Motives and Infinite Primes

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Abstract

In algebraic geometry the notion of variety defined by algebraic equation is very general: all number fields are allowed. One of the challenges is to define the counterparts of homology and cohomology groups for them. The notion of cohomology giving rise also to homology if Poincare duality holds true is central. The number of various cohomology theories has inflated and one of the basic challenges to find a sufficiently general approach allowing to interpret various cohomology theories as variations of the same motive as Grothendieck, who is the pioneer of the field responsible for many of the basic notions and visions, expressed it.

Cohomology requires a definition of integral for forms for all number fields. In p-adic context the lack of well-ordering of p-adic numbers implies difficulties both in homology and cohomology since the notion of boundary does not exist in topological sense. The notion of definite integral is problematic for the same reason. This has led to a proposal of reducing integration to Fourier analysis working for symmetric spaces but requiring algebraic extensions of p-adic numbers and an appropriate definition of the p-adic symmetric space. The definition is not unique and the interpretation is in terms of the varying measurement resolution.

The notion of infinite has gradually turned out to be more and more important for quantum TGD. Infinite primes, integers, and rationals form a hierarchy completely analogous to a hierarchy of second quantization for a super-symmetric arithmetic quantum field theory. The simplest infinite primes representing elementary particles at given level are in one-one correspondence with many-particle states of the previous level. More complex infinite primes have interpretation in terms of bound states.

1. What makes infinite primes interesting from the point of view of algebraic geometry is that infinite primes, integers and rationals at the n:th level of the hierarchy are in 1-1 correspondence with rational functions of n arguments. One can solve the roots of associated polynomials and perform a root decomposition of infinite primes at various levels of the hierarchy and assign to them Galois groups acting as automorphisms of the field extensions of polynomials defined by the roots coming as restrictions of the basic polynomial to planes $x_n = 0$, $x_n = x_{n-1} = 0$, etc...

2. These Galois groups are suggested to define non-commutative generalization of homotopy and homology theories and non-linear boundary operation for which a geometric interpretation in terms of the restriction to lower-dimensional plane is proposed. The Galois group G_k would be analogous to the relative homology group relative to the plane $x_{k-1} = 0$ representing boundary and makes sense for all number fields also geometrically. One can ask whether the invariance of the complex of groups under the permutations of the orders of variables in the reduction process is necessary. Physical interpretation suggests that this is not the case and that all the groups obtained by the permutations are needed for a full description.

3. The algebraic counterpart of boundary map would map the elements of G_k identified as analog of homotopy group to the commutator group $[G_{k-2}, G_{k-2}]$ and therefore to the unit element of the abelianized group defining cohomology group. In order to obtains something analogous to the ordinary homology and cohomology groups one must however replaces Galois groups by their group algebras with values in some field or ring. This allows to define the analogs of homotopy and homology groups as their abelianizations. Cohomotopy, and cohomology would emerge as duals of homotopy and homology in the dual of the group algebra.

4. That the algebraic representation of the boundary operation is not expected to be unique turns into blessing when on keeps the TGD as almost topological QFT vision as the guide line. One can include all boundary homomorphisms subject to the condition that the anticommutator $\delta^i_k \delta^j_{k-1} + \delta^j_k \delta^i_{k-1}$ maps to the group algebra of the commutator group $[G_{k-2}, G_{k-2}]$. By adding dual generators one obtains what looks like a generalization of anticommutative fermionic algebra and what comes in mind is the spectrum of quantum states of a SUSY algebra spanned by bosonic states realized as group algebra elements and fermionic states realized in terms of homotopy and cohomotopy and in abelianized version in terms of homology and cohomology. Galois group action allows to organize quantum states into multiplets of Galois groups acting as symmetry groups of physics. Poincare duality would map the analogs of fermionic creation operators to annihilation operators and vice versa and the counterpart of pairing of k:th and $n-k$:th homology groups would be inner product analogous to that given by Grassmann integration. The interpretation in terms of fermions turns however to be wrong and the more appropriate interpretation is in terms of Dolbeault cohomology applying to forms with homomorphic and antiholomorphic indices.

5. The intuitive idea that the Galois group is analogous to 1-D homotopy group which is the only non-commutative homotopy group, the structure of infinite primes analogous to the braids of braids of braids of ... structure, the fact that Galois group is a subgroup of permutation group, and the possibility to lift permutation group to a braid group suggests a representation as flows of 2-D plane with punctures giving a direct connection with topological quantum field theories for braids, knots and links. The natural assumption is that the flows are induced from transformations of the symplectic group acting on $\delta M^2_\pm \times CP_2$ representing quantum fluctuating degrees of freedom associated with WCW ("world of classical worlds"). Discretization of WCW and cutoff in the number of modes would be due to the finite measurement resolution. The outcome would be rather far reaching: finite measurement resolution would allow to construct WCW spinor fields explicitly using the machinery of number theory and algebraic geometry.

6. A connection with operads is highly suggestive. What is nice from TGD perspective is that the non-commutative generalization homology and homotopy has direct connection to the basic structure of quantum TGD almost topological quantum theory where braids are basic objects and also to hyper-finite factors of type II_1. This notion of Galois group makes sense only for the algebraic varieties for which coefficient field is algebraic extension of some number field. Braid group approach however allows to generalize the approach to completely general polynomials since the braid group make sense also when the ends points for the braid are not algebraic points (roots of the polynomial).

This construction would realize thge number theoretical, algebraic geometrical, and topological content in the construction of quantum states in TGD framework in accordance with TGD as almost TQFT philosophy, TGD as infinite-D geometry, and TGD as generalized number theory visions.

This picture leads also to a proposal how p-adic integrals could be defined in TGD framework.

1. The calculation of twistorial amplitudes reduces to multi-dimensional residue calculus. Motivic integration gives excellent hopes for the p-adic existence of this calculus and braid representation would give space-time representation for the residue integrals in terms of the braid points representing poles of the integrand: this would conform with quantum classical correspondence. The power of 2π appearing in multiple residue integral is problematic unless it disappears from scattering amplitudes. Otherwise one must allow an extension of p-adic numbers to a ring containing powers of 2π.

2. Weak form of electric-magnetic duality and the general solution ansatz for preferred extremals reduce the Kähler action defining the Kähler function for WCW to the integral of Chern-Simons 3-form. Hence the reduction to cohomology takes places at space-time level and since p-adic cohomology exists there are excellent hopes about the existence of p-adic variant of Kähler action. The existence of the exponent of Kähler gives additional powerful constraints on the value of the Kähler fuction in the intersection of real and p-adic worlds consisting of algebraic partonic 2-surfaces and allows to guess the general form of the Kähler action in p-adic context.

3. One also should define p-adic integration for vacuum functional at the level of WCW. p-Adic thermodynamics serves as a guideline leading to the condition that in p-adic sector exponent of Kähler action is of form $(m/n)^r$, where m/n is divisible by a positive power of p-adic prime p. This implies that one has sum over contributions coming as powers of p and the challenge is to calculate the integral for K= constant surfaces using the integration measure defined by an infinite power of Kähler form of WCW reducing the integral to cohomology which should make sense also p-adically. The p-adicization of the WCW integrals has been discussed already earlier using an approach based on harmonic analysis in symmetric spaces and these two approaches should be equivalent. One could also consider a more general quantization of Kähler action as sum $K = K_1 + K_2$ where $K_1 = rlog(m/n)$ and $K_2 = n$, with n divisible by p since $exp(n)$ exists in this case and one has $exp(K) = (m/n)^r \times exp(n)$. Also transcendental extensions of p-adic numbers involving $n + p - 2$ powers of $e^{1/n}$ can be considered.

4. If the Galois group algebras indeed define a representation for WCW spinor fields in finite measurement resolution, also WCW integration would reduce to summations over the Galois groups involved so that integrals would be well-defined in all number fields.

p-Adic physics is interpreted as physical correlate for cognition. The so called Stone spaces are in one-one correspondence with Boolean algebras and have typically 2-adic topologies. A generalization to p-adic case with the interpretation of p pinary digits as physically representable

Boolean statements of a Boolean algebra with $2^n > p > p^{n-1}$ statements is encouraged by p-adic length scale hypothesis. Stone spaces are synonymous with profinite spaces about which both finite and infinite Galois groups represent basic examples. This provides a strong support for the connection between Boolean cognition and p-adic space-time physics. The Stone space character of Galois groups suggests also a deep connection between number theory and cognition and some arguments providing support for this vision are discussed.

1 Introduction

The construction of twistor amplitudes has led to the realization that the work of Grothendieck related to motivic cohomology simplifies enormously the calculation of the integrals of holomorphic forms over sub-varieties of the projective spaces involved. What one obtains are integrals of multivalued functions known as Grassmannian poly-logarithms generalizing the notion of poly-logarithm [2] and Goncharov has given a simple formula for these integrals [5] using methods of motivic cohomology [22] in terms of classical polylogarithms $Li_k(x)$, $k = 1, 2, 3, ...$ This suggests that motivic cohomology might have applications in quantum physics also as a a conceptual tool. One could even hope that quantum physics could provide fresh insights algebraic geometry and topology.

Ordinary theoretical physicist probably does not encounter the notions of homotopy, homology, and cohomology in his daily work and Grothendieck's work looks to him (or at least me!) like a horrible abstraction going completely over the head. Perhaps it is after all good to at least try to understand what this all is about. The association of new ideas with TGD is for me the most effective manner to gain at least the impression that I have managed to understand something and I will apply this method also now. If anything else, this strategy makes the learning of new concepts an intellectual adventure producing genuine surprises, reckless speculations, and in some cases perhaps even genuine output. I do not pretend of being a real mathematician and I present my humble apologies for all misunderstandings unavoidable in this kind enterprise. One should take the summary about the basics of cohomology theory just as a summary of a journalist. I still hope that these scribblings could stimulate mathematical imagination of a real mathematician.

While trying to understand Wikipedia summaries about the notions related to the motivic cohomology I was surprised in discovering how similar the goals and basic ideas about how to achieve them of quantum TGD and motive theory are despite the fact that we work at totally different levels of mathematical abstraction and technicality. I am however convinced that TGD as a physical theory represents similar high level of abstraction and therefore dare hope that the interaction of the these ideas might produce something useful. As a matter fact, I was also surprised that TGD indeed provides a radically new approach to the problem of constructing topological invariants for algebraic and even more general surfaces.

1.1 What are the deep problems?

In motivic cohomology one wants to relate and unify various cohomologies defined for a given number field and its extensions and even for different number fields if I have understood correctly. In TGD one would like to fuse together real and various p-adic physics and this would suggest that one must relate also the cohomology theories defined in different number fields. Number theoretical universality [15] allowing to relate physics in different number fields is one of the key ideas involved.

Why the generalization of homology [15] and cohomology [5] to p-adic context is so non-trivial? Is it the failure of the notion of boundary does not allow to define homology in geometric sense in p-adic context using geometric approach. The lack of definite integral in turn does not allow to define p-adic counterparts of forms except as a purely local notion so that one cannot speak about values of forms for sub-varieties. Residue calculus provides one way out and various cohomology theories defined in finite and p-adic number fields actually define integration for forms over closed surfaces (so that the troublesome boundaries are not needed), which is however much less than genuine integration. In twistor approach to scattering amplitudes one indeed encounters integrals of forms for varieties in projective spaces.

Galois group [12] is defined as the group leaving invariant the rational functions of roots of polynomial having values in the original field. A modern definition is as the automorphism group of the algebraic extension of number field generated by roots with the property that it acts trivially in the original field.

1. Some examples Galois group in the field or rationals are in order. The simplest example is second order polynomial in the field of rationals for which the group is Z_2 if roots are not rational numbers. Second example is $P(x) = x^n - 1$ for which the group is cyclic group $S(n)$ permuting the roots of unity which appear in the elementary symmetric functions of the roots which are rational. When the roots are such that all their products except the product of all roots are irrational numbers, the situation is same since all symmetric functions appearing in the polynomial must be rational valued. Group is smaller if the product for two or more subsets of roots is real. Galois group generalizes to the situation when one has a polynomial of many variables: in this case one obtains for the first variable ordinary roots but polynomials appearing as arguments. Now one must consider algebraic functions as extension of the algebra of polynomial functions with rational coefficients.

2. Galois group permutes branches of the graph $x = (P_n^{-1})(y, ...)$ of the inverse function of the polynomial analogous to the group permuting sheets of the covering space. Galois group is therefore analogous to first homotopy group. Since Galois group is subgroup of permutation group, since permutation group can be lifted to braid group acting as the first homotopy group on plane with punctures, and since the homotopies of plane can be induced by flows, this analogy can be made more precise and leads to a connection with topological quantum field theories for braid groups.

3. Galois group makes sense also in padic context and for finite fields and its abelianization by mapping commutator group to unit element gives rise to the analog of homology group and by Poincare duality to cohomology group. One can also construct p-adic and finite field representations of Galois groups.

These observations motivate the following questions. Could Galois group be generalized to so that they would give rise to the analogs of homotopy groups and homology and cohomology groups as their abelianizations? Could one find a geometric representation for boundary operation making sense also in p-adic context?

1.2 TGD background

The visions about physics as geometry and physics as generalized number theory suggest that number theoretical formulation of homotopy-, homology-, and cohomology groups might be possible in terms of a generalization of the notion of Galois group, which is the unifying notion of number theory. Already the observations of Andre Weil suggesting a deep connection between topological characteristics of a variety and its number theoretic properties indicate this kind of connection and this is what seems to emerge and led to Weil cohomology formulated. The notion of motivic Galois group is an attempt to realize this idea.

