

FOREWORD

When I wrote my original document *Números Primos de Sophie Germain, Demostración de su Infinitud* (There Are Infinitely Many Sophie Germain Prime Numbers), I used the '**Breusch Interval**' in order to prove theorems and to prove that if i is any positive odd integer, then there are infinitely many prime numbers of the form $2p+i$ (where p is a prime number) and infinitely many prime numbers of the form $2p-i$.

I took the Breusch Interval from the following websites created by **Carlos Giraldo Ospina** (Lic. Matemáticas, USC, Cali, Colombia):

<http://www.matematicainsolita.8m.com/>
<http://matematicainsolita.8m.com/Archivos.htm>
<http://numerosprimos.8m.com/Documentos.htm>

C. G. Ospina utilized the Breusch Interval in some of his documents to prove some specific conjectures. This author makes reference to the article *Números Primos, Balance de Nuestra Ignorancia* written a long time ago by Joaquín Navarro, where this person mentions the Breusch Interval and that it was proved that there is always a prime number in this interval.

One of the people who read my document told me that they had never heard about a so-called 'Breusch Interval' before, and that they were unable to find any information on this interval in any Mathematics book.

For this reason, I wrote another document where I proved that for every positive integer n there is always a prime number in the Breusch Interval. To do this, I used another interval which I call '**Iwaniec-Pintz Interval**'.

Rather than rewriting my document *Números Primos de Sophie Germain, Demostración de su Infinitud* using the Iwaniec-Pintz Interval, it was easier for me to prove that there is always a prime number in the Breusch Interval for every positive integer n .

The present document is much more general, because here I prove that if a and b are two positive integers which are coprime and also have different parity, then there are infinitely many prime numbers of the form $ap+b$ and infinitely many prime numbers of the form $ap-b$. In order to prove this, I use the Breusch Interval. This is the reason why this document starts with giving a proof that the Breusch Interval contains at least one prime number for every positive integer n .

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Breusch Interval

The interval $\left[n, \frac{9(n+3)}{8} \right]$, where ' n ' is a positive integer, is what we call '**Breusch Interval**'.

We are going to prove that for every positive integer n there is always a prime number in the interval mentioned above. Probably, this was already proved a long time ago, but in this document we are going to give a clear proof that this is true.

In order to achieve the goal, we are going to use another interval, which we will call '**Iwaniec-Pintz Interval**'.

Iwaniec-Pintz Interval

The **Iwaniec-Pintz Interval** is the interval $\left[n - n^\theta, n \right]$, where $\theta = 23/42$. Iwaniec and Pintz proved that there is always a prime number in the interval $\left[n - n^{\frac{23}{42}}, n \right]$. The proof that this is true was given in their document *Primes in Short Intervals*.

In the Iwaniec-Pintz Interval, n must be considered as a positive integer greater than 1. This is because in the interval $\left[1 - 1^{\frac{23}{42}}, 1 \right]$ there is no prime number.

To sum up, there is always a prime number in the interval $\left[n - n^{\frac{23}{42}}, n \right]$ for every positive integer $n > 1$.

Modified Iwaniec-Pintz Interval

Working with an exponent like $\frac{23}{42}$ may be very difficult. For this reason, we define the

'**Modified Iwaniec-Pintz Interval**' as the interval $\left[n - n^{\frac{2}{3}}, n \right]$.

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We know that $\frac{2}{3} > \frac{23}{42}$, because $0.\overline{6} > 0.5476190\dots$. Now, if $n=1$, then $n^{\frac{2}{3}} = n^{\frac{23}{42}} = 1$,

but if $n > 1$ then $n^{\frac{2}{3}} > n^{\frac{23}{42}}$, as we will prove now:

$$n^{\frac{2}{3}} > n^{\frac{23}{42}} \Rightarrow \frac{n^{\frac{2}{3}}}{n^{\frac{23}{42}}} > \frac{n^{\frac{23}{42}}}{n^{\frac{23}{42}}} \Rightarrow n^{\frac{5}{42}} > 1 \Rightarrow n > \sqrt[42]{1} \Rightarrow n > 1$$

The reciprocal is also correct:

$$n^{\frac{2}{3}} > n^{\frac{23}{42}} \Leftarrow \frac{n^{\frac{2}{3}}}{n^{\frac{23}{42}}} > \frac{n^{\frac{23}{42}}}{n^{\frac{23}{42}}} \Leftarrow n^{\frac{5}{42}} > 1 \Leftarrow n > \sqrt[42]{1} \Leftarrow n > 1$$

For this reason, it is better to say that

$$n^{\frac{2}{3}} > n^{\frac{23}{42}} \Leftrightarrow \frac{n^{\frac{2}{3}}}{n^{\frac{23}{42}}} > \frac{n^{\frac{23}{42}}}{n^{\frac{23}{42}}} \Leftrightarrow n^{\frac{5}{42}} > 1 \Leftrightarrow n > \sqrt[42]{1} \Leftrightarrow n > 1$$

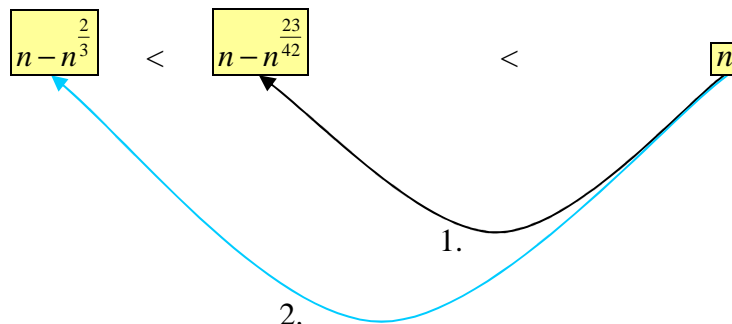
In general, to prove that an inequality is correct, we can solve that inequality step by step. If we get a result which is obviously correct, then we can start with that correct result, 'work backwards from there' and prove that the initial statement is true.

