Particle Mass Ratios

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Abstract

Based on the developments in a previous paper [1], this paper presents straightforward explanation of particle mass ratios, and the specific values for some well known particles. The additional nuclear modes postulated, are similar to the Schrödinger modes in the atom, and the mass ratios calculated for elementary particles have been calculated within the experimental values. Those particles are the Proton, Neutron, Muon, Tauon, and Pion+/-/0.¹

Introduction

This section deals with an interesting approach to calculating mass ratios. It is well known that complex particles have internal multimode oscillations and a complex structure. For example the leptons, though considered “elementary” must be more complex than just a single electron, and have often been considered as higher energy versions of the electrons. This paper will focus on defining modes of oscillation that geometrically couple with the Compton oscillation, such that there are an integral number of cycles completed in the same time frame as the Compton oscillation that contribute to the rest mass of the particle. It will be shown that all complex particles Leptons, Bosons, & Mesons, have a similar multimode structure, and that the characteristic differences of the particles are perhaps in the values of the modes.

The concept is not unlike developed by Dirac in establishing a radial mode of oscillation for the Muon [2], and in fact a radial mode is part of the makeup of the Muon developed here.

The residual values between the modes calculated values and the experimental values have been determined to have a mathematical relation that appears to be related to the

¹ *Previous versions of this paper have included residuals that indicated a systematic error in the calculated mass values. The inclusion of the spin contribution to the mass has removed a large part of that systematic error*
QFT path modification due to perturbation effects, and thus changes the mass of the particle.

**Single Mode**

Starting with the Systemfunction for a single mode particle (electron) from [1] we have:

\[ \Theta = e^{\pm \text{im} \mathcal{R} \phi \pm 1/2} \]

The \( \frac{1}{2} \) in this expression is the value of the spin projection along the z axis. This well describes the dynamic particle properties, but for internal structure the spin must be defined in terms of the Pauli spin vector matrix:

\[ \sigma = \sigma_1 + \sigma_2 + \sigma_3 \]

With:

\[ \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \]

Thus Eq.(1), is restated in more detail as:

\[ \tilde{\Theta} = e^{\pm \text{im} \mathcal{R} \phi \pm \sigma} \]

The radius of the universe \( \mathcal{R} \) becomes just the local value \( \mathcal{R} = cT \rightarrow ct \) for the imaginary terms. The mass \( m = mc / \hbar \), is the reciprocal of the Compton radius, and the velocity is in Feynman slash notation (For general conventions see appendix I). From Eq.(4), the deBroglie and Compton frequencies are:

\[ \omega_d = mc |\vec{v}| \quad \& \quad \omega_c = mc \]

The focus of this paper is particle rest mass and thus the velocity, and thus the deBroglie frequency will be set to zero. Simplifying this is then:

\[ \tilde{\Theta} = e^{\pm \text{m}^2 \mathcal{R}^2 \pm \text{im} c t \sigma + \frac{1}{4} \sigma^2} \]
Normal Modes General Concept

From analogy with mechanical systems with normal coordinates, in general the solution of a complex coupled system is a sum of the different solutions of the various modes, thus if a particle is a composition of coupled normal modes:

$$\tilde{\Theta} = a_1 e^{f_1(\omega)} + a_2 e^{f_2(\omega)}$$  \hspace{1cm} (7)

with \( a_1 \) and, \( a_2 \) being scale factors proportional to the particle masses.

From analogy, also with coupled mechanical systems, we will propose for the internal coupling of a multimode particle the relation among the frequencies such that:

$$\omega = \omega_c \pm 2\sqrt{\omega_c \omega_1 - \omega_c \omega_2}$$  \hspace{1cm} (8)

A review of normal mode coupling in mechanical systems illustrates that this is not an unreasonable form.