Physics as a generalized number theory involves three threads.

1. The fusion of real and p-adic number fields to a larger structure requires number theoretical universality in some sense and leads to a generalization of the notion of number by fusion reals and p-adic number fields together along common rationals (roughly) [15].

2. There are good hopes that the classical number fields could allow to understand standard model symmetries and there are good hopes of understanding $M^4 \times CP_2$ and the classical dynamics of space-time number theoretically [16].

3. The construction of infinite primes having interpretation as a repeated second quantization of an supersymmetric arithmetic QFT having very direct connections with physics is the third thread [14]. The hierarchy has many interpretations: as a hierarchy of space-time sheets for many-sheeted space with each level of hierarchy giving rise to elementary fermions and bosons as bound states of lower level bosons and fermions, hierarchy of logics of various orders realized as statements about statements about..., or a hierarchy of polynomials of several variables with a natural ordering of the arguments.

   This approach leads also to a generalization of the notion of number by giving it an infinitely complex number theoretical anatomy implied by the existence of real units defined by the ratios of infinite primes reducing to real units in real topology. Depending one one's tastes one can speak about number theoretic Brahman=Atman identity or algebraic holography. This picture generalizes to the level of quaternionic and octonionic primes and leads to the proposal that standard model quantum numbers could be understand number theoretically. The proposal is that the number theoretic anatomy could allow to represent the "world of classical worlds" (WCW) as sub-manifolds of the infinite-dimensional space of units assignable to single point of space-time and also WCW spinor fields as quantum superpositions of the units. One also ends up with he idea that there is an evolution associated with the points of the imbedding space as an increase of number theoretical complexity. One could perhaps say that this space represents "Platonia".

1.3 Homology and cohomology theories based on groups algebras for a hierarchy of Galois groups assigned to polynomials defined by infinite primes

The basic philosophy is that the elements of homology and cohomology should have interpretation as states of supersymmetric quantum field theory just as the infinite primes do have. Even more, TGD as almost topological QFT requires that these groups should define quantum states in the Universe predicted by quantum TGD. The basic ideas of the proposal are simple.

1. One can assign to infinite prime at n:th level of hierarchy of second quantizations a rational function and solve its polynomial roots by restricting the rational function to the planes $x_n, ...x_k = 0$. At the lowest level one obtains ordinary roots as algebraic number. At each level one can assign Galois group and to this hierarchy of Galois groups one wants to assign homology and cohomology theories. Geometrically boundary operation would correspond to the restriction to the plane $x_k = 0$. Different permutations for the restrictions would define non-equivalent sequences of Galois groups and the physical picture suggests that all these are needed to characterize the algebraic variety in question.

2. The boundary operation applied to G_k gives element in the commutator subgroup $[G_{k-2}, G_{k-2}]$. In abelianization this element goes to zero and one obtains ordinary homology theory. Therefore one has the algebraic analog of homotopy theory,

3. In order to obtain both homotopy and cohomotopy and cohomology and homology as their abelizations plus a resemblance with ordinary cohomology one must replace Galois groups by their group algebras. The elements of the group algebras have a natural interpretation as bosonic wave functions. The dual of group algebra defines naturally cohomotopy and cohomology theories. One expects that there is a large number of boundary homomorphisms and the assumption is that these homomorphisms satisfy anticommutation relations with anticommutor equal to an element of commutator subgroup $[G_{k-2}, G_{k-2},]$ so that in abelianization one obtains ordinary anticommutation relations. The interpretation for the boundary and coboundary operators would be in terms of fermionic annihilation (creation) operators is suggesti so that homology and cohomology would represent quantum states of super-symmetric QFT. Poincare duality would correspond to hermitian conjugation mapping fermionic creation operators to annihilation operators and vice versa. It however turns out that the analogy with Dolbeault cohomology with several exterior derivatives is more approrpriate.

4. In quantum TGD states are realized as many-fermion states assignable to intersections of braids with partonic 2-surfaces. Braid picture is implied by the finite measurement resolution implying discretization at space-time level. Symplectic transformations in turn act as fundamental symmetries of quantum TGD and given sector of WCW corresponds to symplectic group as far as quantum fluctuating degrees of freedom are considered. This encourages the hypothesis that the hierarchy of Galois groups assignable to infinite prime (integer/rational) having interpretation in terms of repeated second quantization can be mapped to a braid of braids of .... The Galois group elements lifted to braid group elements would be realized as symplectic flows and boundary homomorphism would correspond to symplectic flow induced at given level in the interior of sub-braids and inducing action of braid group. In this framework the braided Galois group cohhomology would correspond to the states of WCW spinor fields in "orbital" degrees of freedom in finite measurement resolution realized in terms of number theoretical discretization.

If this vision is correct, the construction of quantum states in finite measurement resolution would have purely number theoretic interpretation and would conform with the interpretation of quantum TGD as almost topological QFT. That the groups characterize algebraic geometry than mere topology would give a concrete content to the overall important "almost" and would be in accordance with physics as infinite-dimensional geometry vision.

2 Some backgbround about homology and cohomology

Before representing layman's summary about the motivations for the motivic cohomology it is good to introduce some basic ideas of algebraic geometry [41].

2.1 Basic ideas of algebraic geometry

In algebraic geometry one considers surfaces defined as common zero locus for some number $m \leq n$ of functions in n-dimensional space and therefore having dimension $n-m$ in the generic case and one wants to find homotopy invariants for these surfaces: the notion of variety is more precise concept in algebraic geometry than surface. The goal is to classify algebraic surfaces represented as zero loci of collections of polynomials.

The properties of the graph of the map $y = P(x)$ in (x,y)-plane serve as an elementary example. Physicists is basically interested on the number of roots x for a given value of y. For polynomials one can solve the roots easily using computer and the resulting numbers are in the generic case algebraic numbers. Galois group is the basic object and permutes the roots with each other. It is analogous to the first homotopy group permuting the points of the covering space of graph having various branches of the many-valued inverse function $x = P^{-1}(y)$ its sheets. Clearly, Galois group has topological meaning but the topology is that of the imbedding or immersion.

There are invariants related to the internal topology of the surface as well as invariants related to the external topology such as Galois group. The generalization of the Galois group for polynomials of single variable to polynomials of several variables looks like an attractive idea. This would require an assignment of sequence of sub-varieties to a given variety. One can assign algebraic extensions also to polynomials and it would seem that these groups must be involved. For instance, the absolute Galois group associated with the algebraic closure of polynomials in algebraically closed field is free group of rank equal to the cardinality of the field (rank is the cardinality of the minimal generating set).

Homotopy [16], homology [16], and cohomology [16] characterize algebraically the shape of the surface as invariant not affected by continuous transformations and by homotopies. The notion of continuity depends on context and in the most general case there is no need to restrict the consideration to rational functions or polynomials or make restrictions on the coefficient field of these functions. For algebraic surfaces one poses restrictictions on coefficient field of polynimials and the ordinary real number based topology is replaced with much rougher Zariski topology for which algebraic surfaces define closed sets. Physicists might see homology and cohomology theories as linearizations of nonlinear notions of manifold and surface obtained by gluing together linear manifolds. This linearization allows to gain information about the topology of manifolds in terms of linear spaces assignable to surfaces of various dimensions.

In homology one considers formal sums for these surfaces with coefficients in some field and basically algebraizes the statement that boundary has no boundary. Cohomology is kind of dual of homology and in differential geometry based cohomology forms having values as their integrals over surfaces of various dimensions realize this notion.

Betti cohomology or singular cohomology [1] defined in terms of simplicial complexes is probably familiar for physicists and even more so the de Rham cohomology [7] defined by n-forms as also the Dolbeault cohomology [8] using forms characterized by m holomorphic and n antiholomorphic indices. In this case the role of continuous maps is taken by holomorphic maps. For instance, the classification of the moduli of 2-D Riemann surfaces involves in an essential manner the periods of one forms on 2-surfaces and plays important role in the TGD based explanation of family replication phenomenon [4].

In category theoretical framework homology theory can be seen as a http://en.wikipedia.org/wiki/functorfunctor [11] that assigns to a variety (or manifold) a sequence of homology groups characterized by the dimension of corresponding sub-manifolds. One considers formal sums of surfaces. The basic operation is that of taking boundary which has operation δ as algebraic counterpart. One identifies cycles as those sums of surfaces for which algebraic boundary vanishes. This is identically true for exact cycles defined as a boundaries of cycles since boundary of boundary is empty. Only those cycles with are not exact matter and the homology group is defines as the coset space of the kernel at n:th level with respect to the image of the $n+1$:th level two spaces. Cohomology groups can be defined in a formally similar manner and for de Rham cohomology Poincare duality maps homology group H_k to H^{n-k}. The correspondence between covariant with vanishing exterior derivative and contravariant antisymmetric tensors with vanishing divergence is the counterpart of homology-cohomology correspondence in Riemann manifolds.

The calculation of homology and cohomology groups relies on general theorems which are often raised to the status of axioms in generalizations of cohomology theory.

1. Exact sequences [10] of Abelian groups define an important calculational tool. So called short exact sequence $0 \rightarrow B \rightarrow C \rightarrow 0$ of chain complexes gives rise to long exact sequence $H_n(A) \rightarrow H_n(B) \rightarrow H_n(C) \rightarrow H_{n-1}(A) \rightarrow H_{n-1}(B) \rightarrow H_{n-1}(C)$....

   One example of short exact sequence is $0 \rightarrow H \rightarrow G \rightarrow G/H \rightarrow 0$ holding true when H is normal subgroup so that also G/H is group. This condition allows to express the homology groups of G as direct sums of those for H and G/H. In relative cohomology inclusion and δ define exact sequences allowing to express relative cohomology groups [29] $H_n(X, A \subset X)$ in terms of those for X and A. Mayer-Vietoris sequence relates the cohomologies of sets A, B and $X = A \cup B$.

2. Künneth theorem [20] allows to calculated homology groups for Cartesian product as convolution of those for the factors with respect to direct sum.

Steenrod-Eilenberg axioms [32] axiomatize cohomology theory in the category of topological spaces: cohomology theory in this category is a functor to graded abelian groups, satisfying the Eilenberg-Steenrod axioms: functoriality, naturality of the boundary homomorphism, long exact sequence, homotopy invariance, and excision. In algebraic cohomology the category is much more restricted: algebraic varieties defined in terms of polynomial equations and these axioms are not enough. In this case Weil cohomology [36] defines a possible axiomatization consisting of finite generation, vanishing outside the range $[0, dim(X)]$, Poincare duality, Künneth product formula, a cycle class map, and the weak and strong Lefschetz axioms.

In p-adic context sets do not have boundaries since p-adic numbers are not well-ordered so that the statement that boundary has vanishing boundary should be formulated using purely algebraic language. Also cohomology is problematic since definite integral is ill-defined for the same reason. This forces to question either the notion of cohomology and homology groups or the definition of geometric boundary operation and inspires the question whether Galois groups might be a more appropriate notion.

Perhaps it is partially due to the lack of a geometric realization of the boundary operation in the case of general number field that there are very many cohomology theories: the brief summary by Andreas Holmstrom written when he started to work with his thesis, gives some idea about how many!

2.2 Algebraization of intersections and unions of varieties

There are several rather abstract notions involved with cohomology theories: categories, functoriality, sheaves, schemes, abelian rings. Abelian ring is essentially the ring of polynomial functions generated by the coordinates in the open subset of the variety.

1. The spectrum of ring consists of its proper prime ideals of this function algebra. Ideal is subset of functions s closed under sum and multiplication by any element of the algebra and proper ideal is subspace of the entire algebra. In the case of the abelian ring defined on algebraic variety maximal ideals correspond to functions vanishing at some point. Prime ideals correspond to functions vanishing in some sub-variety, which does not reduce to a union of sub-varieties (meaning that one has product of two functions of ring which can separately vanish). Thus the points in spectrum correspond to sub-varieties and product of functions correspond to a union of sub-varieties.

2. What is extremely nice that the product of functions represents in general union of disjoint surfaces: for physicist this brings in mind many boson states created by bosonic creation operators with particles identified as surfaces. Therefore union corresponds to a product of ideals defining a non-prime ideal. The notion of ideal is needed since there is enormous gauge invariance involved in the sense that one can multiply the function defining the surface by any everywhere non-vanishing function.

3. The intersection of varieties in turn corresponds to the condition that the functions defining the varieties vanish separately. If one requires that all sums of the functions belonging to the corresponding ideals vanish one obtains the same condition so that one can say that intersection corresponds to vanishing condition for the sum for ideals. The product of cohomology elements corresponds by Poincare duality [25] the intersection of corresponding homology elements interpreted as algebraic cycles so that a beautiful geometric interpretation is possible in real context at least.

Remark: For fermionic statistics the functions would be anti-commutative and this would prevent automatically the powers of ideals. In fact, the possibility of multiple roots for polynomials of several variables implying what is known as ramification [28] represents a non-generic situation and one of the technical problems of algebraic geometry. For ordinary integers ramification means that integer contains in its composition to primes a power of prime which is higher than one. For the extensions of rationals this means that rational prime is product of primes of extension with some roots having multiplicity larger than one. One can of course ask whether higher multiplicity could be interpreted in terms of many-boson state becoming possible at criticality: in quantum physics bosonic excitations (Goldstone bosons) indeed emerge at criticality and give rise to long range interactions. In fact, for infinite primes allowing interpretation in terms of quantum states of arithmetic QFT boson many particle states corresponds to powers of primes so that the analogy is precise.