If $n^{\frac{2}{3}} > n^{\frac{23}{42}}$, then $n - n^{\frac{2}{3}} < n - n^{\frac{23}{42}}$.

Proof:

$$n^{\frac{2}{3}} > n^{\frac{23}{42}} \Rightarrow n^{\frac{2}{3}} / (-1) < n^{\frac{23}{42}} / (-1) \Rightarrow -n^{\frac{2}{3}} < -n^{\frac{23}{42}} \Rightarrow n - n^{\frac{2}{3}} < n - n^{\frac{23}{42}}$$

Now, let us make a graphic of both the Iwaniec-Pintz Interval and the Modified Iwaniec-Pintz Interval for every positive integer $n > 1$:



1. Iwaniec-Pintz Interval
2. Modified Iwaniec-Pintz Interval

For every positive integer n greater than 1, the interval $\left[n - n^{\frac{23}{42}}, n \right]$ contains at least one prime number. Therefore, the interval $\left[n - n^{\frac{2}{3}}, n \right]$ also contains at least one prime number, according to the graphic on the previous page.

Summary: If n is any positive integer greater than 1, then the Modified Iwaniec-Pintz Interval $\left[n - n^{\frac{2}{3}}, n \right]$ contains at least one prime number. This true statement will be called **Theorem A**.

Value of n

Now, we need to know what value n needs to have so that a Breusch Interval can contain at least one Modified Iwaniec-Pintz Interval. We are going to use **Theorem A**, which we have just proved.

We do not have any way of knowing if $\frac{9(n+3)}{8}$ (upper endpoint of the Breusch Interval) is an integer or not if we do not know the value of n first. The problem is that we use the interval $\left[n - n^{\frac{2}{3}}, n \right]$ with positive **integers** only.

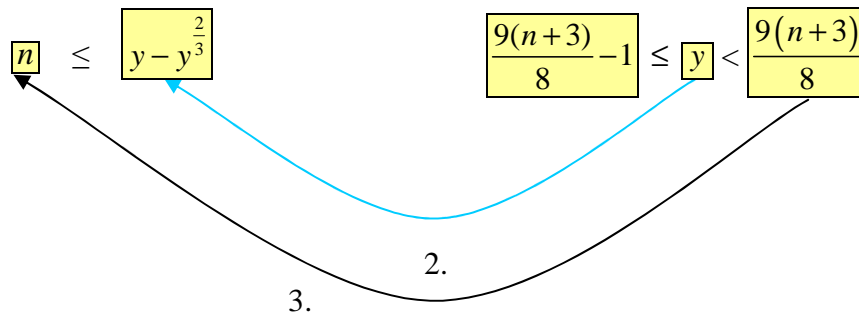
Let us assume y is the integer immediately preceding the number $\frac{9(n+3)}{8}$. In other words, y will be the largest integer that is less than $\frac{9(n+3)}{8}$:

- If $\frac{9(n+3)}{8}$ is an integer, then $\frac{9(n+3)}{8} - d = y$, where $d = 1$.
- If $\frac{9(n+3)}{8}$ is not an integer, then $\frac{9(n+3)}{8} - d = y$, and in this case we know for sure that $0 < d < 1$, but we do not have any way of knowing the exact value of d if we do not know the value of $\frac{9(n+3)}{8}$ first.

All things considered, we have $\frac{9(n+3)}{8} - 1 \leq y$ and $y < \frac{9(n+3)}{8}$. This means that

$$\frac{9(n+3)}{8} - 1 \leq y < \frac{9(n+3)}{8}$$

It is correct to say that if $y - y^{\frac{2}{3}} \geq n$, then a Breusch Interval will contain at least one Modified Iwaniec-Pintz Interval. Let us see the graphic below:



- 2. Modified Iwaniec-Pintz Interval
- 3. Breusch Interval

Let us calculate the value n needs to have so that $y - y^{\frac{2}{3}} \geq n$:

$$y - y^{\frac{2}{3}} \geq n$$

$$\left(\frac{9(n+3)}{8} - d\right) - \left(\frac{9(n+3)}{8} - d\right)^{\frac{2}{3}} \geq n \quad (\text{we know that } y = \frac{9(n+3)}{8} - d)$$

$$\left(\frac{9n}{8} + \frac{9}{8} \times 3 - d\right) - \left(\frac{9n}{8} + \frac{9}{8} \times 3 - d\right)^{\frac{2}{3}} \geq n$$

$$\left(\frac{9n}{8} + \underbrace{\frac{27}{8} - d}_g\right) - \left(\frac{9n}{8} + \underbrace{\frac{27}{8} - d}_g\right)^{\frac{2}{3}} \geq n$$

In order to make the process easier, the number $\frac{27}{8} - d$ will be called g . This number will always be positive, because $\frac{27}{8} = 3.375$, and d can not be greater than 1.

