Prime Distribution in Pythagorean Triples (1)

(the greatest problem in mathematics)

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Abstract. Using Jiang function we study the prime distribution in Pythagorean triples.

Pythagorean triples

\[ a^2 + b^2 = c^2, \]  \hspace{1cm} (1)

in comprime integers must be of the form

\[ a = x^2 - y^2, \quad b = xy, \quad c = x^2 + y^2, \]  \hspace{1cm} (2)

where \( x \) and \( y \) ae coprime integers.

**Theorem 1.** From (2) we have

\[ a = (x + y)(x - y) \]  \hspace{1cm} (3)

Let \( x - y = 1 \) and \( a = x + y = P_1 \), we have,

\[ P_1^2 = (x + y)^2 = x^2 + y^2 + 2xy = c + b, \]  \hspace{1cm} (4)

\[ 1 = (x - y)^2 = x^2 + y^2 - 2xy = c - b \]  \hspace{1cm} (5)

From (4) and (5) we have

\[ a = P_1, \quad b = \frac{P_1^2 - 1}{2}, \quad c = \frac{P_1^2 + 1}{2} = P_2 \]  \hspace{1cm} (6)

There are infinitely many primes \( P_1 \) such that \( P_2 \) is a prime.

**Proof.** We have Jiang function \[1\]

\[ J_2(\omega) = \prod_{P \geq 2} (P - 1 - \chi(P)), \]  \hspace{1cm} (7)

where \( \omega = \prod_{P \geq 2} P, \) \( \chi(P) \) is the number of solutions of congruence

\[ q^2 + 1 \equiv 0 (\text{mod} \ P), \quad q = 1, \ldots, P - 1. \]  \hspace{1cm} (8)

From (8) we have

\[ \chi(P) = 1 + (-1)^\frac{P - 1}{2} \]  \hspace{1cm} (9)

Substituting (9) into (7) we have
\[ J_2(\omega) = \prod_{p>2} \left( P - 2 - \left( -1 \right)^{\frac{P-1}{2}} \right) \neq 0 \quad (10) \]

Since \( J_2(\omega) \neq 0 \), we prove that there are infinitely many prime \( p_1 \) such that \( p_2 \) is a prime.

We have the best asymptotic formula [1]

\[ \pi_2(N, 2) = \left\lvert \{ P_1 \leq N : P_2 = \text{prime} \} \right\rvert \sim \frac{J_2(\omega)\omega}{2\phi^2(\omega)} \frac{N}{\log^2 N} = \left( 1 - \frac{1 + P(-1)^{\frac{P-1}{2}}}{(P-1)^2} \right) \frac{N}{\log^2 N}, \quad (11) \]

where \( \phi(\omega) = \prod_{p>2} (P-1) \).

**Theorem 2.** Let \( x + y = p_1 \) and \( x - y = p_1 - 2 \), we have \( a = p_1(p_1 - 2) \) and

\[ p_1^2 = (x + y)^2 = c + b, \quad (12) \]
\[ (p_1 - 2)^2 = (x - y)^2 = c - b. \quad (13) \]

From (12) and (13) we have

\[ a = p_1(p_1 - 2), \quad b = \frac{p_1^2 - (p_1 - 2)^2}{2}, \quad c = \frac{p_1^2 + (p_1 - 2)^2}{2} = p_2. \quad (14) \]

There are infinitely many primes \( p_1 \) such that \( p_2 \) is a prime.

**Proof.** We have Jiang function [1]

\[ J_2(\omega) = \prod_{p>2} \left( P - 1 - \chi(P) \right), \quad (15) \]

where \( \chi(P) \) is the number of solutions of congruence

\[ q^2 + (q - 2)^2 \equiv 0 (\text{mod } P), \quad q = 1, \ldots, P - 1. \quad (16) \]

From (16) we have

\[ \chi(P) = 1 + (-1)^{\frac{P-1}{2}} \quad (17) \]

Substituting (17) into (15) we have

\[ J_2(\omega) = \prod_{p>2} \left( P - 2 - \left( -1 \right)^{\frac{P-1}{2}} \right) \neq 0 \quad (18) \]

Since \( J_2(\omega) \neq 0 \), we prove that there are infinitely many prime \( p_1 \) such that \( p_2 \) is a prime.

We have the best asymptotic formula [1]
\[
\pi_2(N, 2) = \left\lfloor \sum_{P_1 \leq N : P_2 = \text{prime}} \right\rfloor \sim \left(1 - \frac{1 + P(-1) \frac{P-1}{2}}{(P-1)^2}\right) \frac{N}{\log^2 N}, \quad (19)
\]

**Theorem 3.** Let \( x - y = 1 \) and \( a = x + y = P_1^2 \), we have,

\[
a = P_1^2, \quad b = \frac{P_1^4 - 1}{2}, \quad c = \frac{P_1^4 + 1}{2} = P_2.
\]

There are infinitely many primes \( P_1 \) such that \( P_2 \) is a prime.

**Proof.** We have Jiang function [1]

\[
J_2(\omega) = \prod_{P \neq 2} (P - 1 - \chi(P)),
\]

where \( \chi(P) \) is the number of solutions of congruence

\[
q^4 + 1 \equiv 0 (\text{mod} \ P), \quad q = 1, \ldots, P - 1.
\]

From (22) we have

\[
\chi(P) = 4 \quad \text{if} \quad 8 \mid P - 1, \quad \chi(P) = 0 \quad \text{otherwise}.
\]

Since \( J_2(\omega) \neq 0 \), we prove that there are infinitely many prime \( P_1 \) such that \( P_2 \) is a prime.

We have the best asymptotic formula [1]

\[
\pi_2(N, 2) = \left\lfloor \sum_{P_1 \leq N : P_2 = \text{prime}} \right\rfloor \sim \frac{J_2(\omega)\omega}{4\phi^2(\omega)} \frac{N}{\log^2 N}, \quad (24)
\]

These results are in wide use in biological, physical and chemical fields.

**Reference**

[1] Chun-Xuan Jiang, Jiang function \( J_{\pi_2}(\omega) \) in prime distribution,

http://vixra.org/pdf/0812.0004v2.pdf;

http://www.wbabin.net/math/xuan2.pdf