2.3 Motivations for motives

In the following I try to clarify for myself the motivations for the motivic cohomology which as a general theory is still only partially existent. There is of course no attempt to say anything about the horrible technicalities involved. I just try to translate the general ideas as I have understood (or misunderstood) them to the simple language of mathematically simple minded physicist.

Grothendieck has carried out a monumental work in algebraizing cohomology which only mathematician can appreciate enough. The outcome is a powerful vision and mathematical tools allowing to develop among other things the algebraic variant of de Rham cohomology, etale cohomology having values in p-adic fields different from the p-adic field defining the values of cohomology, and crystalline cohomology [6].

As the grand unifier of mathematics Grothendieck posed the question whether there good exists a more general theory allowing to deduce various cohomologies from single grand cohomology. These cohomology theories would be like variations of the same them having some fundamental core element -motive- in common.

Category theory [4] and the notion of scheme [31], which assigns to open sets of manifold abelian rings - roughly algebras of polynomial functions- consistent with the algebra of open sets, provide the backbone for this approach. To the mind of physicist the notion of scheme brings abelian gauge theory with non-trivial bundle structure requiring several patches and gauge transformations between them. A basic challenge is to relate to each other the cohomologies associated with algebraic varieties with given number field k manifolds. Category theory is the basic starting point: cohomology theory assigns to each category of varieties category of corresponding cohomologies and functors between these categories allow to map the cohomologies to each other and compare different cohomology theories.

One of the basic ideas underlying the motivic cohomology seems is that one should be able perform a local lifting of a scheme from characteristic p (algebraic variety in p-adic number field or its algebraic extension) to that in characteristic 0 (characteristic is the integer n for which the sum of n units is zero, for rational numbers, p-adic number fields and their extensions characteristic is zero and p for finite fields) that is real or complex algebraic variety, to calculate various cohomologies here as algebraic de Rham cohomology and using the lifting to induce the cohomology to p-adic context. One expects that the ring in which cohomology has naturally values consists of ordinary or p-adic integers or extension of p-adic integers. In the case of crystalline cohomology this is however not enough.

The lifting of the scheme is far from trivial since number fields are different and real cohomology has naturally Z or Q as coefficient ring whereas p-adic cohomology has p-adic integers as coefficient ring. This lift must bring in analytic continuation which is lacking at p-adic side since n particular in p-adic topology two spheres with same radius are either non-intersecting or identical. Analytical continuation using a net of overlapping open sets is not possible.

One could even dream of relating the cohomologies associated with different number fields. I do not know to what extend this challenge is taken or whether it is regarded as sensible at all. In TGD framework this kind of map is needed and leads ot the generalization of the number field obtained by glueing together reals and p-adic numbers among rationals and common algebraic numbers. This glueing together makes sense also for the space of surfaces by identifying the surfaces which correspond to zero loci of rational functions with rational coefficients. Similar glueing makes sense for the spaces of polynomials and rational functions.

Remarks::

1. The possibility of p-adic pseudo-constants in the solutions of p-adic differential and p-adic differential equations reflects this difficulty. This lifting should remove this non-uniqueness in analytical continuation. One can of course ask whether the idea is good: maybe the p-adic pseudo constants have some deep meaning. A possible interpretation would be in terms of non-deterministic character of cognition for which p-adic space-time sheets would be correlates. The p-adic space-time sheets would represent intentions which can be transformed to actions in quantum jumps. If one works in the intersection of real and p-adic worlds in which one allows only rational functions with coefficients in the field or rationals or possibly in some algebraic extension of rationals situation changes and non-uniqueness disappears in the intersection of real and p-adic worlds and one might argue that it is here where the universal cohomology applies or that real and p-adic cohomologies are obtained by some kind of algebraic continuation from this cohomology.

2. The universal cohomology theory brings in mind the challenge encountered in the construction of quantum TGD. The goal is to fuse real physics and various p-adic physics to single coherent whole so that one would have kind of algebraic universality. To achieve this I have been forced to introduce a heuristic generalization of number field by fusing together reals and various p-adic number fields among rationals and common algebraic numbers. The notion of infinite primes is second key notion. The hierarchy of Planck constants involving extensions of p-adic numbers by roots of unity is closely related to p-adic length scale hierarchy and seems to be an essential part of the number theoretical vision.

3 Examples of cohomologies

In the following some examples of cohomologies are briefly discussed in hope of giving some idea about the problems involved. Probably the discussion reflects the gaps in my understanding rather than my understanding.

3.1 Etale cohomology and l-adic cohomology

Etale cohomology [9] is defined for algebraic varieties as analogues of ordinary cohomology groups of topological space. They are defined purely algebraically and make sense also for finite fields. The notion of definite integral fails in p-adic context so that also the notion of form makes sense only locally but not as a map assigning numbers to surfaces. This is cohomological counterpart for the non-existence of boundaries in p-adic realm. Etale cohomology allows to define cohomology groups also in p-adic context as l-adic cohomology groups.

In Zariski topology closed sets correspond to surfaces defined as zero loci for polynomials in given field. The number of functions is restricted only by the dimension of the space. In the real case this topology is much rougher than real topology. In etale cohomology Zariski topology is too rough. One needs more open sets but one does not want to give up Zariski topology.

The category of etale maps is the structure needed and actually generalizes the notion of topology. Instead of open sets one considers maps to the space and effectively replaces the open sets with their inverse images in another space. Etale maps -idempotent are essentially projections from coverings of the variety to variety. One can say that open sets are replaced with open sets for the covering of the space and mapping is replaced with a correspondence (for algebraic surfaces X and Y the correspondence is given by algebraic equations in $X \times Y$) which in general is multi-valued and this leads to the notion of etale topology. The etale condition is formulated in the Wikipedia article in a rather tricky manner telling not much to a physicist trying to assign some meaning to this word. Etale requirement is the condition that would allow one to apply the implicit function theorem if it were true in algebraic geometry: it is not true since the inverse of rational map is not in general rational map except in the ase of birational maps to which one assigns birational geometry [2].

Remarks:

1. In TGD framework field as a map from M^4 to some target space is replaced with a surface in space $M^4 \times CP_2$ and the roles of fields and space are permuted for the regions of space-time representing lines of generalized Feynman diagrams. Therefore the relation between M^4 and CP_2 coordinates is given by correspondence. Many-sheeted space-time is locally a many-sheeted covering of Minkowski space.

2. Also the hierarchy of Planck constant involving hierarchy of coverings defined by same values of canonical momentum densities but different values of time derivatives of imbedding space coordinates. The enormous vacuum degeneracy of Kähler action is responsible for this many-valuedness.

3. Implicit function theorem indeed gives several values for time derivatives of imbedding space coordinates as roots to the conditions fixing the values of canonical momentum densities.

The second heuristic idea is that certain basic cases corresponding to dimensions 0 and 1 and abelian varieties which are also algebraic groups obeying group law defined by regular (analytic and single valued) functions are special and same results should follow in these cases.

Etale cohomologies satisfy Poincare duality and Künneth formula stating that homology groups for Cartesian product are convolutions of homology groups with respect to tensor product. l-adic cohomology groups have values in the ring of l-adic integers and are acted on by the absolute Galois group of rational numbers for which no direct description is known.

3.2 Crystalline cohomology

Crystalline cohomology represents such level of technicality that it is very difficult for physicists without the needed background to understand what is in question. I however make a brave attempt by comparing with analogous problems encountered in the realization of number theoretic universality in TGD framework. The problem is however something like follows.

1. For an algebraically closed field with characteristic p it is not possible to have a cohomology in the ring Z_p of p-adic integers. This relates to the fact that the equation for $x^n = x$ in finite field has only complex roots of unity as its solutions when n is not divisible by p whereas for he integers n divisible by p are exceptional due to the fact that $x^p = x$ holds true for all elements of finite field $G(p)$. This implies that $x^p = x$ has p solutions which are ordinary p-adic numbers rather than numbers in an algebraic extension by a root of unity. p-Adic numbers indeed contain n:th cyclotomic field only if n divides $p-1$. On the other hand, any finite field has order $q = p^n$ and can be obtained as an algebraic extension of finite field $G(p)$ with p elements. Its elements satisfy the Frobenius condition $x^{q=p^n} = x$. This condition cannot be satisfied if the extension contains p:th root of unity satisgying $u^p = 1$ since one would have $(xu)^{p^n} = x \neq xu$. Therefore finite fields do not allow an algebraic extensions allowing p :th root of unity so the extension of p-adic numbers containing p:th root of unity cannot be not induced by the extension of $G(p)$. As a consequence one cannot lift cohomology in finite field $G(p^n)$ to p-adic cohomology.

2. Also in TGD inspired vision about integration $p-1$:th and possibly also p:th roots are problematic. p-Adic cohomology is about integration of forms and the reason why integration necessitates various roots of unity can be understood as follows in TGD framework. The idea is to

reduce integration to Fourier analysis which makes sense even for the p-adic variant of the space in the case that it is symmetric space. The only reasonable definition of Fourier analysis is in terms of discrete plane waves which come as powers of n:th root of unity. This notion makes sense if n is not divisible by p. This leads to a construction of p-adic variants of symmetric spaces G/H obtained by discretizing the groups to some algebraic subgroup and replacing the discretized points by p-adic continuum. Certainly the n:th roots of unity with n dividing $p-1$ are problematic since they do not corresponds to phase factors. It seems however clear that one can construct an extension of p-adic numbers containing p:th roots of unity. If it is however necesssary to assume that the extension of p-adic numbers is induced by that for a finite field, situation changes. Only roots of unity for n not divisible by factors of $p-1$ and possibly also by p can appear in the discretizations. There is infinite number extensions and the interpretation is in terms of a varying finite measurement resolution.

3. In TGD framework one ends up with roots of unity also when one wants to realize p-adic variants of various finite group representations. The simplest case is p-adic representations of angular momentum eigenstates and plane waves. In the construction of p-adic variants of symmetric spaces one is also forced to introduce roots of unity. One obtains a hierarchy of extensions involving increasing number of roots of unity and the interpretation is in terms of number theoretic evolution of cognition involving both the increase of maximal value of n and the largest prime involved. Witt ring could be seen as an idealization in which all roots of unity possible are present.

For $l = p$ l-adic cohomology fails for characteristic p. Crystalline cohomology fills in this gap. Roughly speaking crystalline cohomology is de Rham cohomology of a smooth lift of X over a field k with with characteristic p to a variety so called ring of Witt vectors with characteristic 0 consisting of infinite sequences of the elements of k while de Rham cohomology of X is the crystalline cohomology reduced modulo p.

The ring of Witt vectors for characteristic p is particular example of ring of Witt vectors [37] assignable to any ring as infinite sequences of elements of ring. For finite field G_p the Witt vectors define the ring of p-adic integers. For extensions of finite field one has extensions of p-adic numbres. The algebraically closed extension of finite field contains n:th roots of unity for all n not divisible by p so that one has algebraic closure of finite field with p elements. For maximal extension of the finite field G_p the Witt ring is thus a completion of the maximal unramified extension of p-adic integers and contains n:th roots of unity for n not divisible by p. "Unramified" [28] means that p defining prime for p-adic integers splits in extension to primes in such a manner that each prime of extension occurs only once: the analogy is a polynomial whose roots have multiplicity one. This ring is much larger than the ring of p-adic integers. The algebraic variety is lifted to a variety in Witt ring with characteristic 0 and one calculates de Rham cohomology using Witt ring as a coefficient field.

3.3 Motivic cohomology

Motivic cohomology is a attempt to unify various cohomologies as variations of the same motive common to all of them. In motivic cohomology [22] one encounters pure motives and mixed motives. Pure motives is a category associated with algebraic varieties in a given number field k with a contravariant functor from varieties to the category assigning to the variety its cohomology groups. Only smooth projective varieties are considered. For mixed motives more general varieties are allowed. For instance, the condition that projective variety meaning that one considers only homogenous polynomials is given up.

Chow motives [23] is an example of this kind of cohomology theory and relies on very geometric notion of Chow ring with equivalence of algebraic varieties understood as rational equivalence. One can replace rational equivalence with many variants: birational, algebraic, homological, numerical, etc...

The vision about rationals as common points of reals and p-adic number fields leads to ask whether the intersection of these cohomologies corresponds to the cohomology associated with varieties defined by rational functions with rational coefficients. In both p-adic and real cases the number of varieties is larger but the equivalences are stronger than in the intersection. For a non-professional it is impossible to say whether the idea about rational cohomology in the intersection of these cohomologies makes sense.

Homology and cohomology theories rely in an essential manner to the idea of regarding varieties with same shape equivalent. This inspires the idea that the polynomials or rational functions with rational coefficients could correspond to something analogous to a gauge choice without losing relevant information or bringing in information which is irrelevant. If this gauge choice is correct then real and p-adic cohomologies and homologies would be equivalent apart from modifications coming from the different topology for the real and p-adic integers.