$$\left(\frac{9n}{8} + g\right) - \left(\frac{9n}{8} + g\right)^{\frac{2}{3}} \geq n$$

$$\frac{9n}{8} + g - \left(\frac{9n}{8} + g\right)^{\frac{2}{3}} \geq n$$

$$-\left(\frac{9n}{8} + g\right)^{\frac{2}{3}} \geq n - \frac{9n}{8} - g$$

$$-\left(\frac{9n}{8} + g\right)^{\frac{2}{3}} \geq -\frac{n}{8} - g$$

$$\left[-\left(\frac{9n}{8} + g\right)^{\frac{2}{3}}\right] / (-1) \leq \left[-\frac{n}{8} - g\right] / (-1)$$

$$\left(\frac{9n}{8} + g\right)^{\frac{2}{3}} \leq \frac{n}{8} + g$$

$$\sqrt[3]{\left(\frac{9n}{8} + g\right)^2} \leq \frac{n}{8} + g$$

$$\left(\frac{9n}{8} + g\right)^2 \leq \left(\frac{n}{8} + g\right)^3$$

$$\left(\frac{9n}{8}\right)^2 + 2 \times \frac{9n}{8} \times g + g^2 \leq \left(\frac{n}{8}\right)^3 + 3\left(\frac{n}{8}\right)^2 \times g + 3\left(\frac{n}{8}\right)g^2 + g^3$$

$$\frac{81n^2}{64} + \frac{9ng}{4} + g^2 \leq \frac{n^3}{512} + \frac{3n^2g}{64} + \frac{3ng^2}{8} + g^3$$

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$$\frac{81n^2 + 144ng + 64g^2}{64} \leq \frac{n^3 + 24n^2g + 192ng^2 + 512g^3}{512}$$

$$512 \times \frac{81n^2 + 144ng + 64g^2}{64} \leq n^3 + 24n^2g + 192ng^2 + 512g^3$$

$$8(81n^2 + 144ng + 64g^2) \leq n^3 + 24n^2g + 192ng^2 + 512g^3$$

$$648n^2 + 1152ng + 512g^2 \leq n^3 + 24n^2g + 192ng^2 + 512g^3$$

Now we need to know what value n needs to have so that:

1) $n^3 \geq 648n^2$

2) $24n^2g \geq 1152ng$

3) $192ng^2 \geq 512g^2$

If these three inequalities are all true at the same time, then

$$n^3 + 24n^2g + 192ng^2 + 512g^3 \geq 648n^2 + 1152ng + 512g^2$$

Let us calculate the values of n :

① $n^3 \geq 648n^2$

$$\frac{n^3}{n^2} \geq 648$$

$$n \geq 648$$

Result 1: $n \geq 648$

② $24n^2g \geq 1152ng$

$$\frac{24n^2g}{ng} \geq 1152$$

$$24n \geq 1152$$

$$n \geq \frac{1152}{24}$$

$$n \geq 48$$

Result 2: $n \geq 48$

③ $192ng^2 \geq 512g^2$

$$\frac{192ng^2}{g^2} \geq 512$$

$$192n \geq 512$$

$$n \geq \frac{512}{192}$$

$$n \geq 2.\bar{6}$$

Result 3: $n \geq 2.\bar{6}$

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If $n \geq 648$, then $n > 48$ and $n > 2.\bar{6}$. Therefore, if $n \geq 648$ then the inequalities 1), 2) and 3) are all true at the same time. This means that if $n \geq 648$, then

$$n^3 + 24n^2g + 192ng^2 + 512g^3 \geq 648n^2 + 1152ng + 512g^2$$

If we start with this true statement and we ‘work backwards from here’, we prove that if $n \geq 648$ is an integer, then $y - y^{\frac{2}{3}} \geq n$. To sum up,

$$n \leq y - y^{\frac{2}{3}} < y < \frac{9(n+3)}{8} \quad \forall n \in \mathbb{Z}^+ \text{ such that } n \geq 648$$

Since $n \geq 648$ and $y > n$, then $y > 1$. This is the reason why in the interval $\left[y - y^{\frac{2}{3}}, y \right]$ there is always at least one prime number, according to *Theorem A*.

In other words, for every positive integer $n \geq 648$ the Breusch Interval will contain at least one Modified Iwaniec-Pintz Interval, which will contain at least one prime number. This means that for every positive integer $n \geq 648$ the Breusch Interval contains at least one prime number.

We will also prove that there is at least one prime number in the Breusch Interval for every positive integer n such that $1 \leq n \leq 647$:

n	Example of prime number in the interval:	$\frac{9(n+3)}{8}$
1	2	4,5
2	2	5,625
3	3	6,75
4	5	7,875
5	5	9
6	7	10,125
7	7	11,25
8	11	12,375
9	11	13,5
10	11	14,625

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11	11	15,75
12	13	16,875
13	13	18
14	17	19,125
15	17	20,25
16	17	21,375
17	17	22,5
18	19	23,625
19	19	24,75
20	23	25,875
21	23	27
22	23	28,125
23	23	29,25
24	29	30,375
25	29	31,5
26	29	32,625
27	29	33,75
28	29	34,875
29	29	36
30	31	37,125
31	31	38,25
32	37	39,375
33	37	40,5
34	37	41,625
35	37	42,75
36	37	43,875
37	37	45
38	41	46,125
39	41	47,25
40	41	48,375
41	41	49,5
42	43	50,625
43	43	51,75
44	47	52,875
45	47	54
46	47	55,125
47	47	56,25
48	53	57,375
49	53	58,5
50	53	59,625
51	53	60,75
52	53	61,875
53	53	63
54	59	64,125
55	59	65,25
56	59	66,375
57	59	67,5
58	59	68,625

