Analytical Derivation of the Drag Curve $C_D = C_D(\mathcal{R})$

Armando V.D.B. Assis
Departamento de Física,
Universidade Federal de Santa Catarina,
88040-900 Florianópolis, SC, Brazil

(Dated: December 20, 2011)

Through a convenient mathematical approach for the Navier-Stokes equation, we obtain the quadratic dependence v^2 of the drag force F_D on a falling sphere, and the drag coefficient, C_D, as a function of the Reynolds number. Viscosity effects related to the turbulent boundary layer under transition, from laminar to turbulent, lead to the tensorial integration related to the flux of linear momentum through a conveniently choosen control surface in the falling reference frame. This approach turns out to provide an efficient route for the drag force calculation, since the drag force turns out to be a field of a non-inertial reference frame, allowing an arbitrary and convenient control surface, finally leading to the quadratic term for the drag force.

DEFINING THE MATHEMATICAL PROBLEM

Regarding the application of the Newton second law to a small closed subsystem σ with boundary $\partial(\delta V)$ and volume δV of a continuum fluid in an inertial reference frame, one obtains at an instant t:

$$\int_\sigma d\vec{F}_{ext} = \int_{\delta V} \rho(\vec{r},t)\vec{f}(\vec{r},t)dV + \oint_{\partial(\delta V)} \mathbf{T}\cdot\hat{n}dS, \tag{1}$$

where $\vec{f}(\vec{r},t)$ is a locally external acceleration field, $\rho(\vec{r},t)$ the scalar density field, and $\mathbf{T}$ is the most general tensor due to the effects of the surrounding fluid on σ, being given by:

$$T_{ik} = -p\delta_{ik} + \eta\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_\lambda}{\partial x_\lambda}\right) + \zeta\delta_{ik}\frac{\partial v_\lambda}{\partial x_\lambda}, \tag{2}$$

where p is the local thermodynamic pressure field, being $\mathbf{T}$ written in terms of its components under the summation convention on repeated indices and where $\mathbf{T}$ was obtained from the combination of effects due to strain and shear:

$$\mathbf{\Gamma} = \alpha(\nabla\vec{v})_{ts} + \beta(\nabla\vec{v})_c = \alpha\left[(\nabla\vec{v})_s - \frac{1}{3}\vec{\nabla}\cdot\vec{v}\mathbf{1}\right] + \zeta\,\vec{\nabla}\cdot\vec{v}\mathbf{1}, \tag{3}$$

from which one defines the viscosity coefficients, $\alpha = 2\eta$ (this latter relation following from the coupling to the planar flow case, in which one defines the dynamical viscosity η) and ζ, under an isotropic assumption. Back to the Eq. (1), one obtains the Navier-Stokes equation:

$$\rho(\vec{r},t)\dot{\vec{v}}(\vec{r},t) - \rho(\vec{r},t)\vec{f}(\vec{r},t) + \vec{\nabla}p(\vec{r},t) - \eta\vec{\nabla}^2\vec{v}(\vec{r},t) + \\ -\left(\frac{1}{3}\eta + \zeta\right)\vec{\nabla}\left(\vec{\nabla}\cdot\vec{v}(\vec{r},t)\right) = \vec{0}. \tag{4}$$

Under a divergence-free hypothesis for the velocity field (constant density turns out to be a sufficient condition), one has got, hence, in the ground reference frame, the following mathematical problem:

$$\begin{cases} \rho\dot{\vec{v}} - \rho\vec{g} + \vec{\nabla}p - \eta\vec{\nabla}^2\vec{v} = \vec{0}, \quad \vec{\nabla}\cdot\vec{v} = 0; \\ \lim\limits_{|\vec{r}|\to\infty} \vec{v} = \vec{0}, \quad \vec{v}\,(\partial\, sphere) = \dot{h}(t)\hat{e}_z \ \ nonslip, \end{cases} \tag{5}$$

where $\vec{g}$ is the local gravitational field and $\dot{h}(t)$ is the scalar velocity of the center of a falling sphere within the fluid.