It is easy to determine the connecting relations between the frequencies for this coupling from the coupling matrix. These frequencies have to be eigenvalues, thus.

$$\begin{bmatrix}
\omega \\
\omega_c + 2\sqrt{\omega_c \omega_1 \pm \omega_c \omega_2}
\end{bmatrix}, \hspace{1cm} (9)
$$

or

$$0 = \omega_c \left( \omega_c + 2\sqrt{\omega_c \omega_1 - \omega_c \omega_2} \right) \left( \omega_c - 2\sqrt{\omega_c \omega_1 - \omega_c \omega_2} \right), \hspace{1cm} (10)$$

leading to the requirement that the sum of the frequencies be constant.

$$\left( \omega_c - 4\omega_1 \pm 4\omega_2 \right) = K$$  \hspace{1cm} (11)

Presuming frequency and energy are related, then the proposed coupling is just the condition that the sum of the energy be a constant.

$$E = \left( \omega_c - 4\omega_1 \pm 4\omega_2 \right) \hbar$$  \hspace{1cm} (12)
\( \omega_c = mc \) is the Compton frequency and \( \omega_1 \) and \( \omega_2 \) are the frequencies of the other two modes.

Factoring and presuming a constant ratio of the frequencies gives:

\[
\omega = \omega_c \left( 1 \pm 2 \sqrt{a_i - a_j} \right),
\]

(13)

There has to be a fixed non-temporal ratio of the frequency, otherwise mass ratios would not be constant. It is expected that any oscillation, within a particle nucleus has to have an integral number of wavelengths related to the Compton cycle to exist as a stable particle. The Compton cycle will be regarded as a single azimuthal cycle.

**Extra Modes**

The primary Compton mode for the simple particle i.e. the electron is proposed to be a circumferential azimuthal wave with the effective radius of \( \lambda = h / mc \) and having a circumference of \( \lambda = 2\pi \lambda \). Analogously with an atomic system two other modes of oscillation can be defined in similarity with the radial, and colatitude modes. The first as a spherical radial wave, having an integral number of wavelengths within the Compton radius, thus:

\[
\frac{\lambda_C}{\lambda_R} = j,
\]

(14)

with frequencies:

\[
\frac{\omega_R}{\omega_C} = \frac{\lambda_C}{\lambda_R} = j\pi
\]

(15)

The third mode, is a colatitude oscillation having an effective integral wavelength at a distance from the center of mass of \( r = \lambda_c / 2k \) where \( k \) is an integer, thus:

\[
\frac{\omega_L}{\omega_C} = \frac{\lambda_C}{\lambda_L} = k / \pi
\]

(16)

The radial position and wavelengths of the modes is analogous to the Schrodinger modes for which proper treatments would be the path integral formulation. This choice of the colatitude radius is empirical, but will fit the observed mass values.
With these defined modes, Eq. (13), becomes:

$$\omega = \omega_c \left(1 \pm 2\sqrt{j\pi - k / \pi} \right)$$  \hspace{1cm} (17)

This “general concept” development is intuitive approach to understand the physical results. Replacing the $\sigma$ in Eq. (6) with $\left(1 \pm 2\sqrt{j\pi - k / \pi} \right)$ will give the same results as the development in the following section, but lacks the proper mathematical apparatus.

**Spin Matrix Representation**

The foregoing is based on a mechanical model of coupled normal modes of oscillation, but for a proper mathematical representation, the modes need to be defined in terms of the Pauli spin matrix. For the leptons it is expected that the modes described above can be properly incorporated into Eq. (6) by associating each modes with each of the orthogonal Pauli matrix.

Noting that the $\frac{1}{2}$ term in Eq. (6) is actually the $\frac{1}{2}\hbar$ spin angular momentum projection along the direction of motion, and can be represented by $(\frac{1}{2}\sigma_3)$. For the electron we know that the projections along each of the other axis also have an amplitude of $1/2$. It is proposed that this is not necessarily the case, but that the amplitudes of the other axis can have values associated with integral wavelengths as suggested in the normal modes discussed above. It is asserted that the other modes which are symmetric along the x and y axis can be assigned as amplitudes of the other matrix, and represented by:

$$\sqrt{j\pi \sigma_1 + i\sqrt{k / \pi} \sigma_2 + \frac{1}{2} \sigma_3}$$  \hspace{1cm} (18)

and:

$$\sigma^2 = j\pi + k / \pi + 1 / 4$$  \hspace{1cm} (19)

Putting Eq. (18), into Eq. (6) gives:

$$\tilde{\Theta} = e^{-} \left[ \pm im \mathcal{R} + \sqrt{j\pi \sigma_1 + i\sqrt{k / \pi} \sigma_2 + \frac{1}{2} \sigma_3} \right]^2$$  \hspace{1cm} (20)