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59	59	69,75
60	61	70,875
61	61	72
62	67	73,125
63	67	74,25
64	67	75,375
65	67	76,5
66	67	77,625
67	67	78,75
68	71	79,875
69	71	81
70	71	82,125
71	71	83,25
72	73	84,375
73	73	85,5
74	79	86,625
75	79	87,75
76	79	88,875
77	79	90
78	79	91,125
79	79	92,25
80	83	93,375
81	83	94,5
82	83	95,625
83	83	96,75
84	89	97,875
85	89	99
86	89	100,125
87	89	101,25
88	89	102,375
89	89	103,5
90	97	104,625
91	97	105,75
92	97	106,875
93	97	108
94	97	109,125
95	97	110,25
96	97	111,375
97	97	112,5
98	101	113,625
99	101	114,75
100	101	115,875
101	101	117
102	103	118,125
103	103	119,25
104	107	120,375
105	107	121,5
106	107	122,625

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107	107	123,75
108	109	124,875
109	109	126
110	113	127,125
111	113	128,25
112	113	129,375
113	113	130,5
114	127	131,625
115	127	132,75
116	127	133,875
117	127	135
118	127	136,125
119	127	137,25
120	127	138,375
121	127	139,5
122	127	140,625
123	127	141,75
124	127	142,875
125	127	144
126	127	145,125
127	127	146,25
128	131	147,375
129	131	148,5
130	131	149,625
131	131	150,75
132	137	151,875
133	137	153
134	137	154,125
135	137	155,25
136	137	156,375
137	137	157,5
138	139	158,625
139	139	159,75
140	149	160,875
141	149	162
142	149	163,125
143	149	164,25
144	149	165,375
145	149	166,5
146	149	167,625
147	149	168,75
148	149	169,875
149	149	171
150	151	172,125
151	151	173,25
152	157	174,375
153	157	175,5
154	157	176,625

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155	157	177,75
156	157	178,875
157	157	180
158	163	181,125
159	163	182,25
160	163	183,375
161	163	184,5
162	163	185,625
163	163	186,75
164	167	187,875
165	167	189
166	167	190,125
167	167	191,25
168	173	192,375
169	173	193,5
170	173	194,625
171	173	195,75
172	173	196,875
173	173	198
174	179	199,125
175	179	200,25
176	179	201,375
177	179	202,5
178	179	203,625
179	179	204,75
180	181	205,875
181	181	207
182	191	208,125
183	191	209,25
184	191	210,375
185	191	211,5
186	191	212,625
187	191	213,75
188	191	214,875
189	191	216
190	191	217,125
191	191	218,25
192	193	219,375
193	193	220,5
194	197	221,625
195	197	222,75
196	197	223,875
197	197	225
198	199	226,125
199	199	227,25
200	211	228,375
201	211	229,5
202	211	230,625

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203	211	231,75
204	211	232,875
205	211	234
206	211	235,125
207	211	236,25
208	211	237,375
209	211	238,5
210	211	239,625
211	211	240,75
212	223	241,875
213	223	243
214	223	244,125
215	223	245,25
216	223	246,375
217	223	247,5
218	223	248,625
219	223	249,75
220	223	250,875
221	223	252
222	223	253,125
223	223	254,25
224	227	255,375
225	227	256,5
226	227	257,625
227	227	258,75
228	229	259,875
229	229	261
230	233	262,125
231	233	263,25
232	233	264,375
233	233	265,5
234	239	266,625
235	239	267,75
236	239	268,875
237	239	270
238	239	271,125
239	239	272,25
240	241	273,375
241	241	274,5
242	251	275,625
243	251	276,75
244	251	277,875
245	251	279
246	251	280,125
247	251	281,25
248	251	282,375
249	251	283,5
250	251	284,625

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251	251	285,75
252	257	286,875
253	257	288
254	257	289,125
255	257	290,25
256	257	291,375
257	257	292,5
258	263	293,625
259	263	294,75
260	263	295,875
261	263	297
262	263	298,125
263	263	299,25
264	269	300,375
265	269	301,5
266	269	302,625
267	269	303,75
268	269	304,875
269	269	306
270	271	307,125
271	271	308,25
272	277	309,375
273	277	310,5
274	277	311,625
275	277	312,75
276	277	313,875
277	277	315
278	281	316,125
279	281	317,25
280	281	318,375
281	281	319,5
282	283	320,625
283	283	321,75
284	293	322,875
285	293	324
286	293	325,125
287	293	326,25
288	293	327,375
289	293	328,5
290	293	329,625
291	293	330,75
292	293	331,875
293	293	333
294	307	334,125
295	307	335,25
296	307	336,375
297	307	337,5
298	307	338,625

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299	307	339,75
300	307	340,875
301	307	342
302	307	343,125
303	307	344,25
304	307	345,375
305	307	346,5
306	307	347,625
307	307	348,75
308	311	349,875
309	311	351
310	311	352,125
311	311	353,25
312	313	354,375
313	313	355,5
314	317	356,625
315	317	357,75
316	317	358,875
317	317	360
318	331	361,125
319	331	362,25
320	331	363,375
321	331	364,5
322	331	365,625
323	331	366,75
324	331	367,875
325	331	369
326	331	370,125
327	331	371,25
328	331	372,375
329	331	373,5
330	331	374,625
331	331	375,75
332	337	376,875
333	337	378
334	337	379,125
335	337	380,25
336	337	381,375
337	337	382,5
338	347	383,625
339	347	384,75
340	347	385,875
341	347	387
342	347	388,125
343	347	389,25
344	347	390,375
345	347	391,5
346	347	392,625