GEDANKENEXPERIMENT

One measures the local gravitational field in the non-inertial frame attached to the falling sphere from the following gedankenexperiment: hollow sphere having got mass m, with an internal weighing apparatus (with negligible mass) to measure the normal force $\vec{N}$ that the ground of the hollow sphere exerts on a proof mass m_0. By isolating the system $m + m_0$, and, subsequently, by isolating the system m_0, one obtains:

$$\frac{\vec{N}}{m_0} = \frac{\vec{F}'_{\text{drag}}}{(m+m_0)} = \frac{\vec{F}_{\text{drag}} + \delta\vec{F}_{\text{drag}}(m_0)}{(m+m_0)}, \tag{6}$$

where $\vec{F}_{\text{drag}}$ is the force the fluid exerts on the hollow sphere, without the proof mass m_0, and $\delta\vec{F}_{\text{drag}}(m_0)$ is the increment to this force - due to the consideration of the internal proof mass m_0. Hence, the gravitational field $\vec{g}_0$ within the hollow sphere is given by:

$$\vec{g}_0 = \lim_{m_0\to 0} -\frac{\vec{N}}{m_0} = \lim_{m_0\to 0} -\frac{\vec{F}_{\text{drag}} + \delta\vec{F}_{\text{drag}}(m_0)}{(m+m_0)} = -\frac{\vec{F}_{\text{drag}}}{m}, \tag{7}$$

from which the force we want to calculate turns out to be a property of the non-inertial reference frame attached to the sphere. Adopting this falling reference frame, we have got the mathematical problem:

$$\begin{cases} \rho\dot{\vec{v}} + \rho\dfrac{\vec{F}_{\text{drag}}}{m} + \vec{\nabla}p - \eta\vec{\nabla}^2\vec{v} = \vec{0}, \quad \vec{\nabla}\cdot\vec{v} = \vec{0}; \\ \lim\limits_{|\vec{r}|\to\infty} \vec{v} = -\dot{h}(t)\hat{e}_z, \quad \vec{v}\,(\partial\, sphere) = \vec{0}\ nonslip. \end{cases} \tag{8}$$

Comparing the Eqs. (5) and (8), one infers the force we want to calculate (divided by the sphere mass m) turns out to be an acceleration field in the adopted reference frame, given by the Eq. (7). This turns out to be a field at each point of the fluid in the falling reference frame attached to the sphere, from which one may choose a convenient control surface of integration surrounding the sphere, not only the surface of the sphere. This provides an identity, derived in the next section, from which one may extract a convenient information from the adopted control surface.

CALCULATING $\vec{g}_0$

Applying the continuity equation in its most general form, calculating the instantaneous time rate of linear momentum variation within an arbitrary control volume, fixed and undeformable, one reaches the expression for the calculation of $\vec{g}_0$:

$$-\vec{g}_0 = \frac{\vec{F}_{\text{drag}}}{m} = \frac{1}{\int \rho dV}\left(\oint \Pi\cdot\hat{n}dS - \frac{\partial}{\partial t}\int \rho\vec{v}dV\right), \tag{9}$$

$$\Pi = \left[-\mathbf{1}p + \Gamma - \rho\left(\vec{v}\otimes\vec{v}\right)\right], \tag{10}$$

since $\vec{g}_0$ does not depend on the spatial coordinates within the fluid, once this field equally permeates each point of the fluid in the falling reference frame at any given instant t.

