ON THE FORM OF MERSENNE NUMBERS

PREDRAG TERZICH

Abstract. We present a theorem concerning the form of Mersenne numbers, \( M_p = 2^p - 1 \), where \( p > 3 \). We also discuss a closed form expression which links prime numbers and natural logarithms.

1. Introduction.

The friar Marin Mersenne (1588 – 1648), a member of the order of Minims, was a philosopher, mathematician, and scientist who wrote on music, mechanics, and optics. A Mersenne number is a number of the form \( M_p = 2^p - 1 \) where \( p \) is an integer, however many authors prefer to define a Mersenne number as a number of the above form but with \( p \) restricted to prime values. In this paper we shall use the second definition. Because the tests for Mersenne primes are relatively simple, the record largest known prime number has almost always been a Mersenne prime.

2. A theorem and closed form expression

Theorem 2.1. Every Mersenne number \( M_p \), \( (p > 3) \) is a number of the form \( 6 \cdot k \cdot p + 1 \) for some odd positive integer \( k \).

Proof. In order to prove this we shall use following theorems:

Theorem 2.2. The simultaneous congruences \( n \equiv n_1 \pmod{m_1} \), \( n \equiv n_2 \pmod{m_2} \), .... \( n \equiv n_k \pmod{m_k} \) are only solvable when \( n_i = n_j \pmod{\gcd(m_i,m_j)} \), for all \( i \) and \( j \). \( i \neq j \). The solution is unique modulo \( \text{lcm}(m_1,m_2,\cdots,m_k) \).

Proof. For proof see [1]

Theorem 2.3. If \( p \) is a prime number, then for any integer \( a \) : \( a^p \equiv a \pmod{p} \).

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\begin{proof}
For proof see [2]
\end{proof}

According to the theorem 2.3. it follows that :

\[2^p \equiv 2 \pmod{p} \Rightarrow 2^p - 1 \equiv 1 \pmod{p} \Rightarrow M_p \equiv 1 \pmod{p}\]

Since \(M_p\) is an odd prime number we can write : \(M_p \equiv 1 \pmod{2}\)

\[2 \equiv -1 \pmod{3} \Rightarrow 2^p \equiv (-1)^p \pmod{3}\], for \(p\) odd it follows that :

\[2^p \equiv -1 \pmod{3} \Rightarrow M_p = 2^p - 1 \equiv -1 - 1 \equiv 1 \pmod{3}\]

Since \(M_p \equiv 1 \pmod{2}\), \(M_p \equiv 1 \pmod{3}\) and \(M_p \equiv 1 \pmod{p}\) according to theorem 2.2. we can conclude that :
\(M_p \equiv 1 \pmod{6 \cdot p}\), therefore \(M_p\) is of the form \(6 \cdot k \cdot p + 1\)

Now, let us prove that \(k\) must be an odd number . Suppose that \(k\) is an even number so we can write \(k = 2n\) :

\[2^p - 1 = 6 \cdot 2 \cdot n \cdot p + 1 \Rightarrow \]
\[\Rightarrow 2^p - 1 = 12 \cdot n \cdot p + 1 \Rightarrow \]
\[\Rightarrow 2^p = 12 \cdot n \cdot p + 2 \Rightarrow \]
\[\Rightarrow 2^{p-1} = 6 \cdot n \cdot p + 1 \]

The last equality cannot be true since left hand side is an even number and right hand side is an odd number , therefore \(k\) must be odd number .

\end{proof}

**Closed form expression.** So we have shown that Mersenne numbers greater than 7 are of the form :

\[M_p = 6 \cdot n \cdot p + 1 \Rightarrow 2^p - 1 = 6 \cdot n \cdot p + 1 \Rightarrow 2^{p-1} = 3 \cdot n \cdot p + 1\]

Let us solve this transcendental equation for variable \(p\) :

If we apply following substitution :

\[-t = p + \frac{1}{3n} \Rightarrow p = -t - \frac{1}{3n}\, , \text{ we get :}\]

\[2^{-t-1} - \frac{1}{3n} = 3 \cdot n \cdot (-t - \frac{1}{3n}) + 1 \Rightarrow\]
\[ 2^{-t-1} \cdot \frac{1}{3n} = 3 \cdot n \cdot (-t) - 1 + 1 \Rightarrow \]
\[ 2^{-t-1} \cdot \frac{1}{3n} = -3 \cdot n \cdot t \Rightarrow \]
\[ 2^{-t} \cdot 2^{-1} \cdot \frac{1}{3n} = -3 \cdot n \cdot t \Rightarrow 2^{-1} \cdot \frac{1}{3n} = -t \cdot 2^t \Rightarrow \]
\[ t \cdot 2^t = -\frac{2^{-1} \cdot \frac{1}{3n}}{\ln 2} \Rightarrow \]
\[ t = \frac{W\left(-\frac{2^{-1} \cdot \frac{1}{3n} \ln 2}{\ln 2}\right)}{\ln 2} \]

which yields final solution:

\[ p = -\frac{W\left(-\frac{2^{-1} \cdot \frac{1}{3n} \ln 2}{\ln 2}\right)}{\ln 2} - \frac{1}{3n} \], therefore:

\[ p = -\left(3nW\left(-\frac{2^{-1} \cdot \frac{1}{3n} \ln 2}{n \ln 8}\right) + \ln 2\right) \]

We can check this solution using WolframAlpha, online computational engine, see [3].

If we take in consideration only a lower branch of double-valued Lambert W relation, \(W_{-1}\), see [4] we shall obtain following closed form expression:

\[ p = -\left(3nW_{-1}\left(-\frac{2^{-1} \cdot \frac{1}{3n} \ln 2}{n \ln 8}\right) + \ln 2\right) \]

If the value of this closed form expression is an positive integer then it is a prime number greater than 3 or it is a pseudoprime. In fact, if we take values for \(n\) to be terms of the sequence \(A096060\), see [5] then this expression above generates all prime numbers greater than 3 and certain number (probably infinite) of pseudoprimes. It is an interesting task to determine what is a number of the pseudoprimes in a sense of the asymptotic density.
3. Acknowledgments

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References

1. Proof of theorem 2.2. available at:
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2. Proof of Fermat little theorem available at:
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E-mail address: tersit26@gmail.com