The extra modes probably should not be interpreted as an angular momentum since the values don’t correspond to known values for particles. From the following, it more likely that it represents the amplitude of standing waves along these axes.
Noting that from Eq.(4), the value of the square of the spin vector is “real”, the magnitude of the square of the vector contributes to the scale factors of the function, and if the scale factor of the particle function is proportional to the mass, then the ratio of particle mass values can be determined:

It is asserted that the fundamental reference mass is the electron, and thus the contributions of the modes as noted in Eq.(7), would be such that the ratio of a particles mass to the mass of an electron is:

\[
\frac{m}{m_e} = \left[ e^{\sigma_A^2} + w e^{\sigma_B^2} \right]
\]

(21)

Where \(w\) is the sign of the second term.

After some trial and error, values of \(j\) and \(k\) can be arrived at that give mass values very close to the actual mass values of many particles. The sequence of the \(j, k\) for the particles has patterns that do not appear to be random numbers, but indicate some underlying, yet to be determined, selection rules.

### Calculated Particle Mass Ratio

The values of \(j\) and \(k\) that give the proper values for the mass of the leptons in terms of electron masses will be:

\[
\sigma_A^2 = j_1 \pi + k_1 / \pi + S_a
\]

(22)

\[
\sigma_B^2 = j_2 \pi + k_2 / \pi + S_a
\]

(23)

### Leptons

Interpreting the terms in Eq.(21), to be two coupled coexisting parts of a particle, and for leptons that have \(1/2\) spin, the value of one of the \(s\) term must be zero and the other must be \(S = 1/4\)

**Muon**

Mode amplitude square

\[
\sigma_A^2 = 0 \pi - 2 / \pi + 1 / 4
\]

\[
\sigma_B^2 = 2 \pi - 3 / \pi + 0
\]

(24)
Mass ratio  
<table>
<thead>
<tr>
<th>Experimental</th>
<th>Calculated</th>
<th>Error in parts</th>
</tr>
</thead>
<tbody>
<tr>
<td>206.7682648</td>
<td>206.7575378</td>
<td>1/1850.</td>
</tr>
</tbody>
</table>

Tauon

Mode amplitude square

\[ \sigma_A^2 = 0\pi + 6/\pi + 1/4 \]
\[ \sigma_B^2 = 3\pi - 4/\pi + 0 \] (25)

Mass ratio  
<table>
<thead>
<tr>
<th>Experimental</th>
<th>Calculated</th>
<th>Error in parts</th>
</tr>
</thead>
<tbody>
<tr>
<td>3477.1502</td>
<td>3477.381104</td>
<td>within exp. error</td>
</tr>
</tbody>
</table>

**Quark Matrix Representation**

By noting the mathematical relations between the Pauli matrix and the ud quarks the above procedure can be extended to at least the primary hadrons.

The composition of the hadrons are defined in terms of the SU(3) quark, or Gell-Mann matrix which are generalizations of the Pauli spin matrix, and the first three are identical to the Pauli matrix.

Using the quark wavefunctions noted by Xiangdong, [4] the meson wave function in terms of quarks for SU(2), is by noting the isosinglet \( I = 0 \) combination, is \( q^1\bar{q} \sim \left( u\bar{u} + d\bar{d} \right) / \sqrt{2} \), where \( \bar{q} = (\bar{u}, \bar{d}) \). The isotriplet \( I = 1 \) is \( \phi^i - q^i\bar{q} \) with \( i = 1, 2, 3 \), where \( \tau^i \) are 3 Pauli matrices,

\[ \phi^1 = \left( u\bar{d} + d\bar{u} \right) / \sqrt{2} \]
\[ \phi^2 = i \left( d\bar{u} - u\bar{d} \right) / \sqrt{2} \]
\[ \phi^3 = i \left( u\bar{u} - d\bar{d} \right) / \sqrt{2} \]

Inserting these into Eq.(20), for the Pauli matrix allows:

\[
\tilde{\Theta} = e^{[\pm i m \mathfrak{M} + \sqrt{j\pi} (u\bar{d} + d\bar{u}) + i\sqrt{k/\pi} (d\bar{u} - u\bar{d}) + \frac{1}{2} (u\bar{u} - d\bar{d})]^2} \]

(26)
This displays the function as having meson type modes with integral wave orbital amplitudes. For \( L=0 \) particles, like the proton and neutron, the waves must have zero orbital angular momentum, and thus be standing waves with integral wavelengths around the x and y axis.