OBTAINING THE DRAG FORCE F_D AND THE DRAG COEFFICIENT C_D

Applying the Eqs. (9) and (10) to the control region $FGBAF$ depicted in the Fig. 1, at the stationary flow regime $t\to\infty$, one obtains:

$$\vec{F}^{\infty} = \frac{m}{m+m_{BL}}\left\{-\left[\int_{FG} + \int_{GB} + \int_{BA} + \int_{AF}\right]\mathbf{1}p_{\infty}\cdot\hat{n}dS - \int_{FG}\left[\rho\left(\vec{v}_{\infty}\otimes\vec{v}_{\infty}\right)\right]\cdot\hat{n}dS\right\}, \tag{11}$$

where m_{BL} is the mass of the boundary layer attached to the sphere. The pressure field on FG can be obtained, since this choosen surface (FG) does not violate the laminarity condition for a sufficiently thin boundary layer $\partial p/\partial r \approx 0$, in relation to DC, in virtue of the internal confinement of turbulence within the region $GBAFG$ (AB touching the boundary layer).

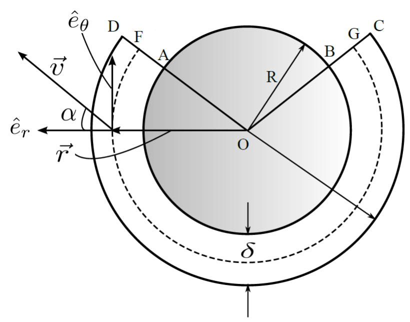

Fig. 1: Figure for the integration.

Hence:

$$p_{\infty}^{FG} = -\frac{\rho}{m}\varphi_{FG}^{\infty} + p_{\infty}^{0} - \frac{9}{8}\rho\left(\dot{h}^{\infty}(t)\right)^2\sin^2\theta, \tag{12}$$

where φ^{∞} is a scalar field due to the vanishing rotational of the force the fluid exerts on the sphere. The pressure field on AB (AB touching the *wake*, the rear region of the flow) is obtained from the condition of broken equilibrium at the separation point $S\approx B$. The obtention of the velocity field profile internal to the boundary layer is accomplished by the time average on the ensemble of turbulence within the boundary layer with the Fourier representation of the velocity field on FG by the step function, since we are firstly interested in the contribution term at high Reynolds number provided a full turbulent flow within the boundary layer at the brink of the drag crisis, where the drag force will suddenly decrease. Hence:

$$p_{\infty}^{S} = -\frac{\rho}{m}\varphi_{GBAF}^{\infty} + p_{\infty}^{0} - \frac{9}{16}\rho\left(\dot{h}^{\infty}(t)\right)^2\sin^2\theta_S, \tag{13}$$

$$\left\langle\left(\vec{v}\otimes\vec{v}\right)_{FG}\cdot\hat{n}\right\rangle_t = \left(\vec{v}_{\infty}\otimes\vec{v}_{\infty}\right)\cdot\hat{n} =$$

$$\left\langle v_{FG}^2\left(R+\delta',\theta,t\right)\left(\cos^2\alpha(t)\,\hat{e}_r + \cos\alpha(t)\sin\alpha(t)\,\hat{e}_\theta\right)\right\rangle_t =$$

$$\frac{9}{16}\left(\dot{h}^{\infty}(t)\right)^{2}\sin^{2}\theta\,\hat{e}_{r},\tag{14}$$

where θ_S is the separation angle, depicted in the Fig. 2.

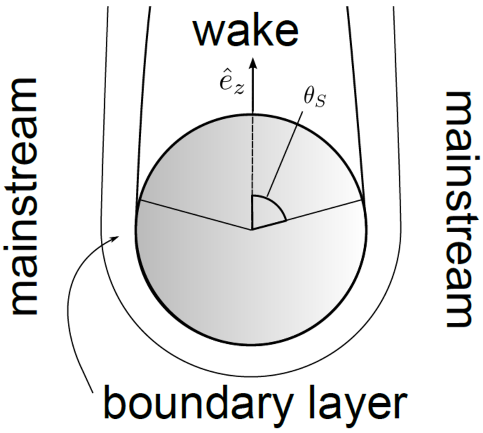

Fig. 2: Depicted elements.