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347	347	393,75
348	349	394,875
349	349	396
350	353	397,125
351	353	398,25
352	353	399,375
353	353	400,5
354	359	401,625
355	359	402,75
356	359	403,875
357	359	405
358	359	406,125
359	359	407,25
360	367	408,375
361	367	409,5
362	367	410,625
363	367	411,75
364	367	412,875
365	367	414
366	367	415,125
367	367	416,25
368	373	417,375
369	373	418,5
370	373	419,625
371	373	420,75
372	373	421,875
373	373	423
374	379	424,125
375	379	425,25
376	379	426,375
377	379	427,5
378	379	428,625
379	379	429,75
380	383	430,875
381	383	432
382	383	433,125
383	383	434,25
384	389	435,375
385	389	436,5
386	389	437,625
387	389	438,75
388	389	439,875
389	389	441
390	397	442,125
391	397	443,25
392	397	444,375
393	397	445,5
394	397	446,625

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395	397	447,75
396	397	448,875
397	397	450
398	401	451,125
399	401	452,25
400	401	453,375
401	401	454,5
402	409	455,625
403	409	456,75
404	409	457,875
405	409	459
406	409	460,125
407	409	461,25
408	409	462,375
409	409	463,5
410	419	464,625
411	419	465,75
412	419	466,875
413	419	468
414	419	469,125
415	419	470,25
416	419	471,375
417	419	472,5
418	419	473,625
419	419	474,75
420	421	475,875
421	421	477
422	431	478,125
423	431	479,25
424	431	480,375
425	431	481,5
426	431	482,625
427	431	483,75
428	431	484,875
429	431	486
430	431	487,125
431	431	488,25
432	433	489,375
433	433	490,5
434	439	491,625
435	439	492,75
436	439	493,875
437	439	495
438	439	496,125
439	439	497,25
440	443	498,375
441	443	499,5
442	443	500,625

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443	443	501,75
444	449	502,875
445	449	504
446	449	505,125
447	449	506,25
448	449	507,375
449	449	508,5
450	457	509,625
451	457	510,75
452	457	511,875
453	457	513
454	457	514,125
455	457	515,25
456	457	516,375
457	457	517,5
458	461	518,625
459	461	519,75
460	461	520,875
461	461	522
462	463	523,125
463	463	524,25
464	467	525,375
465	467	526,5
466	467	527,625
467	467	528,75
468	479	529,875
469	479	531
470	479	532,125
471	479	533,25
472	479	534,375
473	479	535,5
474	479	536,625
475	479	537,75
476	479	538,875
477	479	540
478	479	541,125
479	479	542,25
480	487	543,375
481	487	544,5
482	487	545,625
483	487	546,75
484	487	547,875
485	487	549
486	487	550,125
487	487	551,25
488	491	552,375
489	491	553,5
490	491	554,625

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491	491	555,75
492	499	556,875
493	499	558
494	499	559,125
495	499	560,25
496	499	561,375
497	499	562,5
498	499	563,625
499	499	564,75
500	503	565,875
501	503	567
502	503	568,125
503	503	569,25
504	509	570,375
505	509	571,5
506	509	572,625
507	509	573,75
508	509	574,875
509	509	576
510	521	577,125
511	521	578,25
512	521	579,375
513	521	580,5
514	521	581,625
515	521	582,75
516	521	583,875
517	521	585
518	521	586,125
519	521	587,25
520	521	588,375
521	521	589,5
522	523	590,625
523	523	591,75
524	541	592,875
525	541	594
526	541	595,125
527	541	596,25
528	541	597,375
529	541	598,5
530	541	599,625
531	541	600,75
532	541	601,875
533	541	603
534	541	604,125
535	541	605,25
536	541	606,375
537	541	607,5
538	541	608,625

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539	541	609,75
540	541	610,875
541	541	612
542	547	613,125
543	547	614,25
544	547	615,375
545	547	616,5
546	547	617,625
547	547	618,75
548	557	619,875
549	557	621
550	557	622,125
551	557	623,25
552	557	624,375
553	557	625,5
554	557	626,625
555	557	627,75
556	557	628,875
557	557	630
558	563	631,125
559	563	632,25
560	563	633,375
561	563	634,5
562	563	635,625
563	563	636,75
564	569	637,875
565	569	639
566	569	640,125
567	569	641,25
568	569	642,375
569	569	643,5
570	571	644,625
571	571	645,75
572	577	646,875
573	577	648
574	577	649,125
575	577	650,25
576	577	651,375
577	577	652,5
578	587	653,625
579	587	654,75
580	587	655,875
581	587	657
582	587	658,125
583	587	659,25
584	587	660,375
585	587	661,5
586	587	662,625

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587	587	663,75
588	593	664,875
589	593	666
590	593	667,125
591	593	668,25
592	593	669,375
593	593	670,5
594	599	671,625
595	599	672,75
596	599	673,875
597	599	675
598	599	676,125
599	599	677,25
600	601	678,375
601	601	679,5
602	607	680,625
603	607	681,75
604	607	682,875
605	607	684
606	607	685,125
607	607	686,25
608	613	687,375
609	613	688,5
610	613	689,625
611	613	690,75
612	613	691,875
613	613	693
614	617	694,125
615	617	695,25
616	617	696,375
617	617	697,5
618	619	698,625
619	619	699,75
620	631	700,875
621	631	702
622	631	703,125
623	631	704,25
624	631	705,375
625	631	706,5
626	631	707,625
627	631	708,75
628	631	709,875
629	631	711
630	631	712,125
631	631	713,25
632	641	714,375
633	641	715,5
634	641	716,625