4 Infinite rationals define rational functions of several variables: a possible number theoretic generalization for the notions of homotopy, homology, and cohomology

This section represents my modest proposal for how the generalization of number theory based on infinite integers might contribute to the construction of topological and number theoretic invariants of varieties. I can represent only the primitive formulation using the language of second year math student. The construction is motivated by the notion of infinite prime but applies to ordinary polynomials in which case however the motivation is not so obvious. The visions about TGD as almost topological QFT, about TGD as generalized number theory, and about TGD as infinite-dimensional geometry serve as the main guidelines and allow to resolve the problems that plagued the first version of the theory.

4.1 Infinite rationals and rational functions of several variables

Infinite rationals correspond in natural manner to rational functions of several variables.

1. If the number of variables is 1 one has infinite primes at the first level of the hierarchy as formal rational functions of variable X having as its value as product of all finite primes and one can decompose the polynomial to prime polynomial factors. This amounts to solving the roots of the polynomial by obtained by replacing X with formal variable x which is real variable for ordinary rationals. For Gaussian rationals one can use complex variable.

2. If the roots are not rationals one has infinite prime. Physically this state is the analog of bound state whereas first order polynomials correspond to free many-particle states of supersymmetric arithmetic QFT.

3. Galois group permuting the roots has geometric interpretation as the analog of the group of deck transformations permuting the roots of the covering of the graph of the polynomial y=f(x) at origin. Galois group is analogous to fundamental group whose abelianization obtained as a coset group by dividing with the commutator group gives first homology group. The finiteness of the Galois group does not conform with the view about cohomology and homology, which suggests that it is the group algebra of Galois group which is the correct mathematical structure to consider.

One can find the roots also at the higher levels of the hierarchy of infinite primes. One proceeds by finding the roots at the highest level as roots which are algebraic functions. In other words finds the decomposition

$$P(x_n, ...) = \prod_k (x_n - R_k(x_{n-1}, ...)$$

with R_k expanded in powers series with respect to x_{n-1}. This expansion is the only manner to make sense about the root if x_{n-1} corresponds to infinite prime. At the next step one puts $x_n = 0$ and obtains a product of R_k and performs the same procedure for x_{n-1} and continues down to $n = 1$ giving ordinary algebraic numbers as roots. One therefore obtains a sequence of sub-varieties by restricting the polynomial to various planes $x_i = 0$, $i = k, ..., n$ of dimension $k - 1$. The invariants associated with the intersections with these planes define the Galois groups characterizing the polynomial and therefore also infinite prime itself.

1. The process takes place in a sequential manner. One interprets first the infinite primes at level n+1 as as polynomial function in the variable X_{n+1} with coefficients depending on X_k, $k < n+1$. One expands the roots R in power series in the variable X_n. In p-adic topology this series converges for all primes of the previous levels and the deviation from the value at $X_n = 0$ is infinitesimal in infinite-P p-adic topology.

2. What is new as compared to the ordinary situation is that the necessity of Taylor expansion, which might not even make sense for ordinary polynomials. One can find the roots and one can assign a Galois group to them.

3. One obtains a hierarchy of Galois groups permuting the roots and at the lowest level on obtains roots as ordinary algebraic numbers and can assign ordinary Galois group to them. The Galois group assigned to the collection of roots is direct sum of the Galois groups associated with the individual roots. The roots can be regarded as a power series in the variables X and the deviation from algebraic number is infinitesimal in infinite-p p-adic topology.

4. The interesting possibility is that the infinitesimal deformations of algebraic numbers could be interpreted as a generalization of real numbers. In the construction of motivic cohomology the idea is to lift varieties defined for surfaces in field of characteristic p (finite fields and their extensions) to surfaces in characteristic 0 field (p-adic numbers) in some sense to infinitesimal thickenings of their characteristic 0 counterparts. Something analogous is encountered in the proposed scenario since the roots of the polynomials are algebraic numbers plus multi-p p-adic expansion in terms of infinite-p p-adic numbers representing infinitesimal in infinite-p p-adic topology.

4.2 Galois groups as non-commutative analogs of homotopy groups

What one obtains is a hierarchy of Galois groups and varieties of $n+1$-dimensional space with dimensions $n, n-1, ..., 1, 0$.

1. A suggestive geometric interpretation would be as an analog of first homotopy group permuting the roots which are now surfaces of given dimension k on one hand and as a higher homotopy group π_k on the on the other hand. This and the analogy with ordinary homology groups suggests the replacement of Galois group with their group algebras. Homology groups would be obtained by abelianization of the analogs of homotopy groups with the square of the boundary homomorphism mapping the group element to commutator sub-group. Group algebra allows also definition of cohomotopy and cohomology groups by assigning them to the dual of the group algebra.

2. The boundary operation is very probably not unique and the natural proposal inspired by physical intuition is that the boundary operations form an anticommutative algebra having interpretation in terms of fermionic creation (say) operators. Cohomology would in turn correspond to annilation operators. Poincare duality would be hermitian conjugation mapping fermionic creation operators to annihilation operators and vice versa. Number theoretic vision combined with the braid representation of the infinite primes in turn suggests that the construction actually reduces the construction of quantum TGD to the construction of these homology and cohomology theories.

3. The Galois analogs of homotopy groups and their duals up to the dimension of the algebraic surface would be obtained but not the higher ones. Note that for ordinary homotopy groups all homotopy group π_n, $n > 1$ are Abelian so that the analogy is not complete. The abelianizations of these Galois groups could in turn give rise to higher homology groups. Since the rational functions involved make sense in all number fields this could provide a possible solution to the challenge of constructing universal cohomology theory.

The hierarchy of infinite primes and the hierarchy of Galois groups associated with the corresponding polynomials have as an obvious analogy the hierarchy of loop groups and corresponding homotopy groups.

1. The construction brings in mind the reduction of n-dimensional homotopy to a 1-D homotopy of n-1-D homotopy. Intuitively n-dimensional homotopy indeed looks like a 1-D homotopy of n-1-D homotopy so that everything should reduce to iterated 1-dimensional homotopies by replacing the original space with the space of maps to it.

2. The hierarchical ordering of the variables plays an essential role. The ordering brings strongly in mind loop groups. Loop group $L(X^m, G)$ defined by the maps from space X^n to group G can be also regarded as a loop group from space X^m to the loop group $L(X^{n-m}, G)$ and one obtains $L(X^n, G) = L(X^1, L(X^{n-1})$.

   The homotopy equivalence classes of these maps define homotopy groups using the spaces X^n instead of spheres. Infinite primes at level n would correspond to $L(X^n, G)$. Locally the fundamental loop group is defined by $X = S^1$ which would suggest that homotopy theory using tori might be more natural then the one using spheres. Naively one might hope that this kind of groups could code for all homotopic information about space. As a matter fact, even more general identity $L(X \times Y, G) = L(X, L(Y, G))$ seems to hold true.

3. Note that one can consider also many variants of homotopy theories since one can replace the image of the sphere in manifold with the image of any manyfold and construct corresponding homotopy theory. Sphere and tori define only the simplest homotopy theories.

4.3 Generalization of the boundary operation

The algebraic realization of boundary operation should have a geometric counterpart at least in real case and it would be even better if this were the case also p-adically and even for finite fields.

1. The geometric analog of the boundary operation would replace the k-dimensional variety with its intersection with $x_k = 0$ hyperplane producing a union of $k-1$-dimensional varieties. This operation would make sense in all number fields. The components in the union of the surface would be very much analogous to the lower-dimensional edges of k-simplex so that boundary operation might make sense. What comes in mind is relative homology $H(X, A)$ in which the intersection of X with $A \subset X$ is equivalent with boundary so that its boundary vanishes. Maybe one should interpret the homology groups as being associated with the sequence of relative homologies defined by the sequence of varieties involved as $A_0 \subset A_1 \subset ..$ and relativizing for each pair in the sequence. The ordinary geometric boundary operation is ill-defined in p-adic context but its analog defined in this manner would be number theoretically universal notion making sense also for finite fields.

2. The geometric idea about boundary of boundary as empty set should be realized somehow- at least in the real context. If the boundary operation is consistent with the ordinary homology, it should give rise to a surface which as an element of H_{n-2} is homologically trivial. In relative homology interpretation this is indeed the case. In real context the condition is satisfied if the intersection of the n-dimensional surface with the $x_{n-1} = 0$ hyper-plane consists of closed surface so that the boundary indeed vanishes. This is indeed the case as simplest visualizations in 3-D case demonstrate. Therefore the key geometric idea would be that that the intersection of the surface defined by zeros of polynomial with lower dimensional plane is a closed surface in real context and that this generalizes to p-adic context as algebraic statement at the level of homology.

3. The sequence of slicings could be defined by any permutation of coordinates. The question is whether the permutations lead to identical homologies and cohomologies. The physical interpretation does not encourage this expection so that different permutation would all be needed to characterize the variety using the proposed homology groups.

4.4 Could Galois groups lead to number theoretical generalizations of homology and cohomology groups?

My own humble proposal for a number theoretic approach to algebraic topology is motivated by the above questions. The notion of infinite primes leads to a proposal of how one might assign to a variety a sequence of Galois group [12] algebras defining analogs of homotopy groups assignable to the algebraic extensions of polynomials of many variables obtained by putting the variables of a polynomial of n-variable polynomial one by one to zero and finding the Galois groups of the resulting lower dimensional varieties as Galois groups of corresponding extensions of polynomial fields. The construction of the roots is discussed in detail [13], where infinite primes are compared with non-standard numbers. The earlier idea about the possibility to lift Galois groups to braid groups is also essential and implies a connection with several key notions of quantum TGD.

1. One can assign to infinite primes at the n:th level of hierarchy (n is the number of second quantizations) polynomials of n variables with variables ordered according to the level of the hierarchy by replacing the products $X_k = \pi_i P_i$ of all primes at k:th level with formal variables x_n to obtain polynomial in x_n with coefficients which are rational functions of x_k, $k < n$. Note that X_k is finite in p-adic topologies and infinitesimal in their infinite-P variants.

2. One can construct the root decomposition of infinite prime at n:th level as the decomposition of the corresponding polynomial to a product of roots which are algebraic functions in the extensions of polynomials. One starts from highest level and derives the decomposition by expanding the roots as powers series with respect to x_n. The process can be done without ever mentioning infinite primes. After this one puts $x_n = 0$ to obtain a product of roots at $x_n = 0$ expressible as rational functions of remaining variables. One performs the decomposition with respect to x_{n-1} for all the roots and continues down to $n = 1$ to obtain ordinary algebraic numbers.

3. One obtains a collection of varieties in n-dimensional space. At the highest level one obtains $n-1$-D variety referred to as divisor in the standard terminology, $n-2$-D variety in $x_n = 0$ hyperplane, $n-3$-D surface in $(x_n, x_{n-1}) = (0, 0)$ plane and so on. To each root at given level one can assign polynomial Galois group permuting the polynomial roots at various levels of the hierarchy of infinite primes in correspondence with the branches of surfaces of a many-valued map. At the lowest level one obtains ordinary Galois group relating the roots of an ordinary polynomial. The outcome is a collection of sequences of Galois groups $\{(G_n, G_{n,i}, G_{n,i,j}...)\}$ corresponding to all sequences of roots from $k = n$ to $k = 1$.

   One can also say that at given level one has just one Galois group which is Cartesian product of the Galois groups associated with the roots. Similar situation is encountered when one has a product of irreducible polynomials so that one has two independent sets of roots.

The next question is how to induce the boundary operation. The boundary operation for the analogs of homology groups should be be induced in some sense by the projection map putting one of the coordinates x_k to zero. This suggests a geometric interpretation in terms of a hierarchy of relative homologies $H_k(S_k, S_{k-1})$ defined by the hierarchy of surfaces S_k. Boundary map would map S_k to is intersection at $(x_n = 0, ..., x_k = 0)$ plane. This map makes sense also p-adically. The square of boundary operation would produce an intersection of this surface in $x_{k-1} = 0$ plane and this should correspond to boundary sense for Galois groups.

4.4.1 Algebraic representation of boundary operations in terms of group homomorphisms

The challenge is to find algebraic realizations for the boundary operation or operations in terms of group homomorphisms $G_k \rightarrow G_{k-1}$. One can end up with the final proposal through heuristic ideas and counter arguments and relying on the idea that algebraic geometry should have interpretation in terms of quantum physics as it is described by TGD as almost topological QFT.

1. n-dimensional Galois group is somewhat like a fundamental group acting in the space of n-1-dimensional homotopies so that Grothendieck's intuition that 1-D homotopies are somehow fundamental is realized. The abelianizations of these Galois groups would define excellent candidates for homology groups and Poincare duality would give cohomology groups. The homotopy aspects becomes clearer if one interprets Galois group for n:th order polynomials as subgroup of permutation group and lifts the Galois group to a subgroup of corresponding braid group. Galois groups are also stable againt small changes of the coefficients of the polynomial so that topological invariance is guaranteed.

2. Non-abelian boundary operations $G_k \to G_{k-1}$ must reduce to their abelian counterparts in abelianization so that they their squares defining homomorphisms from level k to $k-2$ must be maps of G_k to the commutator subgroup $[G_{k-2}, G_{k-2}]$.

3. There is however a grave objection. Finite abelianized Galois groups contain only elements with finite order so that in this sense the analogy with ordinary homotopy and homology groups fails. On the other hand, if Galois group is replaced with its group algebra and group algebra is defined by (say) integer valued maps, one obtains something very much analogous to homotopy and homology groups. Also group algebras in other rings or fields can be considered. This replacement would provide the basis of the homotopy and homology groups with an additional multiplicative structure induced by group operation allowing the interpretation as representations of Galois group acting as symmetry groups. The tentative physical interpretation would in terms of quantum states defined by wave functions in groups. Coboundary operation in the dual of group algebra would be induced by the action of boundary operation in group algebra. Homotopy and homology would be associated with the group algebra and and cohomotopy and cohomology with its dual.