Using these results within the Eq. (11), one obtains the quadratic contribution for the drag force via straightforward integration:

$$\left(1+\frac{m_{BL}}{m}\right)\vec{F}^{\infty}=\text{Buoyancy}+\frac{m_{BL}}{m}\vec{F}^{\infty}+$$

$$+\frac{9\pi}{32}\rho\left(\dot{h}^{\infty}(t)\right)^{2}R^{2}\sin^{4}\theta_{S}\,\hat{e}_{z}\Rightarrow$$

$$\vec{F}_{D}=\frac{9\pi}{32}\rho\left(\dot{h}^{\infty}(t)\right)^{2}R^{2}\sin^{4}\theta_{S}\,\hat{e}_{z}.\tag{15}$$

Renaming $\dot{h}^{\infty}(t)\equiv v$, knowing that the drag force points along the $\hat{e}_z$ direction, we simply write for the quadratic drag force contribution, the quadratic scalar component:

$$F_{D}=\frac{9\pi}{32}\rho v^{2}R^{2}\sin^{4}\theta_{S}.\tag{16}$$

One should notice this contribution arises from our consideration regarding the turbulent profile within the boundary layer, from which we see there is not any linear contribution arising at this flow regime. Writing the drag force as a series on v:

$$F_{D}(v)=\sum_{k=0}^{\infty}a_{k}v^{k},\tag{17}$$

we know from the low Reynolds number regime that the linear contribution is given by the Stokes force [4]:

$$a_{0}=0,\;\;a_{1}v=6\pi\eta Rv.\tag{18}$$

Hence, up to the drag crisis, the drag force reads:

$$F_{D}=6\pi\eta Rv+\frac{9\pi}{32}\rho\left(\sin^{4}\theta_{S}\right)R^{2}v^{2}.\tag{19}$$

The drag coefficient, C_D, and the Reynolds number, $\mathcal{R}$, are defined by:

$$C_{D}=\frac{2F_{D}}{\pi\rho R^{2}v^{2}},\;\;\mathcal{R}=\frac{2\rho Rv}{\eta}.\tag{20}$$

Hence, from the Eqs. (19) and (20), one obtains the drag coefficient as a function of the Reynolds number, up to the drag crisis:

$$C_{D}(\mathcal{R})=\frac{24}{\mathcal{R}}+\frac{9}{16}\sin^{4}\theta_{S}.\tag{21}$$

Fig. 3 shows the graph for the Eq. (21), for $\theta_S = 70.4°$. This is the separation angle obtained from the Froessling method [1]. One sees this dependence on the Reynolds number agrees with the experimental one over the entire range of Reynolds numbers up to the drag crisis, as one verify, e.g., in [2] and [3].

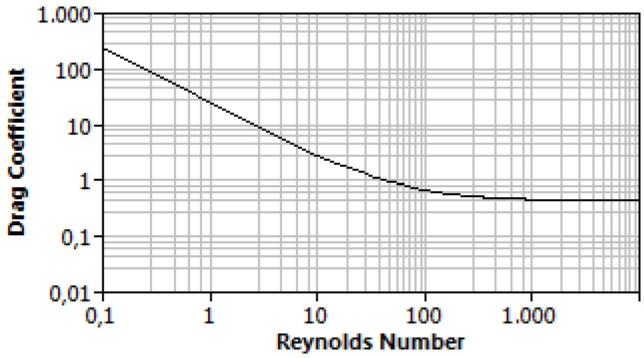

Fig. 3: Drag coefficient vs. Reynolds number, Eq. (21).

ACKNOWLEDGMENTS

A.V.D.B.A is grateful to Y.H.V.H and CNPq for financial support.

[1] H. Schlichting, *Boundary-Layer Theory* (McGraw-Hill Books, 1979), 2nd ed., english Edition.
[2] L. Landau and E. Lifshitz, *Course of Theoretical Physics, Vol.6.* (Pergamon Press, London-Paris-Frankfurt, 1959), 2nd ed., english Edition. 536 pp.
[3] K. S. George Constantinescu, Physics of Fluids **16**, 1449 (2004).
[4] The Oseen force will turn out to be within the quadratic contribution we obtained, viz., included within the Eq. (16). One may infer the Oseen force is a quadratic contribution.