Very accurate values of the mass ratios for the Proton and Neutron can be calculated, by picking certain values of the \( j \), and \( k \) integers.

### Proton amplitude square

\[
\sigma_A^2 = \frac{0 \pi - 1}{\pi + 1/4} \\
\sigma_B^2 = \frac{3 \pi - 6}{\pi + 0}
\]

### Neutron amplitude square

\[
\sigma_A^2 = \frac{0 \pi + 3}{\pi + 1/4} \\
\sigma_B^2 = \frac{3 \pi - 6}{\pi + 0}
\]

### \( \pi^0 \) Meson amplitude square

\[
-\sigma_A^2 = \frac{0 \pi - 3}{\pi + 1/4} \\
\sigma_B^2 = \frac{2 \pi - 3}{\pi + 1/4}
\]

### \( \pi^\pm \) Meson amplitude square

\[
\sigma_A^2 = \frac{0 \pi + 6}{\pi + 1/4} \\
\sigma_B^2 = \frac{2 \pi - 3}{\pi + 1/4}
\]

Note that the \( \pi \) mesons have a spin of zero with two opposite components. The square of each component is \( 1/4 \), and both contribute to the total mass.

For at least the baryons there are other aspects to the wavefunctions, however clearly the defined modes are the main modes that contribute to the mass. Since SU(3) symmetry is not exact, there is a mixing, it is presumed the difference between the calculated and experimental modes can be attributed to the contributions of other modes.

The following is a table listing the values and giving the error in one part per the experimental value. Most of the calculated values are well beyond a probability of random coincidence, and the totality of the accurate values indicates a physical causation.

Putting the mode values into Eq.(21), gives:

---

2 The \( \pi^0 \) is the only presented particle with a negative sign for \( w \) in Eq. (21)
Notable about the values is that the calculated value of the proton is within one part in three million, but still not within experimental error bars. QED effects may also play a role in the mass for the proton. Since this is a differentials approach to the issue, and does not include QED effects, subtle effects such as in the case of the electron gyro magnetic ratio may be significant. There are other issues such as parity, charge and angular momentum that affect mass, but these mode amplitude values seem to be the major effect.

Modes for most other baryons and mesons can be found, but experimental error bars are so wide, several modes can satisfy the mass relation. These selected particle masses are fairly accurate, and only one mode fits, thus more useful in finding selection rules and residuals.

### Residuals

Although the modes discovered for the shown particles produce highly accurate values of the mass ratios, there are still residuals between the calculated value and the experimentally determined values. The Tauon’s experimental mass is not extremely accurate, and the predicted value is within the experimental error bars, thus investigation of the residual values for this particle is not of use. The other evaluated particles are more accurate however, and the residual values to several decimal places are certain.

<table>
<thead>
<tr>
<th>Particle</th>
<th>Exp. value</th>
<th>Mode Calculated value</th>
<th>Residual</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proton</td>
<td>1836.152672</td>
<td>1836.152135</td>
<td>0.00053778(75)</td>
</tr>
<tr>
<td>Muon</td>
<td>206.7682913</td>
<td>206.7575378</td>
<td>0.0107535(35)</td>
</tr>
<tr>
<td>Neutron</td>
<td>1838.68372</td>
<td>1838.55468</td>
<td>0.12903</td>
</tr>
<tr>
<td>Pion ±</td>
<td>273.1320496</td>
<td>273.27954102</td>
<td>-0.14754</td>
</tr>
<tr>
<td>Pion 0</td>
<td>264.1426392</td>
<td>264.1154785</td>
<td>0.02661</td>
</tr>
</tbody>
</table>
The presumption would nominally be that the error is somehow a fraction of the mass and is proportional to the mass. This is not the case however, and there is no such proportionality. Instead the residuals are relatively small and apparently and independent of the total mass of the particle. There appears to be a relation between the masses, and is presumed to be the result of a QFT modification of the mod wavelengths similar to the contribution to the gyro magnetic ratio the multiple possible paths as a result of the interaction of the particles in the nucleus.