4. A further grave objection against the analog of homology theory is there is no reason to expect that the boundary homomorphism is unique. For instance, one can always have a trivial solution mapping G_k to unit element of G_{k-1}. Isomorphism theorem [19] implies that the image of the group G_k in G_{k-1} under homomorphism h_k is $G_k/ker(h_k)$, where $ker(h_k)$ is a normal subgroup of G_k as is easy to see. One must have $h_{k-1}(G_k/ker(h_k)) \subset [G_{k-2}, G_{k-2}]$, which is also a normal subgroup.

   The only reasonable option is to accept all boundary homomorphisms. This collection of boundary homomorphisms would satisfy anticommutation relations inducing similar anticommutation relations in cohomology. Putting all together, one would would obtain the analog of fermionic oscillator algebra. In particular, Poincare duality would correspond to the mapping exchanging fermionic creation and annihilation operators. It however turns out that tis interpretation fails. Rather, braided Galois homology could represent the states of WCW spinor fields in "orbital" degrees of freedom of WCW in finite measurement resolution. A better analogy for braided Galois cohomology is provided by Dolbeault cohomology which also allows complex conjugation.

If this picture makes sense, one would clearly have what category theorist would have suggested from the beginning. TGD as almost topological QFT indeed suggests strongly the interpretation of quantum states in terms of homology and cohomology theories.

4.4.2 Lift of Galois groups to braid groups and induction of braidings by symplectic flows

One can build a tighter connection with quantum TGD by developing the idea about the analogy between homotopy groups and Galois groups.

1. The only homotopy groups [16], which are non-commutative are first homotopy groups π_1 and plane with punctures provides the minimal realization for them. The lift of permutation groups to `http://en.wikipedia.org/wiki/Braid_group`braid groups [3] by giving up the condition that the squares of generating permutations satisfy $s_i^2 = 1$ defines a projective representation for them and should apply also now. There is also analogy with Wilson loops. This leads to topological QFTs for knots and braids [49, 50].

2. In TGD framework light-like 3-surfaces (and also space-like at the ends of causal diamonds) carry braids beginning at partonic 2-surfaces and ending at partonic 2-surfaces at the boundaries of causal diamonds. This realization is highly suggestive now. This also conforms with the general TGD inspired vision about absolute Galois group of rationals as permutation group S_∞ lifted to braiding groups such that its representation always reduce to finite-dimensional ones [21]. This also conforms with the view about the role of hyper-finite factors of type II_1 and the idea about finite measurement resolution and one would obtain a new connection between various mathematical structure of TGD.

3. The physical interpretation of infinite primes represented by polynomials as bound states suggests that infinite prime at level n corresponds to a braid of braids of ... braids such that at given level of hierachy braid group acting on the physical states is associated with covering group realized as subgroup of the permutation group for the objects whose number is the number of roots. This gives also a connection with the the notion of operad [24, 47, 40] which involves also a hierarchy of discrete structures with the action of permutation group inside each and appears also in quantum TGD as a natural notion [2, 5].

4. The assumption that the braidings are induced by flows of the partonic 2-surface could glue the actions of different Galois groups to single coherent whole was originally motivated by the hope that boundary homomorphism could be made unique in this manner. This restriction is however un-necessary and the physical picture does not support it. The basic motivation for the braid representation indeed comes from TGD as an almost topological QFT vision.

5. The role of symplectic transformations in TGD suggests the identification of flows as symplectic flows induced by those of $\delta M^2 \times CP_2$. These flows should map the area enclosed by the sub-braid (of braids) to itself and corresponding Hamiltonian should be constant at the boundary of the area and induce a flow horizontal to the boundary and also continuous at the boundary. The flow would in general be non-trivial inside the area and induce the braiding of the sub-braid of braids. One could assign "Galois spin" to the sub-braids with respect to the higher Galois group and boundary homomorphism would realize unitary action of G_k as spin rotation at k_1:th level. At k_2:th level the "Galois spin" rotation would reduce to that in commutator subgroup and in homology theory would become trivial. The interpretation of the commutator group as the analog of gauge group might make sense. This would conform with an old idea of quantum TGD that the commutator subgroup of symplectic group acts as gauge transformations.

6. It is not necessary to assign the braids at various level of the hierarchy to the same partonic 2-surface. Since the symplectic transformations act on $\delta M^4_\pm \times CP_2$, one can consider also the projections of the braids to the homologically non-trivial 2-sphere of CP_2 or to the 2-sphere at light-cone boundary: both of these spheres play important part in the formulation of quantum TGD and I have indeed assigned the braidings to these surfaces [9].

7. The representation of the hierarchy of Galois groups acting on the braid of braids of... can be understood in terms of the replacement of symplectic group of $\delta M^4_\pm \times CP_2$ -call it G- permuting the points of the braids with its discrete subgroup obtained as a factor group G/H, where H is a normal subgroup of G leaving the endpoints of braids fixed. One must also consider subgroups of the permutation group for the points of the triangulation since Galois group for n:th order polynomial is in general subgroup of S_n One can also consider flows with these properties to get braided variant of G/H.

The braid group representation works also for ordinary polynomials with continuous coefficients in all number fields as also finite fields. One therefore achieves number theoretical universality. The values of the variables x_i appearing in the polynomials can belong to any numer field and the representation spaces of the Galois groups correspond to any number field. Since the Galois groups are stable against small perturbations of coefficients one obtains topological invariance in both real and p-adic sense. Also the representation in all number fields are possible for the Galois groups.

The construction is universal but infinite primes provide the motivation for it and can be regarded as a representation of the generalized cohomology group for surfaces which belong to the intersection of real and p-adic worlds (rational coefficients). In particular, the expansion of the roots in powers series is the only manner to make sense about the roots when x_n is identified with X_n so that convergence takes place if some of the lower level infinite primes appearing in the product defining X_n is interpreted as infinite p-adic prime. All higher powers are infinitesimal in infinite-P p-adic norm. At the lowest level one obtains expansion in X_1 for which X_1^n has norm p^{-n} with respect to any prime p. The value of the product of primes different from p is however not well-defined for given p-adic topology. If it makes sense to speak about multi-p p-adic expansion all powers X_1^n, $n > 0$ would be infinitesimal.

4.4.3 What can one say about the lifting to braid groups?

The generators of symmetry group are given by permutations s_i permuting i:th and $i+1$:th element of n-element set. The permutations s_i and s_j obviously commute for $|i-j| > 2$. It is also easy to see that the identity $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ holds true. Besides this the identity $s_i^2 = 1$ holds true.

Braid group B_n [3] is obtained by dropping the condition $s_i^2 = 1$ and can be regarded as an infinite covering group of the permutation group. For instance, for the simplest non-trivial case $n = 3$ the braid group is universal central extension of the modular group $PSL(2, Z)$. In the general case the braid group is isomorphic to the mapping class group of a punctured disk with n punctures and the realization of the braidings as a symplectic transformations would mean additional restriction to the allowed isotopies inducing the braid group action.

One can decompose any element of braid group B_n to a product of element of symmetric group S_n and of pure braid group P_n consisting of braidings which correspond to trivial permutations. P_n is a normal subgroup of braid group and the following short exact sequence $1 \to F_{n-1} \to P_n \to P_{n-1} \to 1$ allows to decompose P_n to a product of image of free group F_{n-1} and of the image of P_n in P_{n-1}. This leads to a decomposition to a representation of P_n as an iterated semidirect product of free groups.

Concerning the lifting of Galois groups to subgroups of braid groups following observations are relevant.

1. For n:th order polynomial of single variable Galois group can be regarded as a subgroup of permutation group S_n. The identification is probably not completely unique (at least inner automorphisms make the identification non-unique) but I am unable to say whether this has significance in the recent context.

2. The natural lifting of Galois group to its braided version is as a product of corresponding subgroup of S_n with with pure braid group of n braids so that pure braidings would allow also braidings of all permutations as intermediate stages. Pure braid group is normal subgroup trivially. Whether also more restricted braidings are possible is not clear to me. Braid group has a subgroup obtained by coloring braid strands with a finite number of colors and allowing only the braidings which induce permutations of braids of same color. Clearly this group is a good candidate for the minimal group decomposable to a product of subgroups of symmetric subgroups containing braided Galois group. Different colors would correspond to the decomposition of S_n to a product of permutation groups. Note that one can have cyclic subgroups of permutation sub-groups.

One might hope that it is enough to lift the boundary homomorphisms between Galois groups G_k and G_{k-1} to homomorphisms between corresponding braided groups. Life does not look so simple.

1. The group algebra of Galois group is replaced with an infinite-dimensional group algebra of braid groups so that the number of physical states is expected to become much larger and the interpretation could be in terms of many-boson states.

2. The square of the boundary homomorphism must map braided Galois group $B(G_k)$ to $[B(G_{k-2}), B(G_{k-2})]$. The obvious question is whether this conditions reduces to corresponding conditions for Galois group and pure braided groups. In other words, does the braiding commute with the formation of commutator sub-group: $[B(G_k), B(G_k)] = B([G_k, G_k])$? In this case the decomposition of the braided Galois group to a product of Galois group and pure braid group would allow to realize the braided counterpart of boundary homomorphism as a product of Galois group homomorphism and homomorphism acting on the pure braid group. Direct calculation however shows that this is not the case so that the problem is considerably more complicated.

4.4.4 More detailed view about braided Galois homology

Consider next a more detailed view about the braided Galois homology.

1. One can wonder whether the application of only single boundary operator creates a state which represents gauge degree of freedom or whether boundaries correspond to "full" boundaries obtained by applying maximum number of boundary operations, which k:th level is k. "Full boundary" would correspond to what one obtains by applying at most k boundary operators to the state, and many combinations are possible if the number of boundary homomorphisms is larger than k. The physical states as elements of homology group would be analogous many-fermion states bu would differ from them in the sense that they would be annihilated by all

fermionic creation operators. In particular, full Fermi spheres at k:th level would represent gauge degrees of freedom.

Homologically non-trivial states are expected to be rather rare, especially so if already single boundary operation creates gauge degree of freedom. Certainly the existence of constraints is natural since infinite primes corresponding to irreducible polynomials of degree higher are interpreted as bound states. Homological non-triviality would most naturally express bound state property in bosonic degrees of freedom. In any case, one can argue that fermionic analogy is not complete and that a more natural interpretation is as an analog of cohomology with several exterior derivatives.

2. The analogy with fermionic oscillator algebra makes also the realization of bosonic oscillator operator algebra suggestive. Pointwise multiplication of group algebra elements regarded as functions in group looks the most plausible option since for continuous groups like $U(1)$ this implies additivity of quantum numbers. Many boson states for given mode would correspond to powers of group algebra element with respect to pointwise multiplication. If the commutator for the analogs of the bosonic oscillator operators is defined as

$$[B_1, B_2] \equiv \sum_{g_1,g_2} B_1(g_1)B_2(g_2)[g_1, g_2] \quad , [g_1, g_2] \equiv g_1g_2g_1^{-1}g_2^{-1} \quad ,$$

it is automatically in the commutator sub-group. This condition is not consistent with fermionic anti-commutation relations. The consistency requires that the commutator is defined as

$$[B_1, B_2] \equiv \sum_{g_1,g_2} (B_1(g_1)B_2(g_2)[[g_1, g_2] \quad , [g_1, g_2] \equiv g_1g_2 - g_2g_1 \quad . \tag{4.1}$$

The commutator must belong to the group algebra of the commutator subgroup. In this case the commutativity conditions are non-trivial. Bosonic commutation relations would put further constraints on the homology.

A delicacy related to commutation and anti-commutation relations should be noticed. One could fermionic creation (annihilation) operators as elements in the dual of group algebra. If group algebra and its dual are not identified (this might not be possible) then the anti-commutator is element of the field of ring in which group algebra elements have values. In the bosonic case the conjugate of the bosonic group algebra element should be treated in the same manner as a pointwise multiplication operator instead of an exterior derivative like operator.

3. One could perhaps interpret the commutation and anti-commutation relations modulo commutator subgroup in terms of finite measurement resolution realized by the transition to homology implying that observables commute in the standard sense. The connection of finite measurement resolution with inclusions of hyper-finite factors of type II_1 implying a connection with quantum groups and non-commutative geometry conforms also with the vision that finite measurement resolution means commutativity modulo commutator group.

4. The alert reader has probably already asked why one could not define also diagonal homology for G_k via diagonal boundary operators $\delta_k : G_k \to H_k$, where H_k is subgroup of G_k. The above argument would suggest interpretation for this cohomology in terms of finite measurement resolution. If one allows this the Galois cohomology groups would be labelled by two integers. Similar situation is encountered in motivic cohomology [22].

4.4.5 Some remarks

Some remarks about the proposal are in order.

1. The proposal makes as such sense if the polynomials with rational coefficients define a subset of more general function space able to catch the non-commutative homotopy and homology and their duals terms of Galois groups associated with rational functions with coefficients. One could however abstract the construction so that it applies to polynomials with coefficients in real and

p-adic fields and forget infinite primes altogether. One can even consider the replacement of algebraic surfaces with more general surfaces as along as the notion of Galois group makes sense since braiding makes sense also in more general situation. This picture would conform with the idea of number theoretical universality based on algebraic continuation from rationals to various number fields. In this case infinite primes would characterize the rational sector in the intersection of real and p-adic worlds.

