5)(3 4)(6 11)(7 12)(8 13). The left-right swap (0 5)(1 6)(2 10)(3 9)(4 13)(7 8) is given by the operation $g_1 g_2 g_1$.

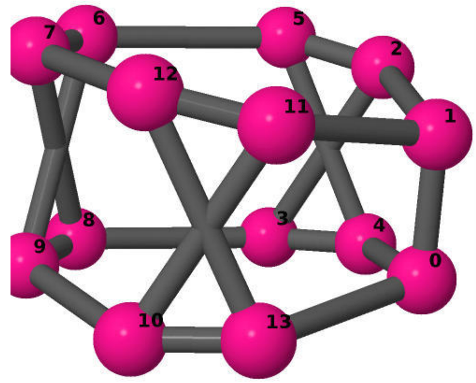

FIG. 15. 21-j symbol $[-4, 3, 5, -4, -3, 3, 5; -]$.

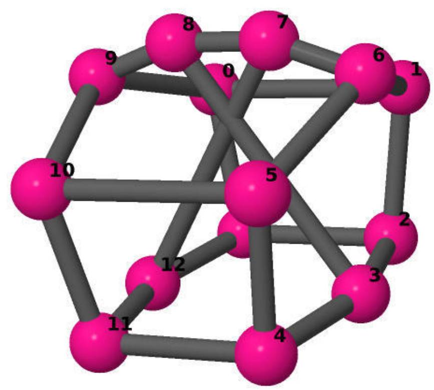

FIG. 16. 21-j symbol $[-3, 5, 7, -5, 3, 5, -5]^2$.

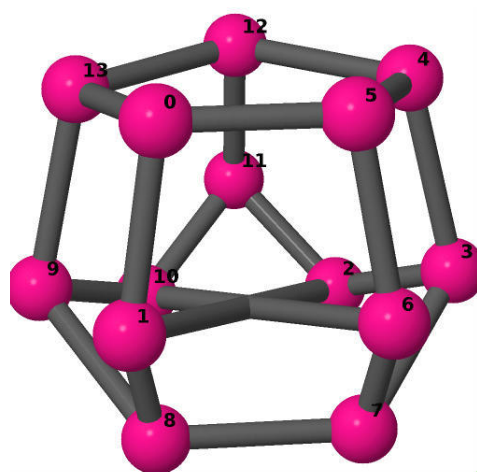

FIG. 17. 21-j symbol $[-5, 4, -4, 7, 4, -4, 5, 6, -4, 5, 7, -5, 4, -6]$.

E. Order 12

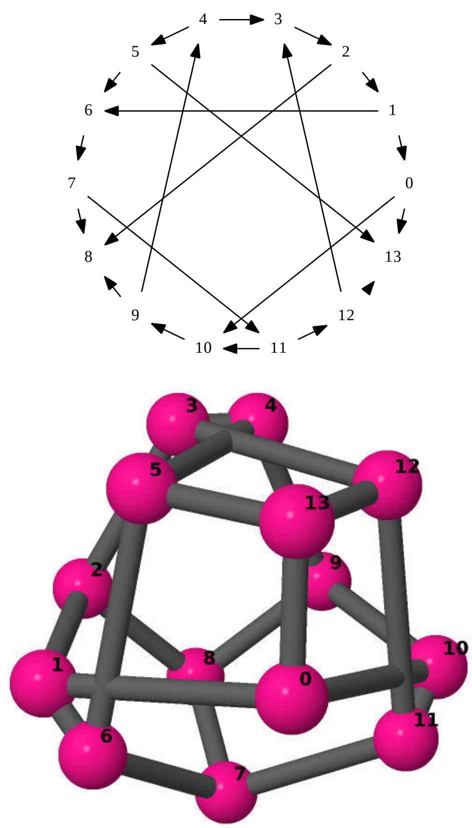

FIG. 18. 21-j symbol of Di-Leva and Ponzano $[-4,5,6,-4,5,6,-5;-]$ [7].

There are two 21-j symbols with 12 symmetries; only one is considered here. A symbol with an automorphism group of order 12, isomorphic to D_{12} is shown in Figure 18, generated by two generators, (1 10)(2 9)(3 4)(5 12)(6 11) and (0 2)(3 10)(4 11)(5 7)(8 13)(9 12). This implies that one of the three equalities in [7, (25)] is redundant—at least, if sign flips are left aside. The two generators are mirror operations along 2 of 3 equivalent planes which are easily recognized in the 3D view: the 4 vertices that are fixed and not noted in the permutation are defining the mirror plane.

V. SUMMARY

We dissassembled the symmetry operations of the 18-j symbol commonly labeled as H. We added one symmetry to Ponzano's 21-j symbol number (3), and one symmetry to Ponzano's 21-j symbol number (8).

ACKNOWLEDGMENTS

The planar views have been plotted with the `neato` program of `graphviz`. The ball-and-stick models are screenshots exported from the `Jmol` viewer.

Members of permutation groups have been factorized into generator products with `GAP4`.

Appendix A: Errata

The lower k_2' in the fourth column of the second term in [6, (6)] should read k_2.

The lower k_2 in the second column of the second term in [6, (6)] should read k_2'.

[1] R. J. Mathar, arXiv:1109.2358 [math-ph] (2011).
[2] A. P. Yutsis, I. B. Levinson, V. V. Vanagas, and A. Sen, *Mathematical Apparatus of the Theory of Angular Momentum* (Israel program for scientific translations, 1962).
[3] B. D. McKay, Congr. Numerantium **30**, 45 (1981).
[4] B. D. McKay and G. F. Royle, Ars Combin. **21A**, 129 (1986).
[5] K.-N. Huang, Rev. Mod. Phys. **51**, 215 (1979).
[6] G. Ponzano, Nuovo Cimento **36**, 385 (1965).
[7] A. Di Leva and G. Ponzano, Nuovo Cimento **51A**, 1107 (1967).