The following is a proposal for calculation the mass ratios that fits within the proposed theory. The values and functions can be determined, but the calculations will have to await the understanding of the internal particle mechanics.

The general expression for a spin ½ particle mass as illustrated above is:

$$m = m_e \left[ e^{(J \pi - \frac{L}{\pi} + 0)} + \text{w} \times e^{(\pi - K/\pi + 1/4)} \right]$$  \hspace{1cm} (31)

Presuming that the modes represent particle circulation in the nucleus then each mode has a path, and each path would be modified by the effects of QFT. The modes would then be:

$$J \pi + f_j(\alpha) \hspace{1cm} \frac{L}{\pi} + f_L(\alpha)$$  \hspace{1cm} (32)

This term is comparative small and can be factored out such that:

$$m = m_e \left[ e^{(J \pi - \frac{L}{\pi} + 0)} + \text{w} \times e^{(\pi - K/\pi + 1/4)} \right]$$

$$+ \left[ f_j(\alpha) + f_L(\alpha) \right] + \text{w} \left[ f_j(\alpha) + f_L(\alpha) \right]$$  \hspace{1cm} (33)

It has been determined, by trial and error, that the operative function is a combination of the series expansion of:

$$\frac{1}{1 - \alpha \pi} = 1 + \alpha \pi + (\alpha \pi)^2 + (\alpha \pi)^3,$$

In numbers this is:

$$\frac{1}{1 - \alpha \pi} = 1 + 0.0229253091 + 0.000525570 + 0.000012049 = 1.0234629277$$

This series is reminiscent of the anomalous magnetic moment corrections, and presumably has a similar origin.
Accurate calculations and evaluation shows the residuals are:

Proton: 
\[ \left( \alpha \pi \right)^2 + \left( \alpha \pi \right)^3 = 0.000537619^* \]

Muon: 
\[ 20 \times \left( \alpha \pi \right)^2 + \left( \alpha \pi \right)^3 = 0.01075238^* \]

Neutron: 
\[ 12 \times 20 \times \left( \alpha \pi \right)^2 + \left( \alpha \pi \right)^3 = 0.12902856^* \]

Pion\(^0\): 
\[ \pi \alpha + \frac{\alpha}{2} = 0.0265739854^* \]

Pion\(^\pm\): 
\[ 2\pi \times \left[ \pi \alpha + \left( \pi \alpha \right)^2 + \left( \pi \alpha \right)^3 \right] = 0.1474219226^* \]

*These values put the calculated value of the mass ratio within the current experimental error.

Summary

<table>
<thead>
<tr>
<th>Particle</th>
<th>Experimental Min</th>
<th>Experimental Max</th>
<th>Calculated Min</th>
<th>Calculated Max</th>
<th>Accuracy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proton</td>
<td>1836.1526717</td>
<td>1836.1526732</td>
<td>1836.1526722</td>
<td>1.22E+09</td>
<td></td>
</tr>
<tr>
<td>Muon</td>
<td>206.7682845</td>
<td>206.7682981</td>
<td>206.7682902</td>
<td>2.95E+07</td>
<td></td>
</tr>
<tr>
<td>Neutron</td>
<td>1838.68368</td>
<td>1838.68374</td>
<td>1838.68372</td>
<td>3.06E+07</td>
<td></td>
</tr>
<tr>
<td>Pion(^+)</td>
<td>273.1313</td>
<td>273.1327</td>
<td>273.1321</td>
<td>1.99E+05</td>
<td></td>
</tr>
<tr>
<td>Pion(^0)</td>
<td>264.1414</td>
<td>264.1428</td>
<td>264.1427</td>
<td>1.89E+05</td>
<td></td>
</tr>
</tbody>
</table>

The position of the calculated value can be placed in the experimental error bars as noted by the particle data group. and all of the evaluated particles are near the center of the value.
Some Speculation

The 12 and 240 values of the multiplier for the Muon and Neutron are exact multiples of the proton error. It is noted that the differences in the sum of the K and L modes (1/π modes) between the proton and respective particles is 2 and 4

<table>
<thead>
<tr>
<th>K+L</th>
<th>Δ</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proton</td>
<td>7</td>
</tr>
<tr>
<td>Muon</td>
<td>5</td>
</tr>
<tr>
<td>Neutron</td>
<td>3</td>
</tr>
</tbody>
</table>

Putting these values into the relation:

\[
\text{Muon Multiplier} = \left[ +2^\Delta + \left( 2^\Delta \right)^2 \right] = 20 \tag{34}
\]

And

\[
\text{Proton Multiplier} = \left[ -2^\Delta + \left( 2^\Delta \right)^2 \right] = 12 \times 20 = 240 \tag{35}
\]

generates the respective values.

