ON THE FORM OF MERSENNE NUMBERS

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Abstract. We present a theorem concerning the form of Mersenne numbers , $M_p = 2^p - 1$, where $p > 3$. We also discuss a closed form expression which links prime numbers and natural logarithms
.

1. Introduction.

The friar Marin Mersenne $(1588 - 1648)$, a member of the order of Minims, was a philosopher, mathematician, and scientist who wrote on music, mechanics, and optics . A Mersenne number is a number of the form $M_p = 2^p - 1$ where p is an integer , however many authors prefer to define a Mersenne number as a number of the above form but with p restricted to prime values . In this paper we shall use the second definition . Because the tests for Mersenne primes are relatively simple, the record largest known prime number has almost always been a Mersenne prime .

2. A theorem and closed form expression

Theorem 2.1. *Every Mersenne number* M_p , $(p > 3)$ *is a number of the form* $6 \cdot k \cdot p + 1$ *for some odd possitive integer* k *.*

Proof. In order to prove this we shall use following theorems :

Theorem 2.2. *The simultaneous congruences* $n \equiv n_1 \pmod{m_1}$ *,*
$n \equiv n_2 \pmod{m_2}$*, ... ,*$n \equiv n_k \pmod{m_k}$ *are only solvable when*
$n_i = n_j \pmod{\gcd(m_i, m_j)}$ *, for all* i *and* j *.* $i \neq j$ *. The solution is unique modulo* $\operatorname{lcm}(m_1, m_2, \cdots, m_k)$ *.*

Proof. For proof see [1]

□

Theorem 2.3. *If* p *is a prime number, then for any integer* a *:*
$a^p \equiv a \pmod{p}$ *.*

Date: January 4 , 2012.

Proof. For proof see [2]

□

According to the theorem 2.3. it follows that :

$2^p \equiv 2 \pmod{p} \Rightarrow 2^p - 1 \equiv 1 \pmod{p} \Rightarrow M_p \equiv 1 \pmod{p}$

Since M_p is an odd prime number we can write : $M_p \equiv 1 \pmod{2}$

$2 \equiv -1 \pmod{3} \Rightarrow 2^p \equiv (-1)^p \pmod{3}$, for p odd it follows that :

$2^p \equiv -1 \pmod{3} \Rightarrow M_p = 2^p - 1 \equiv -1 - 1 \equiv 1 \pmod{3}$

Since $M_p \equiv 1 \pmod{2}$, $M_p \equiv 1 \pmod{3}$ and $M_p \equiv 1 \pmod{p}$ according to theorem 2.2. we can conclude that :
$M_p \equiv 1 \pmod{6 \cdot p}$, therefore M_p is of the form $6 \cdot k \cdot p + 1$

Now , let us prove that k must be an odd number .
Suppose that k is an even number so we can write $k = 2n$:

$2^p - 1 = 6 \cdot 2 \cdot n \cdot p + 1 \Rightarrow$
$\Rightarrow 2^p - 1 = 12 \cdot n \cdot p + 1 \Rightarrow$
$\Rightarrow 2^p = 12 \cdot n \cdot p + 2 \Rightarrow$
$\Rightarrow 2^{p-1} = 6 \cdot n \cdot p + 1$

The last equality cannot be true since left hand side is an even number and right hand side is an odd number , therefore k must be odd number .

□

Closed form expression. So we have shown that Mersenne numbers greater than 7 are of the form :

$M_p = 6 \cdot n \cdot p + 1 \Rightarrow 2^p - 1 = 6 \cdot n \cdot p + 1 \Rightarrow 2^{p-1} = 3 \cdot n \cdot p + 1$

Let us solve this transcendental equation for variable p :

If we apply following substitution :

$-t = p + \frac{1}{3n} \Rightarrow p = -t - \frac{1}{3n}$, we get :

$2^{-t-1-\frac{1}{3n}} = 3 \cdot n \cdot (-t - \frac{1}{3n}) + 1 \Rightarrow$

$$\Rightarrow 2^{-t-1-\frac{1}{3n}} = 3 \cdot n \cdot (-t) - 1 + 1 \Rightarrow$$

$$\Rightarrow 2^{-t-1-\frac{1}{3n}} = -3 \cdot n \cdot t \Rightarrow$$

$$\Rightarrow 2^{-t} \cdot 2^{-1-\frac{1}{3n}} = -3 \cdot n \cdot t \Rightarrow \frac{2^{-1-\frac{1}{3n}}}{3n} = -t \cdot 2^{t} \Rightarrow$$

$$\Rightarrow t \cdot 2^{t} = -\frac{2^{-1-\frac{1}{3n}}}{3n} \Rightarrow$$

$$\Rightarrow t = \frac{W\left(-\frac{2^{-1-\frac{1}{3n}} \cdot \ln 2}{3n}\right)}{\ln 2}$$

which yields final solution :

$$p = -\frac{W\left(-\frac{2^{-1-\frac{1}{3n}} \cdot \ln 2}{3n}\right)}{\ln 2} - \frac{1}{3n} \text{ , therefore :}$$

$$p = -\left(\frac{3n \cdot W\left(-\frac{2^{-1-\frac{1}{3n}} \cdot \ln 2}{3n}\right) + \ln 2}{n \cdot \ln 8}\right)$$

We can check this solution using WolframAlpha , online computational engine ,see [3] .

If we take in consideration only a lower branch of double-valued Lambert W relation ,W_{-1} , see [4] we shall obtain following closed form expression :

$$p = -\left(\frac{3n \cdot W_{-1}\left(-\frac{2^{-1-\frac{1}{3n}} \cdot \ln 2}{3n}\right) + \ln 2}{n \cdot \ln 8}\right)$$

If the value of this closed form expression is an positive integer then it is a prime number greater than 3 or it is a pseudoprime .In fact , if we take values for n to be terms of the sequence $A096060$, see [5] then this expression above generates all prime numbers greater than 3 and certain number (probably infinite) of pseudoprimes .It is an interesting task to determine what is a number of the pseudoprimes in a sense of the asymptotic density .

3. Acknowledgments

I wish to express my gratitude to Bojan Terzich for grammatical improvement of the text .

References

1. Proof of theorem 2.2. available at :
   $http : //www.cut - the - knot.org/blue/chinese.shtml$
2. Proof of Fermat little theorem available at :
   $http : //en.wikipedia.org/wiki/Fermat's\_little\_theorem$
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5. Sequence $A096060$ available at :
   $https : //oeis.org/$

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