635	641	717,75
636	641	718,875
637	641	720
638	641	721,125
639	641	722,25
640	641	723,375
641	641	724,5
642	643	725,625
643	643	726,75
644	647	727,875
645	647	729
646	647	730,125
647	647	731,25

Breusch Interval Conclusion

It has been proved that for every positive integer 'n' there is at least one prime number

in the Breusch Interval $\left[n, \frac{9(n+3)}{8} \right]$.

$$\forall n \in \mathbb{Z}^+, \quad \pi \left[n, \frac{9(n+3)}{8} \right] \geq 1$$

Infinitely many prime numbers of the form $ap \pm b$

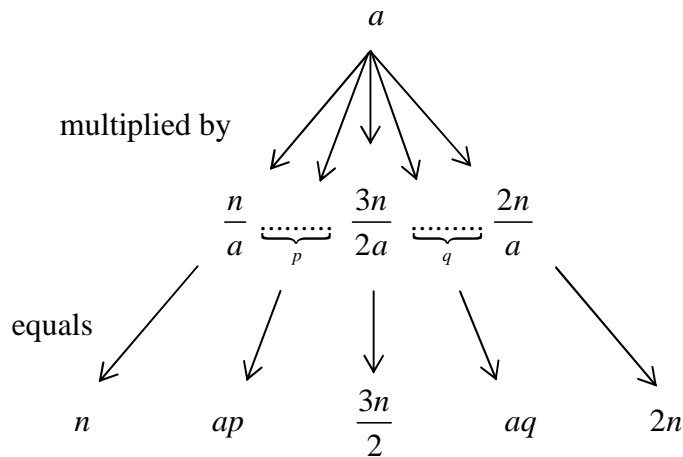
We just proved that for every positive integer n there is always a prime number in the

Breusch Interval $\left[n, \frac{9(n+3)}{8} \right]$.

Note: In this document, whenever we say that a number b is **between** a number a and a number c , it will mean that $a < b < c$, which means that b will never be equal to a or c (the same rule will be applied to intervals). Moreover, the number n that we will use in this document will always be a positive **integer**.

Let us suppose that a is a positive integer. We need to know what value n needs to have so that there exist a prime number p and a prime number q such that

$\frac{n}{a} < p < \frac{3n}{2a} < q < \frac{2n}{a}$. Let us look at the graphic on the next page:



In order to achieve the goal, we need to know what value n needs to have so that there exist at least one Breusch Interval (and therefore a prime number) between $\frac{n}{a}$ and $\frac{3n}{2a}$ and at least one Breusch Interval between $\frac{3n}{2a}$ and $\frac{2n}{a}$.

■ The integer immediately following the number $\frac{n}{a}$ will be called $\frac{n}{a} + d$. In other words, $\frac{n}{a} + d$ will be the smallest integer that is greater than $\frac{n}{a}$:

- If $\frac{n}{a}$ is an integer, then $\frac{n}{a} + d = \frac{n}{a} + 1$, because in this case we have $d = 1$.
- If $\frac{n}{a}$ is not an integer, then in the expression $\frac{n}{a} + d$ we have $0 < d < 1$, but we do not have any way of knowing the exact value of d if we do not know the value of $\frac{n}{a}$ first.

The reason why we work in this way is because the number n that we will use in the Breusch Interval will always be a positive **integer**.

Now, let us make the calculation:

$$\frac{9\left(\frac{n}{a} + d + 3\right)}{8} < \frac{3n}{2a}$$

$$9\left(\frac{n}{a} + d + 3\right) < 8 \times \frac{3n}{2a}$$

$$9\left(\frac{n}{a} + d + 3\right) < \frac{24n}{2a}$$

$$9\left(\frac{n}{a} + d + 3\right) < \frac{12n}{a}$$

$$\frac{9n}{a} + 9d + 27 < \frac{12n}{a}$$

$$9d + 27 < \frac{12n}{a} - \frac{9n}{a}$$

$$9d + 27 < \frac{3n}{a}$$

We need to take the largest possible value of d , which is $d = 1$ (if $9 \times 1 + 27 < \frac{3n}{a}$, then

$9d + 27 < \frac{3n}{a}$ for all d such that $0 < d \leq 1$):

$$9 \times 1 + 27 < \frac{3n}{a}$$

$$9 + 27 < \frac{3n}{a}$$

$$36 < \frac{3n}{a}$$

$$\frac{3n}{a} > 36$$

$$\frac{n}{a} > \frac{36}{3}$$

$$\frac{n}{a} > 12$$

Result 1: $n > 12a$

So, if $n > 12a$ then there is at least one prime number p such that $\frac{n}{a} < p < \frac{3n}{2a}$. This means that if $n > 12a$, then there exists a prime number p such that

$$n < ap < \frac{3n}{2} \quad (\text{we multiply the previous numbers by } a)$$

■ Now, we need to calculate what value n needs to have so that there is at least one Breusch Interval (and therefore a prime number) between $\frac{3n}{2a}$ and $\frac{2n}{a}$. The integer immediately following the number $\frac{3n}{2a}$ will be called $\frac{3n}{2a} + d$. In other words, $\frac{3n}{2a} + d$ will be the smallest integer that is greater than $\frac{3n}{2a}$. Let us take into account that $d = 1$ or $0 < d < 1$ depending on whether $\frac{3n}{2a}$ is an integer or not, respectively.