2. The above discussion is for the rational primes only. Each algebraic extension of rationals however gives rise to its own primes. In particular, one obtains also complex integers and Gaussian primes. Each algebraic extension gives to its own notion of infinite prime. One can also consider quaternionic and octonionic primes and their generalization to infinite primes and this generalization is indeed one of the key ideas of the number theoretic vision [14]. Note that already for quaternions Galois group defined by the automorphisms of the arithmetics is continuous Lie group.

3. The decomposition of infinite primes to primes in extension of rational or polynomials is analogous to the decomposition of hadron to quarks in higher resolution and suggests that reduction of the quantum system to its basic building bricks could correspond number theoretically to the introduction of higher algebraic extensions of various kinds of number fields. The emergence of higher extensions would mean emergence of algebraic complexity and have interpretation as evolution of cognition in TGD inspired theory of consciousness.

This picture conforms with the basic visions of quantum TGD about physics as infinite-dimensional geometry on one hand and physics as generalized number theory on one hand implying that algebraic geometry reduces in some sense to number theory and one can also regard quantum states as representations of algebraic geometric invariants in accordance with the vision about TGD as almost topological QFT.

Infinite primes form a discrete set since all the coefficients are rational (unless one allows even algebraic extensions of infinite rationals). Physically infinite primes correspond to elementary particle like states so that elementary particle property corresponds to number theoretic primeness. Infinite integers define unions of sub-varieties identifiable physically as many particle states. Rational functions are in turn interpreted in zero energy ontology as surfaces assignable to initial and final states of physical event such that positive energy states correspond to the numerator and negative energy states to the denominator of the polynomial. One also poses the additional condition that the ratio equals to real unit in real sense so that real units in this sense are able to represent zero energy state and the number theoretic anatomy of single space-time point might be able to represent arbitrary complex quantum states.

The generalization of the notion of real point has been already mentioned as also the fact that the number theoretic anatomy could in principle allow to code for zero energy states if they correspond to infinite rationals reducing to unit in real sense. Also space-time surfaces could by quantum classical correspondence represent in terms of this anatomy as I have proposed. Single space-time point could code in its structure not only the basic algebraic structure of topology as proposed but represent Platonia. If the above arguments really maks sense then this number theoretic Brahman=Atman identify would not be a mere beautiful philosophical vision but would have also practical consequences for mathematics.

4.5 What is the physical interpretation of the braided Galois homology

The resulting cohomology suggests either the interpretation in terms of many-fermion states or as a generalization of de Rham cohomology involving several exterior derivative operators. The arguments below show that fermionic interpretation does not make sense and that the more plausible interpretation in concordance with finite measurement resolution is in terms of "orbital" WCW degrees of freedom represented by the symplectic group assignable to the product of light-cone boundary and CP_2.

4.5.1 What the restriction to the plane $x_k = 0$ could correspond physically?

The best manner to gain a more detailed connection between physics and homology is through an attempt to understand what operation putting $x_k = 0$ could mean physically.

1. Given infinite prime at level n corresponds to single particle state characterized by Galois group G_n. The fermionic part of the state corresponds to its small part and purely bosonic part multiplies X_{n-1} factors as powers of primes not dividing the fermionic part of the state. Therefore the finite part of the state contains information about fermions and bosons labelled by fermionic primes. When one puts $x_n = 0$, the information about the bosonic part is lost.

   One can of course divide the polynomial by a suitable infinite integer of previous level so that its highest term is just power of X_n with a unit coefficient. Bosonic part appears in this case in the denominator of the finite part of the infinite prime and does not contribute to zeros of the resulting rational function at $n-1$:th level: it of course affects the zeros at n:th level. Hence the information about bosons at $n-1$:th level is lost also now unless one considers also the Galois groups assignable to the poles of the resulting rational function at $n-1$:th level.

2. What could this loss of information about bosons correspond geometrically and physically? To answer this question must understand how the polynomial of many variables can be represented physically in TGD Universe.

   The proposal has been that a union of hierarchically ordered partonic 2-surfaces gives rise to a local representation of n-fold Cartesian power for a piece of complex plane. A more concrete realization would be in terms of wormhole throats at the end of causal diamond at 3-surfaces topologically condensed at each other. The operation $x_n = 0$ would corresponding to the basic reductionistic step destroying the bound state by removing the largest space-time sheets so that one would have many-particle state rather than elementary particle at the lower level of the hierarchy of space-time sheet. This loss of information would be unavoidable outcome of the reductionistic analysis.

One can consider two alternative geometric interpretations depending on whether one identifies to infinite primes connected 3-surfaces or connected 2-surfaces.

1. If infinite primes correspond to connected 3-surfaces having hierarchical structure of topological condensate the disappearing bosons could correspond to the wormhole throats connecting smaller space-time sheet to the largest space-time sheet involved. Wormhole throats would carry bosonic quantum numbers and would be removed when the largest space-time sheet disappears. Many-fermion state at highest level represented by the "finite" part of the infinite prime would correspond to "half" wormhole throats- CP_2 type vacuum extremals topological condensed at smaller space-time sheets but not at the highest one.

2. If elementary particles/infinite primes correspond to connected partonic 2-surfaces (this is not quite not the case since tangent space data about partonic 2-surfaces matters), one must replace 3-D topological condensation by its 2-dimensional version. Infinite prime would correspond to single wormhole throat asa partonic 2-surface at which smaller wormhole throats would have suffered topological condensation. Topological condensation would correspond to a formation of a connection by flux tube like structure between the 2-surfaces considered. The disappearance of this highest level would mean decay to a many particle state containing several wormhole contacts. The formation of anyonic many-particle states could be interpreted in terms of build-up of higher level infinite primes.

3. What ever the interpretation is, it should be consistent with the idea that braiding as induced by symplectic flow. If the symplectic flow is defined by the inherent symplectic structure of the partonic 2-surface only the latter option works. If the symplectic flow acts at the level of the imbedding space - as is natural to assume- both interpretations make sense.

4.5.2 The restriction to $x_k = 0$ plane cannot correspond to homological boundary operation

Can one model the restriction to $x_k = 0$ plane as boundary operation in the sense of generalized homology? There are several objections.

1. There are probably several homological boundary operations δ_i at given level whereas the restriction $x_k = 0$ is a unique operation (recall however the possibility to permute the arguments in the case of polynomial).

2. The homology is expected to contain large number of generators whereas the state defined by infinite prime is unique as are also the states resulting via restriction operations.

3. It is not possible to assign fermion number to $x_k = 0$ operation since fermion number is not affected: this would not allow to assign fermion number to the homological boundary operators.

Although the interpretation as many-fermion states does not make sense, one must notice that the structure of homology is highly analogous to the space of states of super-symmetric QFT and of the set of infinite primes. Only the infinite primes $X_n \pm 1$, where X_n is the product of all primes at level n, correspond to states containing no fermions and have interpretation as Dirac sea and vacuum state. In the same manner the elements of braided Galois homology in general are obtained by applying the analogs of fermionic annihilation (creation) operators to a full Fermi sphere (Fock vacuum). Also the identification of *all* physical states as many-fermion states in quantum TGD where all known elementary bosons are identified as fermion pairs conforms with this picture.

A more natural interpretation of the restriction operation is as an operation making possible to assign to a given state in fermionic sector the space of possible states in WCW degrees of freedom characterized in terms of Galois cohomology represented in terms of the symplectic group of acting as isometries of WCW. The transition from Lie algebra description natural for continuum situation to discrete subgroup is natural due to the discretization realizing the finite measurement resolution.

One cannot however avoid a nasty question. What about the lower level bosonic primes associated with the infinite prime? What is their interpretation if they do not correspond to WCW degrees of freedom? Maybe one could identify the bosonic parts of infinite prime as super-partners of fermions behaving like bosons. The addition of a right handed neutrino to a given quantum state could represent this supersymmetry.

4.5.3 Braided Galois group homology and construction of quantum states in WCW degrees of freedom in finite measurement resolution

The above arguments fix the physical interpretation of infinite primes and corresponding group cohomology to quite high degree.

1. From above it is clear that the restriction operation cannot correspond directly to homological boundary operation. Single infinite prime corresponds to an entire spectrum of states. Hence the assignment of fermion number to the boundary operators is not correct thing to do and one must interpret the coboundary operations as analogs of exterior derivatives and various states as bosonic excitations of a given state analogous to states assignable to closed forms of various degrees in topological or conformal quantum field theories.

2. The natural interpretation of Galois homology is as a homology assignable to a discrete subgroup hierarchy of the symplectic group acting as isometries of WCW and therefore as the space of wave functions in WCW degrees of freedom in finite measurement resolution. Infinite primes would code for fermionic degrees of freedom identifiable as spinor degrees of freedom at the level of WCW.

3. The connection between infinite primes and braided Galois homology would basically reflect the supersymmetry relating these degrees of freedom at the level of WCW geometry where WCW Hamiltonians correspond to bosonic generators and contractions of WCW gamma matrices with symplectic currents to the fermionic generators of the super-symmetry algebra. If this identification is correct, it would solve the problem of constructing the modes of WCW spinor fields in finite measurement resolution. An especially well-come feature would be the reduction of WCW integration to summations in braided Galois group algebra allowing an easy realization of number theoretical universality. If the picture is correct it should also have connections to the realization of finite measurement resolution in terms of inclusions of hyper-finite factors of type II_1 [6] for which fermionic oscillator algebra provides the basic realization.

4. Of course, it is far from clear whether it is really possible to reduce spin, color and electroweak quantum numbers to number theoretic characteristics of infinite primes and it might well be that the proposed construction does not apply to center of mass degrees of freedom of the partonic 2-surface. I have considered these questions for the octonionic generalization of infinite primes

and suggested how standard model quantum numbers could be understood in terms of subset of infinite octonionic primes [14].

4.6 Is there a connection with the motivic Galois group?

The proposed generalized of Galois group brings in mind he notion of motivic Galois group, which is one possible generalization for the notion of zero-dimensional Galois group associated with algebraic extensions of number fields to the level of algebraic varieties.

One of the many technical challenges of the motivic cohomology theory is the non-uniqueness of the imbedding of the algebraic extension as a subfield in the algebraic closure of k. The number of these imbeddings is however finite and absolute Galois group associated with the algebraic closure of k acts in the set of the imbeddings. Which of them one should choose?

Quantum physicist would solve this problem by saying that there is no need to choose: one could introduce quantum superpositions of different choices and "Galois spin" regarding the different imbeddings as analogs of different spin components. Absolute Galois group would act on the quantum states regarded as superpositions of different imbeddings by permuting them. In TGD framework this kind of representation could emerge in p-adic context raise Galois group to a role of symmetry group acting on quantum states: indeed absolute Galois group is very natural notion in TGD framework. I have proposed this kind of interpretation for some years ago in a chapter [10] about Langlands program [43, 21, 44, 42].

If I have understood correctly, the idea of the motivic Galois theory is to generalize this correspondence so that the varieties in field k are replaced the varieties in the extension of k imbedded to the algebraic closure of k, the number of which is finite. Whether the number of the lifts for varieties is finite seems to depends on the situation.

1. If the imbedding is assumed to be same for all points of the variety the situation seems to reduce to the imbeddings of k to the algebraic completion of rationals and one would have quantum superposition of varieties in the union of finite number of representatives of the algebraic extension to which the absolute Galois group acts.

2. Physicist could however ask whether the invariance under the action of Galois group could be local in some sense. The selection of separable extension could indeed be only pseudo-constant in p-adic case and thus depend on finite number of pinary digits of the k-valued coordinates of the point of the algebraic variety. Local gauge invariance would say that any pseudo constant element of local absolute Galois group acts as a symmetry. This would suggest that one can introduce Galois connection. Since Lie algebra is not defined now one should introduce the connection as parallel translations by Galois group element for paths in the algebraic variety.

One key result is that pure motives using numerical equivalence are equivalent with the category of representations of an algebraic group called motivic Galois group which has Lie algebra and is thus looks like a continuous group.

1. Lie algebra structure for something apparently discrete indeed makes sense for profinite groups (synonymous to Stone spaces). Spaces with p-adic topology are basic examples of this kind of spaces. For instance, 2-adic integers is a Stone space obtained as the set of all bit sequences allowed to contain infinite number of non-vanishing digits. This implies that real discreteness transforms to p-adic continuity and the notion of Lie algebra makes sense. For polynomials this would correspond to polynomials with strictly infinite degree unless one considers the absolute Galois group associated with the algebraic extension of rationals associated with an ordinary polynomial. For infinite primes this would correspond to many-fermion states containing infinite number of fermions kicked out from the Dirac sea and from the point of view of physics would look like an idealization.

2. Motivic Galois group does not obviously correspond to the Galois groups as they are introduced above. Absolute Galois group for the extension of say rationals however emerges if one performs the lift to the algebraic completion and this might be how one ends up with motivic Galois group and also with p-adic physics. One can perhaos say that the Galois groups as introduced above make sense in the intersection of real and p-adic worlds.

