Generalized Fermat primes $p$ Such That 3 is a Primitive Root Modulo $p$

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Abstract: We explore property of generalized Fermat primes of the form:

$$F_n(2 \cdot q) = (2 \cdot q)^{2^n} + 1,$$

where $n > 1$ and $q$ is an odd prime number.

1. Introduction

Generalized Fermat numbers of the form $a^{2^n} + 1$ with $a > 2$, see [1], are a generalization of usual Fermat numbers $2^{2^n} + 1$. Generalized Fermat numbers can be prime only for even $a$, because if $a$ is odd then every generalized Fermat number will be divisible by 2. Many of the known largest prime numbers are generalized Fermat primes. A primitive root of a prime $p$ is an integer $g$ such that $g \pmod{p}$ has modulo order $p-1$, see [2]. It is known that 3 is a primitive root modulo $p$ for every Fermat prime $F_n$ with $n > 0$. In this paper we explore for which conditions 3 is a primitive root modulo $p$ for generalized Fermat primes $F_n(a)$ with $n > 1$.

2. Conjectures

2.1. Definition: Let $p$ be a prime number of the form $p = F_2(2 \cdot q) = (2 \cdot q)^{2^2} + 1$, where $q$ is an odd prime number.

Consider the output of the following Maple code:
2.1. **Conjecture**: If $q$ is a greater than 3 then 3 is a primitive root modulo $p$ for all $p$.

2.2. **Definition**: Let $p$ be a prime number of the form $p = \mathcal{F}_3(2 \cdot q) = (2 \cdot q)^{23} + 1$, where $q$ is an odd prime number.

Consider the output of the following Maple code:

```maple
with(numtheory):
i:=0:
n:=2:
for q from 1 to 8640200 do
if isprime(q) then
if isprime((2*q)^(2^n)+1) then
i:=i+1:
if not (primroot((2*q)^(2^n)+1)=3) then
print(q);
end if;
end if;
end if;
end do;
i;
```

```
3
48997
```

2.2. **Conjecture**: If $q$ is a greater than 271 then 3 is a primitive root modulo $p$ for all $p$.
2.3. **Definition**: Let $p$ be a prime number of the form $p = 2^q + 1$, where $q$ is an odd prime number. Consider the output of the following Maple code:

```maple
with(numtheory):
i:=0:
n:=4:
for q from 1 to 640200 do
  if isprime(q) then
    if isprime((2*q)^(2^n)+1) then
      i:=i+1:
      if not(primroot((2*q)^(2^n)+1)=3) then
        print(q);
      end if;
    end if;
  end if;
end do;
i;
```

2.3. **Conjecture**: 3 is a primitive root modulo $p$ for all $p$.

2.4. **Definition**: Let $p$ be a prime number of the form $p = 2^q + 1$, where $q$ is an odd prime number. Consider the output of the following Maple code:

```maple
with(numtheory):
i:=0:
n:=5:
for q from 1 to 840200 do
  if isprime(q) then
    if isprime((2*q)^(2^n)+1) then
      i:=i+1:
      if not(primroot((2*q)^(2^n)+1)=3) then
        print(q);
      end if;
    end if;
  end if;
end do;
i;
```

2.3. **Conjecture**: 3 is a primitive root modulo $p$ for all $p$.
2.4. **Conjecture**: 3 is a primitive root modulo $p$ for all $p$.

**References**

1. Definition of generalized Fermat number available at:
   http://mathworld.wolfram.com/GeneralizedFermatNumber.html
2. Definition of primitive root modulo $p$ available at:
   http://en.wikipedia.org/wiki/Primitive_root_modulo_n

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