$$\frac{9\left(\frac{3n}{2a} + d + 3\right)}{8} < \frac{2n}{a}$$

$$9\left(\frac{3n}{2a} + d + 3\right) < 8 \times \frac{2n}{a}$$

$$9\left(\frac{3n}{2a} + d + 3\right) < \frac{16n}{a}$$

$$\frac{27n}{2a} + 9d + 27 < \frac{16n}{a}$$

$$9d + 27 < \frac{16n}{a} - \frac{27n}{2a}$$

$$9d + 27 < \frac{2 \times 16n - 27n}{2a}$$

$$9d + 27 < \frac{32n - 27n}{2a}$$

$$9d + 27 < \frac{5n}{2a}$$

Now, we need to take the largest possible value of d , which is $d = 1$ (if $9 \times 1 + 27 < \frac{5n}{2a}$, then $9d + 27 < \frac{5n}{2a}$ for all d such that $0 < d \leq 1$):

$$9 \times 1 + 27 < \frac{5n}{2a}$$

$$9 + 27 < \frac{5n}{2a}$$

$$36 < \frac{5n}{2a}$$

$$\frac{5n}{2a} > 36$$

$$5n > 36 \times 2a$$

$$5n > 72a$$

$$n > \frac{72a}{5}$$

Result 2: $n > 14.4a$

So, if $n > 14.4a$ then there is at least one prime number q such that $\frac{3n}{2a} < q < \frac{2n}{a}$. This means that if $n > 14.4a$, then there exists a prime number q such that

$$\frac{3n}{2} < aq < 2n \quad (\text{we multiply the previous numbers by } a)$$

■ So far we have these two results:

Result 1: $n > 12a$

Result 2: $n > 14.4a$

If $n > 14.4a$, then $n > 12a$. To sum up, we have a general result:

General Result: $n > 14.4a$

✚ Consequently, if a is a positive integer, then for every positive integer n such that $n > 14.4a$ there is always a pair of prime numbers p and q such that $n < ap < \frac{3n}{2} < aq < 2n$. This true statement will be called **Theorem 1**.

✚ In the case where $a = 1$ we can say that if $n > 14.4$, then there always exist prime numbers r and s such that $n < r < \frac{3n}{2} < s < 2n$. This statement will be called **Theorem 2**.

If $n > 14.4a$, then there is always a pair of prime numbers p and q such that

$$n < ap < \frac{3n}{2} < aq < 2n, \text{ according to } \mathbf{Theorem 1}$$

If $n > 14.4$, then there is always a pair of prime numbers r and s such that

$$n < r < \frac{3n}{2} < s < 2n, \text{ according to } \mathbf{Theorem 2}$$

If $n > 14.4a$ and $a > 1$, then $n > 14.4$. As a consequence, we will state Theorem 3.

✚ **Theorem 3:** If a is any positive integer greater than 1, then for every positive integer $n > 14.4a$ there are always at least four prime numbers p , q , r and s such that $n < ap < \frac{3n}{2} < aq < 2n$ and simultaneously $n < r < \frac{3n}{2} < s < 2n$.

Dirichlet's Theorem states that if a and b are two positive integers which are coprime, then the arithmetic progression $an + b$ contains infinitely many prime numbers. In the expression $an + b$ the number n is considered as a positive integer by some authors ($n \geq 1$), and it is considered as a non-negative integer by others ($n \geq 0$). In this document, n is considered as a positive integer.

In the expression $an + b$, the numbers a and b must be coprime, but these coprime numbers do not need to have different parity (they can have the same or different parity), because the number n can be even as well as odd.

In order for $ap + b$ (where p is a prime number) to be able to generate infinitely many prime numbers, the numbers a and b must be coprime and they must also have different parity. This is because even if a and b are coprime, if these numbers are both odd then $ap + b$ can generate at most one prime number (only when $p = 2$).

When we say that the numbers a and b must have different *parity*, we mean that:

- If a is even, then b must be odd.
- If a is odd, then b must be even.

Note: The numbers a and b must be coprime, because if they are not, then neither $ap + b$ nor $ap - b$ can generate infinitely many prime numbers, since only composite numbers will be generated or at most a finite number of primes will be generated. The numbers a and b must also have different parity, otherwise only composite numbers will be generated or at most only one prime number will be generated, as explained before.

In the article ***PRIMOS GEMELOS, DEMOSTRACIÓN KMELLIZA***, **Carlos Giraldo Ospina** proves that if k is any positive even integer, then there are infinitely many prime numbers p such that $p + k$ is also a prime number. **In particular, this author proves that there are infinitely many twin prime numbers, solving one of the unsolved problems in Mathematics.**

We are going to prove that if a and b are two positive integers which are coprime and have different parity, then there are infinitely many prime numbers of the form $ap + b$ (where p is a prime number) and infinitely many prime numbers of the form $ap - b$.

In order to achieve our goal, we are going use the same method that C. G. Ospina used in his paper.

Proof:

According to **Theorem 3**, if $a > 1$ is an integer, then for every positive integer $n > 14.4a$ there exist prime numbers p and s , such that

$$n < ap < \frac{3n}{2} < s < 2n$$

The numbers ap and s form what we will call '**pair (ap , prime) of order b** '. This is because:

- We have a pair of numbers: a number ap and a prime number s .
- The number ap is followed by the prime number s . In other words, $ap < s$.
- We say that $ap + b = s$. In other words, we say the pair (ap , prime) is 'of order b ' because the difference between the numbers forming this pair is b .

The set made up of all positive integers z such that $z > 14.4a$ will be called **Set A(a)**. Obviously, Set A(a) changes as the value of a changes, that is to say, Set A(a) depends on the value of a .