3. The choice of algebraic extension might be encountered also in the construction of roots for the polynomials associated with infinite primes and since this choice is not unique it seems that one must use quantum superposition of the different choices and must introduce the action of an appropriate absolute Galois group. This group would be absolute Galois group for algebraic extension of polynomials of n variables at n:th level and ordinary Galois group at the lowest level of hierarchy which should be or less the same as the Galois group introduced above. This could bring in additional spin like degrees of freedom in which the absoltey Galois group acts.

   The fascinating question is whether one could regard not only the degrees of freedom associated with the finite Galois groups but even those associated with the absolute Galois group as physical. Physically the analogs of color quantum numbers whose net values vanish for confined states would be in question. To sum up, it seems that number theory could contain implicitly an incredible rich spectrum of physics.

5 Motives and twistor approach applied to TGD

Motivic cohomology has turned out to pop up in the calculations of the twistorial amplitudes using Grassmannian approach [2, 5]. The amplitudes reduce to multiple residue integrals over smooth projective sub-varieties of projective spaces. Therefore they represent the simplest kind of algebraic geometry for which cohomology theory exists. Also in Grothendieck's vision about motivic cohomology [48] projective spaces are fundamental as spaces to which more general spaces can be mapped in the construction of the cohomology groups (factorization).

5.1 Number theoretic universality, residue integrals, and symplectic symmetry

A key challenge in the realization of the number theoretic universality is the definition of p-adic definite integral. In twistor approach integration reduces to the calculation of multiple residue integrals over closed varieties. These could exist also for p-adic number fields. Even more general integrals identifiable as integrals of forms can be defined in terms of motivic cohomology.

Yangian symmetry [38], [6] is the symmetry behind the successes of twistor Grassmannian approach [4] and has a very natural realization in zero energy ontology [17]. Also the basic prerequisites for twistorialization are satisfied. Even more, it is possible to have massive states as bound states of massless ones and one can circumvent the IR difficulties of massless gauge theories. Even UV divergences are tamed since virtual particles consist of massless wormhole throats without bound state condition on masses. Space-like momentum exchanges correspond to pairs of throats with opposite sign of energy.

Algebraic universality could be realized if the calculation of the scattering amplitudes reduces to multiple residue integrals just as in twistor Grassmannian approach. This is because also p-adic integrals could be defined as residue integrals. For rational functions with rational coefficients field the outcome would be an algebraic number apart from power of 2π, which in p-adic framework is a nuisance unless it is possible to get rid of it by a proper normalization or unless one can accepts the infinite-dimensional transcendental extension defined by 2π. It could also happen that physical predictions do not contain the power of 2π.

Motivic cohomology defines much more general approach allowing to calculate analogs of integrals of forms over closed varieties for arbitrary number fields. In motivic integration [46] - to be discussed below - the basic idea is to replace integrals as real numbers with elements of so called scissor group whose elements are geometric objects. In the recent case one could consider the possibility that $(2\pi)^n$ is interpreted as torus $(S^1)^n$ regarded as an element of scissor group which is free group formed by formal sums of varieties modulo certain natural relations meaning.

Motivic cohomology allows to realize integrals of forms over cycles also in p-adic context. Symplectic transformations are transformation leaving areas invariant. Symplectic form and its exterior powers define natural volume measures as elements of cohomology and p-adic variant of integrals over closed and even surfaces with boundary might make sense. In TGD framework symplectic transformations indeed define a fundamental symmetry and quantum fluctuating degrees of freedom reduce to a symplectic group assignable to $\delta M^4 \pm \times CP_2$ in well-defined sense [3]. One might hope that they could allow to define scissor group with very simple canonical representatives- perhaps even polygons-so that integrals could be defined purely algebraically using elementary area (volume) formulas and allowing continuation to real and p-adic number fields. The basic argument could be that varieties with rational symplectic volumes form a dense set of all varieties involved.

5.2 How to define the p-adic variant for the exponent of Kähler action?

The exponent of Kähler function defined by the Kähler action (integral of Maxwell action for induced Kähler form) is central for quantum at least in the real sector of WCW. The question is whether this exponent could have p-adic counterpart and if so, how it should be defined.

In the real context the replacement of the exponent with power of p changes nothing but in the p-adic context the interpretation is affected in a dramatic manner. Physical intuition provided by p-adic thermodynamics [11] suggest that the exponent of Kähler function is analogous to Bolzmann weight replaced in the p-adic context with non-negative power of p in order to achieve convergence of the series defining the partition function not possible for the exponent function in p-adic context.

1. The quantization of Kähler function as $K = rlog(m/n)$, where r is integer, $m > n$ is divisible by a positive power of p and n is indivisible by a power of p, implies that the exponent of Kähler function is of form $(m/n)^r$ and therefore exists also p-adically. This would guarantee the p-adic existence of the vacuum functional for any prime dividing m and for a given prime p would select a restricted set of p-adic space-time sheets (or partonic 2-surfaces) in the intersection of real and p-adic worlds. It would be possible to assign several p-adic primes to a given space-time sheet (or partonic 2-surface). In elementary particle physics a possible interpretation is that elementary particle can correspond to several p-adic mass scales differing by a power of two [12]. One could also consider a more general quantization of Kähler action as sum $K = K_1 + K_2$ where $K_1 = rlog(m/n)$ and $K_2 = n$, with n divisible by p since $exp(n)$ exists in this case and one has $exp(K) = (m/n)^r \times exp(n)$. Also transcendental extensions of p-adic numbers involving $p + n - 2$ powers of $e^{1/n}$ can be considered.

2. The natural continuation to p-adic sector would be the replacement of integer coefficient r with a p-adic integer. For p-adic integers not reducing to finite integers the p-adic norm of the vacuum functional would however vanish and their contribution to the transition amplitude vanish unless the number of these space-time sheets increases with an exponential rate making the net contribution proportional to a finite positive power of p. This situation would correspond to a critical situation analogous to that encounted in string models as the temperature approaches Hagedorn temperature [3] and the number states with given energy increases as fast as the Boltzmann weight. Hagedorn temperature is essentially due to the extended nature of particles identified as strings. Therefore this kind of non-perturbative situation might be encountered also now.

3. Rational numbers m/n with n not divisible by p are also infinite as real integers. They are somewhat problematic. Does it make sense to speak about algebraic extensions of p-adic numbers generated by $p^{1/n}$ and giving $n-1$ fractional powers of p in the extension or does this extension reduce to something equivalent with the original p-adic number field when one redefines the p-adic norm as $|x|_p \rightarrow |x|^{1/n}$? Physically this kind of extension could have a well defined meaning. If this does not make sense, it seems that one must treat p-adic rationals as infinite real integers so that the exponent would vanish p-adically.

4. If one wants that Kähler action exists p-adically a transcendental extension of rational numbers allowing all powers of $log(p)$ and $log(k)$, where $k < p$ is primitive $p - 1$:th root of unity in $G(p)$. A weaker condition would be an extension to a ring with containing only $log(p)$ and $log(k)$ but not their powers. That only single $k < p$ is needed is clear from the identity $log(k^r) = rlog(k)$, from primitive root property, and from the possibility to expand $log(k^r + pn)$, where n is p-adic integer, to powers series with respect to p. If the exponent of Kähler function is the quantity coding for physics and naturally required to be ordinary p-adic number, one could allow $log(p)$ and $log(k)$ to exists only in symbolic sense or in the extension of p-adic numbers to a ring with minimal dimension.

   Remark: One can get rid of the extension by $log(p)$ and $log(k)$ if one accepts the definition of p-adic logarithm as $log(x) = log(p^{-k}x/x_0)$ for $x = p^k(x_0 + py)$, $|y|_p < 1$. To me this definition looks somewhat artificial since this function is not strictly speaking the inverse of exponent function but might have a deeper justification.

5. What happens in the real sector? The quantization of Kähler action cannot take place for all real surfaces since a discrete value set for Kähler function would mean that WCW metric is not defined. Hence the most natural interpretation is that the quantization takes place only in the intersection of real and p-adic worlds, that is for surfaces which are algebraic surfaces in some sense. What this actually means is not quite clear. Are partonic 2-surfaces and their tangent space data algebraic in some preferred coordinates? Can one find a universal identification for the preferred coordinates- say as subset of imbedding space coordinates selected by isometries?

If this picture inspired by p-adic thermodynamics holds true, p-adic integration at the level of WCW would give analog of partition function with Boltzman weight replaced by a power of p reducing a sum over contributions corresponding to different powers of p with WCW integra.l over space-time sheets with this value of Kähler action defining the analog for the degeneracy of states with a given value of energy. The integral over space-time sheets corresponding to fixed value of Kähler action should allow definition in terms of a symplectic form defined in the p-adic variant of WCW. In finite-dimensional case one could worry about odd dimension of this sub-manifold but in infinite-dimensional case this need not be a problem. Kähler function could defines one particular zero mode of WCW Kähler metric possessing an infinite number of zero modes.

One should also give a meaning to the p-adic integral of Kähler action over space-time surface assumed to be quantized as multiples of $log(m/n)$.

1. The key observation is that Kähler action for preferred extrememals reduces to 3-D Chern-Simons form by the weak form of electric-magnetic duality. Therefore the reduction to cohomology takes place and the existing p-adic cohomology gives excellent hopes about the existence of the p-adic variant of Kähler action. Therefore the reduction of TGD to almost topological QFT would be an essential aspect of number theoretical universality.

2. This integral should have a clear meaning also in the intersection of real and p-adic world. Why the integrals in the intersection would be quantized as multiple of $log(m/n)$, m/n divisible by a positive power of p? Could $log(m/n)$ relate to the integral of $\int_1^p dx/x$, which brings in mind $\oint dz/z$ in residue calculus. Could the integration range $[1, m/n]$ be analogous to the integration range $[0, 2\pi]$. Both multiples of 2π and logarithms of rationals indeed emerge from definite integrals of rational functions with rational coefficients and allowing rational valued limits and in both cases $1/z$ is the rational function responsible for this.

3. $log(m/n)$ would play a role similar to 2π in the approach based on motivic integration where integral has geometric objects as its values. In the case of 2π the value would be circle. In the case of $log(m/n)$ the value could be the arc between the points $r = m/n > 1$ and $r = 1$ with r identified the radial coordinate of light-cone boundary with conformally invariant length measures dr/r. One can also consider the idea that $log(m/n)$ is the hyperbolic angle analogous to 2π so that these two integrals could correspond to hyper-complex and complex residue calculus respectively.

4. TGD as almost topological QFT means that for preferred extremals the Kähler action reduces to 3-D Chern-Simons action, which is indeed 3-form as cohomology interpretation requires, and one could consider the possibility that the integration giving $log(m/n)$ factor to Kähler action is associated with the integral of Chern-Simons action density in time direction along light-like 3-surface and that the integral over the transversal degrees of freedom could be reduced to the flux of the induced CP_2 Kähler form. The logarithmic quantization of the effective distance between the braid end points the in metric defined by modified gamma matrices has been proposed earlier [7].

Since p-adic objects do not possess boundaries, one could argue that only the integrals over closed varieties make sense. Hence the basic premise of cohomology would fail when one has p-adic integral over braid strand since it does not represent closed curve. The question is whether one could identify the end points of braid in some sense so that one would have a closed curve effectively or alternatively relative cohomology. Periodic boundary conditions is certainly one prerequisite for this kind of identification.

1. In one of the many cohomologies known as quantum cohomology [27] one indeed assumes that the intersection of varieties is fuzzy in the sense that two surfaces for which points are connected by what is called pseudo-holomorphic curve can be said to intersect at these points. As a special case pseudo-holomorphic curve reduce to holomorphic curve defined by a holomorphic map of 2-D Kähler manifold to complex manifold with Kähler structure. The question arises what "pseudoholomorphic curve connects points" really means. In the recent case a natural analog would be 2-D string world sheets or partonic 2-surfaces so that complex numbers are replaced by hyper-complex numbers effectively. The boundaries of string world sheets would be 1-D braid strands at wormhole throats and at the end of space-time sheet at boundaries of CD. In spirit of algebraic geometry one could also call the 1-D braid strands holomorphic curves connecting points of the partonic 2-surfaces at the two light-like boundaries of CD. In the similar manner space-like braid strands would connect points of partonic 2-surface at same end of CD.

2. In the construction of the solutions of the modified Dirac equation one assumes periodic boundary conditions so that in physical sense these points are identified [7]. This assumption actually reduces the locus of solutions of the modified Dirac equation to a union of braids at light-like 3-surfaces so that finite measurement resolution for which discretization defines space-time correlates becomes an inherent property of the dynamics. The coordinate varying along the braid strands is light-like so that the distance in the induced metric vanishes between its end points (unlike the distance in the effective metric defined by the modified gamma matrices): therefore also in metric sense the end points represent intersection point. Also the effective 2-dimensionality means are effectively one and same point.

3. The effective metric 2-dimensionality of the light-like 2-surfaces implies the counterpart of conformal invariance with the light-like coordinate varying along braid strands so that it might make sense to say that braid strands are pseudo-holomorphic curves. Note also that the end points of a braid along light-like 3-surface are not causally independent: this is why M-matrix in zero energy ontology is non-trivial. Maybe the causal dependence together with periodic boundary conditions, light-likeness, and pseudo-holomorphy could imply a variant of quantum cohomology and justify the p-adic integration over the braid strands.