This same series is suggested by the anomalous magnetic moment corrections, in that for a delta of 3 the value is 72, the number of Feynman diagrams in the three loop correction for the electron.

The implication is that the difference in the number of K & L modes between the Proton and the other two particles has a QFT correction factor related to a number loops and Feynman diagrams for the internal motion of constituent particles.

This is clearly speculation for the purpose of offering some suggestions for those who would pursue the task.

which are the 1/π

CONCLUSION

This paper has presented a plausible, relatively straightforward explanation of particle mass ratios, and the specific values for some well-known particles. The additional nuclear modes postulated, are similar to the Schrödinger modes in the atom. This paper should still be considered speculative, but the mass ratios for the Proton, Neutron, Muon, Tauon, and Pions, are all calculated within experimental error, and for the proton this accuracy is within parts per billion. With the absence of a mathematical rela-
tion between the residuals there would be a concern that the series mode values could produce random numbers close to the mass values. The results are that, even if the source of this mathematical relation is speculative, the belief that the modes are random is put to rest.

Références:


Appendix I

Definitions and Conventions

The radius of particle universe
\[ R = cT = R_0 + ct \]

Four velocity
\[ \gamma^\mu \left( \frac{v}{c} \right)_\mu = p \quad \text{unitless} \]
Three velocity\n\[ \gamma^k(\mathbf{v})_k = \mathbf{v} = \gamma^k \cdot \mathbf{v} \]

Null unit vector\n\[ \gamma^\mu = \gamma^0 + \bar{\eta} \cdot \mathbf{v} \]
\[ \bar{\eta} \cdot \bar{\eta} = -1 \]

Mass in this paper\n\[ m = \frac{mc}{\hbar} = \frac{1}{\bar{\lambda}} \]

Rest mass\n\[ m_0 = \frac{mc}{\hbar} \]

Compton radius\n\[ \bar{\lambda} = \frac{\hbar}{mc} \]

Vector 4 potential\n\[ \mathbf{\bar{A}} \]

Charge sign of m particle\n\[ m \pm \]

Shortened derivatives\n\[ \gamma^\mu \frac{\partial}{\partial(x^\mu)} = \gamma^\mu \partial^\mu \]

The Dirac gamma matrix convention:
\[
\begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & -1 \\
\end{bmatrix}
\]

\[
\gamma^1 = \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & -1 \\
\end{bmatrix}
\gamma^2 = \begin{bmatrix}
i & 0 & 0 & -i \\
0 & i & 0 & 0 \\
0 & 0 & i & 0 \\
-i & 0 & 0 & i \\
\end{bmatrix}
\gamma^3 = \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-1 & 0 & 0 & 1 \\
\end{bmatrix}
\gamma^0 = \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[ \gamma^1 \gamma^1 = -1, \quad \gamma^2 \gamma^2 = -1, \quad \gamma^3 \gamma^3 = -1, \quad \gamma^0 \gamma^0 = +1. \]

Pauli spin matrix:
\[
\sigma_1 = \begin{bmatrix}
0 & 1 \\
1 & 0 \\
\end{bmatrix}
\sigma_2 = \begin{bmatrix}
0 & -i \\
i & 0 \\
\end{bmatrix}
\sigma_3 = \begin{bmatrix}
1 & 0 \\
0 & -1 \\
\end{bmatrix}
\]

Feynman slash notation:
\[ \mathcal{K} = \gamma^\mu A_\mu \]
\[ \mathcal{A} = \mathcal{A}^\mu A_\mu = a^2 \]
\[ \mathcal{A} \mathcal{A} + \mathcal{A} \mathcal{A} = 2 \mathcal{A} \cdot \mathcal{A} \]
\[ \mathcal{P} = \left( \gamma^1 v_x + \gamma^2 v_y + \gamma^3 v_z + \gamma^0 c \right) / c \]