Examples:

- Set A(2) is the set of all positive integers z such that $z > 14.4 \times 2$. In other words, Set A(2) is made up of all positive integers z such that $z > 28.8$.
- Set A(3) is the set of all positive integers z such that $z > 14.4 \times 3$. In other words, Set A(3) is made up of all positive integers $z > 43.2$.

1. Let us choose any positive integer $a > 1$ and let us suppose that in Set A(a) there are no pairs (ap, prime) of order $b < u$ starting from $n = u$.

The numbers n and u are positive integers which belong to Set A(a), and the number b is a positive integer such that a and b are coprime and have different parity.

2. Between u and $2u$, $n = u$ (and therefore in the interval $[u, 2u]$) there is at least one pair (ap, prime) , according to **Theorem 3**.
3. The difference between a number ap and a prime number which are located between u and $2u$ is $b < u$.
4. Between u and $2u$, $n = u$ (and therefore in the interval $[u, 2u]$) there is at least one pair (ap, prime) of order $b < u$, according to statements 2. and 3.
5. In Set A(a), starting from $n = u$ there is at least one pair (ap, prime) of order $b < u$, according to statement 4.
6. Statement 5. contradicts statement 1.
7. Therefore, no kind of pair (ap, prime) of any order b can be finite, according to statement 6.

This means that if $a > 1$ is an integer, then for every positive integer b such that a and b are coprime and have different parity there are infinitely many prime numbers of the form $ap + b$, where p is a prime number. As a consequence, we can state Theorem 4 now.

Theorem 4: If $a > 1$ and b are two positive integers which are coprime and have different parity, then there are infinitely many prime numbers of the form $ap + b$, where p is a prime number.

Theorem 4 proves that there are infinitely many prime numbers of the form $2p+1$, where p is a prime number. This is because the numbers 2 and 1 are coprime and they also have different parity.

This proves that there are infinitely many prime numbers p such that $2p+1$ is also a prime number, which proves there are infinitely many Sophie Germain Prime Numbers. Consequently, one of the unsolved problems in Mathematics is now solved.

Next, we are going to prove another very important theorem.

According to **Theorem 3**, if $a > 1$ is an integer, then for every positive integer $n > 14.4a$ there exist prime numbers r and q such that

$$n < r < \frac{3n}{2} < aq < 2n$$

The numbers r and aq form what we will call '*pair (prime, aq) of order b* ', because

- We have a pair of numbers: a prime number r and a number aq .
- The prime number r is followed by the number aq . In other words, $r < aq$.
- We say that $r + b = aq$. In other words, we say the pair (prime, aq) is 'of order b ' because the difference between the numbers forming this pair is b .

Again, the set made up of all positive integers z such that $z > 14.4a$ will be called **Set $A(a)$** . As we said before, Set $A(a)$ changes as the value of a changes.

1. Let us choose any positive integer $a > 1$ and let us suppose that in Set $A(a)$ there are no pairs (prime, aq) of order $b < u$ starting from $n = u$.

The numbers n and u are positive integers which belong to Set $A(a)$, and the number b is a positive integer such that a and b are coprime and have different parity.

2. Between u and $2u$, $n = u$ (and therefore in the interval $[u, 2u]$) there is at least one pair (prime, aq), according to **Theorem 3**.
3. The difference between a prime number and a number aq which are located between u and $2u$ is $b < u$.
4. Between u and $2u$, $n = u$ (and therefore in the interval $[u, 2u]$) there is at least one pair (prime, aq) of order $b < u$, according to statements 2. and 3.
5. In Set $A(a)$, starting from $n = u$ there is at least one pair (prime, aq) of order $b < u$, according to statement 4.

6. Statement 5. contradicts statement 1.
7. Therefore, no kind of pair (prime, aq) of any order b can be finite, according to statement 6.

This means that if $a > 1$ is an integer, then for every positive integer b such that a and b are coprime and have different parity there are infinitely many prime numbers r such that $r + b = aq$. This means that there are infinitely many prime numbers r which are prime numbers of the form $aq - b$, where q is, as we said before, a prime number.

In the expression $aq - b$, we will change letter q for letter p and we will state Theorem 5.

Theorem 5: If $a > 1$ and b are two positive integers which are coprime and have different parity, then there are infinitely many prime numbers of the form $ap - b$, where p is a prime number.

If we consider the case where $a = 1$ and we follow the procedures described on pages 28 to 30, we can easily prove that for every positive even integer k there are infinitely many prime numbers p such that $p + k$ is also a prime number. This is basically how C. G. Ospina proved his theorem in his paper.

In the expression $p + k$, the number 1 and the positive even integer k are coprime and have different parity ($p + k = 1p + k$). For this reason, **theorems 4 and 5 can also be applied to the case where $a = 1$.**

Final Conclusion

We will restate the two most important theorems that were proved in this document.

Theorem 4: If a and b are two positive integers which are coprime and have different parity, then there are infinitely many prime numbers of the form $ap + b$, where p is a prime number.

Theorem 5: If a and b are two positive integers which are coprime and have different parity, then there are infinitely many prime numbers of the form $ap - b$, where p is a prime number.

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See also:

- ❖ *Solution to One of Landau's Problems*
- ❖ *Números primos, fórmula precisa. Original paper: Cálculo de la cantidad de números primos que hay por debajo de un número dado (How to calculate the amount of prime numbers that are less than a given number)*
- ❖ *Números Primos de Sophie Germain, Demostración de su Infinitud (There Are Infinitely Many Sophie Germain Prime Numbers)*