5.3 Motivic integration

While doing web searches related to motivic cohomology I encountered also the notion of motivic measure [46] proposed first by Kontsevich. Motivic integration is a purely algebraic procedure in the sense that assigns to the symbol defining the variety for which one wants to calculate measure. The measure is not real valued but takes values in so called scissor group, which is a free group with group operation defined by a formal sum of varieties subject to relations. Motivic measure is number theoretical universal in the sense that it is independent of number field but can be given a value in particular number field via a homomorphism of motivic group to the number field with respect to sum operation.

Some examples are in order.

1. A simple example about scissor group is scissor group consisting operations needed in the algorithm transforming plane polygon to a rectangle with unit edge. Polygon is triangulated; triangles are transformed to rectangle using scissors; long rectangles are folded in one half; rectangles are rescaled to give an unit edge (say in horizontal direction); finally the resulting rectangles with unit edge are stacked over each other so that the height of the stack gives the area of the polygon. Polygons which can be transformed to each other using the basic area preserving building bricks of this algorithm are said to be congruent.

   The basic object is the free abelian group of polygons subject to two relations analogous to second homology group. If P is polygon which can be cut to two polygons P_1 and P_2 one has $[P] = [P_1] + [P_2]$. If P and P' are congruent polygons, one has $[P] = [P']$. For plane polygons the scissor group turns out to be the group of real numbers and the area of polygon is the area of the resulting rectangle. The value of the integral is obtained by mapping the element of scissor group to a real number by group homomorphism.

2. One can also consider symplectic transformations leaving areas invariant as allowed congruences besides the slicing to pieces as congruences appearing as parts of the algorithm leading to a standard representation. In this framework polygons would be replaced by a much larger space of varieties so that the outcome of the integral is variety and integration means finding a simple representative for this variety using the relations of the scissor group. One might hope that a symplectic transformations singular at the vertices of polygon combined with with scissor transformations could reduce arbitrary area bounded by a curve into polygon.

3. One can identify also for discrete sets the analog of scissor group. In this case the integral could be simply the number of points. Even more abstractly: one can consider algebraic formulas defining algebraic varieties and define scissor operations defining scissor congruences and scissor group as sums of the formulas modulo scissor relations. This would obviously abstract the analytic calculation algorithm for integral. Integration would mean that transformation of the formula to a formula stating the outcome of the integral. Free group for formulas with disjunction of formulas is the additive operation [48]. Congruence must correspond to equivalence of some kind. For finite fields it could be bijection between solutions of the formulas. The outcome of the integration is the scissor group element associated with the formula defining the variety.

4. For residue integrals the free group would be generated as formal sums of even-dimensional complex integration contours. Two contours would be equivalent if they can be deformed to each other without going through poles. The standard form of variety consists of arbitrary small circles surrounding the poles of the integrand multiplied by the residues which are algebraic numbers for rational functions. This generalizes to rational functions with both real and p-adic coefficients if one accepts the identification of integral as a variety modulo the described equivalence so that $(2\pi)^n$ corresponds to torus $(S^1)^n$. One can replace torus with 2π if one accepts an infinite-dimensional algebraic extension of p-adic numbers by powers of 2π. A weaker condition is that one allows ring containing only the positive powers of 2π.

5. The Grassmannian twistor approach for two-loop hexagon Wilson gives dilogarithm functions $L_k(s)$ [5]. General polylogarithm is defined by obey the recursion formula:

$$Li_{s+1}(z) = \int_0^z Li_s(t)\frac{dt}{t} \quad .$$

   Ordinary logarithm $Li_1(z) = -log(1-z)$ exists p-adically and generates a hierarchy containing dilogarithm, trilogarithm, and so on, which each exist p-adically for $|x| < 1$as is easy to see. If one accepts the general definition of logariths one finds that the entire function series exists p-adically for integer values of s. An interesting question is how strong constraints p-adic existence gives to the thetwistor loop integrals and to the underlying QFT.

6. The ring having p-adic numbers as coefficients and spanned by transcendentals $log(k)$ and $log(p)$, where k is primitive root of unity in $G(p)$ emerges in the proposed p-adicization of vacuum functional as exponent of Kähler action. The action for the preferred extremals reducing to 3-D Chern-Simons action for space-time surfaces in the intersection of real and p-adic worlds would be expressible p-adically as a linear combination of $log(p)$ and $log(k)$. $log(m/n)$ expressible in this manner p-adically would be the symbolic outcome of p-adic integral $\int dx/x$ between rational points. x could be identified as a preferred coordinate along braid strand. A possible identification for x earlier would be as the length in the effective metric defined by modified gamma matrices appearing in the modified Dirac equation [7] .

5.4 Infinite rationals and multiple residue integrals as Galois invariants

In TGD framework one could consider also another kind of cohomological interpretation. The basic structures are braids at light-like 3-surfaces and space-like 3-surfaces at the ends of space-time surfaces. Braids intersects have common ends points at the partonic 2-surfaces at the light-like boundaries of a causal diamond. String world sheets define braid cobordism and in more general case 2-knot [9]. Strong form of holography with finite measurement resolution would suggest that physics is coded by the data associated with the discrete set of points at partonic 2-surfaces. Cohomological interpretation would in turn would suggest that these points could be identified as intersections of string world sheets and partonic 2-surface defining dual descriptions of physics and would represent intersection form for string world sheets and partonic 2-surfaces.

Infinite rationals define rational functions and one can assign to them residue integrals if the variables x_n are interpreted as complex variables. These rational functions could be replaced with a hierarchy of sub-varieties defined by their poles of various dimensions. Just as the zeros allow realization as braids or braids also poles would allow a realization as braids of braids. Hence the n-fold residue integral could have a representation in terms of braids. Given level of the braid hierarchy with n levels would correspond to a level in the hierarchy of complex varieties with decreasing complex dimension.

One can assign also to the poles (zeros of polynomial in the denominator of rational function) Galois group and obtains a hierarchy of Galois groups in this manner. Also the braid representation would exists for these Galois groups and define even cohomology and homology if they do so for the zeros. The intersections of braids with of the partonic 2-surfaces would represent the poles in the preferred coordinates and various residue integrals would have representation in terms of products of complex points of partonic 2-surface in preferred coordinates. The interpretation would be in terms of quantum classical correspondence.

Galois groups transform the poles to each other and one can ask how much information they give about the residue integral. One would expect that the n-fold residue integral as a sum over residues expressible in terms of the poles is invariant under Galois group. This is the case for the simplest integrals in plane with n poles and probably quite generally. Physically the invariance under the hierarchy of Galois group would mean that Galois groups act as the symmetry group of quantum physics. This conforms with the number theoretic vision and one could justify the formula for the residue integral also as a *definition* motivated by the condition of Galois invariance. Of course, all symmetric functions of roots would be Galois invariants and would be expected to appear in the expressions for scattering amplitudes.

The Galois groups associated with zeros and poles of the infinite rational seem to have a clear physical significance. This can be understood in zero energy ontology if positive (negative) physical states are indeed identifiable as infinite integers and if zero energy states can be mapped to infinite rationals which as real numbers reduce to real units. The positive/negative energy part of the zero energy state would correspond to zeros/poles in this correspondence. An interesting question is how strong correlations the real unit property poses on the two Galois groups hierarchies. The asymmetry between positive and negative energy states would have interpretation in terms of the thermodynamic arrow of geometric time [1] implied by the condition that either positive or negative energy states correspond to state function reduced/prepared states with well defined particle numbers and minimum amount of entanglement.

5.5 Twistors, hyperbolic 3-manifolds, and zero energy ontology

While performing web searches for twistors and motives I have begun to realize that Russian mathematicians have been building the mathematics needed by quantum TGD for decades by realizing the vision of Grothendieck. One of the findings was the article Volumes of hyperbolic manifolds and mixed Tate motives [45] by Goncharov- one of the great Russian mathematicians involved with the drama- about volumes of hyperbolic n-manifolds and motivic integrals.

Hyperbolic n-manifolds [18] are n-manifolds equipped with complete Riemann metric having constant sectional curvature equal to -1 (with suitable choice of length unit) and therefore obeying Einstein's equations with cosmological constant. They are obtained as coset spaces on proper-time constant hyperboloids of n+1-dimensional Minkowski space by dividing by the action of discrete subgroup of SO(n,1), whose action defines a lattice like structure on the hyperboloid. What is remarkable is that the volumes of these closed spaces are homotopy invariants in a well-define sense.

What is even more remarkable that hyperbolic 3-manifolds [17] are completely exceptional in that there are very many of them. The complements of knots and links in 3-sphere are often cusped hyperbolic 3-manifolds (having therefore tori as boundaries). Also Haken manifolds are hyperbolic. According to Thurston's geometrization conjecture, proved by Perelman (whom we all know!), any closed, irreducible, atoroidal 3-manifold with infinite fundamental group is hyperbolic. There is an analogous statement for 3-manifolds with boundary. One can perhaps say that very many 3-manifolds are hyperbolic.

The geometrization conjecture of Thurston [13] allows to see hyperbolic 3-manifolds in a wider framework. The theorem states that compact 3-manifolds can be decomposed canonically into sub-manifolds that have geometric structures. It was Perelman who sketched the proof of the conjecture. The prime decomposition with respect to connected sum reduces the problem to the classification of prime 3-manifolds and geometrization conjecture states that closed 3-manifold can be cut along tori such that the interior of each piece has a geometric structure with finite volume serving as a topological invariant. There are 8 possible geometric structures in dimension three and they are characterized by the isometry group of the geometry and the isotropy group of point.

Important is also the behavior under Ricci flow [30] $\partial_t g_{ij} = -2R_{ij}$: here t is not space-time coordinate but a parameter of homotopy. If I have understood correctly, Ricci flow is a dissipative flow gradually polishing the metric for a particular region of 3-manifold to one of the 8 highly symmetric metrics defining topological invariants. This conforms with the general vision about dissipation as the source of maximal symmetries. For compact n-manifolds the normalized Ricci flow $\partial_t g_{ij} = -2R_{ij} + (2/n)Rg_{ij}$ preserving the volume makes sense. Interestingly, for $n = 4$ the right hand side is Einstein tensor so that the solutions of vacuum Einstein's equations in dimension four are fixed points of normalized Ricci flow. Ricci flow expands the negatively curved regions and contracts the positively curved regions of space-time time. Hyperbolic geometries represent one these 8 geometries and for the Ricci flow is expanding. The outcome is amazingly simple and gives also support for the idea that the preferred extremals of Kähler action could represent maximally symmetries 4-geometries defining topological or algebraic geometric invariants: the preferred extremals would be maximally symmetric representatives - kind of archetypes- for a given topology or algebraic geometry.

The volume spectrum for hyperbolic 3-manifolds forms a countable set which is however not discrete: some reader might understand what the statement that one can assign to them ordinal ω^ω could possibly mean for the man of the street. What comes into my simple mind is that p-adic integers and more generally, profinite spaces which are not finite, are something similar: one can enumerate them by infinitely long sequences of pinary digits so that they are countable (I do not know whether also infinite p-adic primes must be allowed). They are totally disconnected in real sense but do not form a discrete set since since can connect any two points by a p-adically continuous curve.

What makes twistor people excited is that the polylogarithms emerging from twistor integrals and making sense also p-adically seems to be expressible in terms of the volumes of hyperbolic manifolds. What fascinates me is that the moduli spaces for causal diamonds or rather for the double light-cones associated with their M^4 projections with second tip fixed are naturally lattices of the3-dimensional hyperbolic space defined by all positions of the second tip and 3-dimensional hyperbolici spaces are the most intersting ones! At least in the intersection of the real and p-adic worlds number theoretic discretization requires discretization and volume could be quantized in discrete manner.

For $n = 3$ the group defining the lattice is a discrete subgroup of the group of SO(3,1) which equals to $PSL(2, C)$ obtained by identifying $SL(2, C)$ matrices with opposite sign. The divisor group defining the lattice and hyperbolic spaces as its lattice cell is therefore a subgroup of $PSL(2, Z_c)$, where Z_c denotes complex integers. Recall that $PSL(2, Z_c)$ acts also in complex plane (and therefore on partonic 2-surfaces) as discrete Möbius transformations wheras $PSL(2, Z)$ correspond to 3-braid group. Reader is perhaps familiar with fractal like orbits of points under iterated Möbius transformations. The lattice cell of this lattice obtained by identifying symmetry related points defines hyperbolic 3-manifolds. Therefore zero energy ontology realizes directly the hyperboliic manifolds whose volumes should somehow represent the polylogarithms.

The volumes, which are topological invariants, are said to be highly transcendental. In the intersection of real and p-adic worlds only algebraic volumes are possible unless one allows extension by say finite number of roots of e (e^p is p-adic number). The p-adic existence of polylogarithms suggests that also p-adic variants of hyperbolic spaces make sense and that one can assign to them volume as topological invariance although the notion of ordinary volume integral is problematic. In fact, hyperbolic spaces are symmetric spaces and general arguments allow to imagine what the p-adic variants of real symmetric spaces could be.

6 Could Gromov-Witten invariants and braided Galois homology together allow to construct WCW spinor fields?

The challenge of TGD is to understand the structure of WCW spinor fields both in the zero modes which correspond to symplectically invariant degrees of freedom not contributing to the WCW Kähler metric and in quantum fluctuating degrees of freedom parametrized by the symplectic group of $\delta M^4_\pm \times CP_2$. The following arguments suggest that an appropriate generalization of Gromov-Witten invariants to covariants combined with braid Galois homology could allow do construct WCW spinor fields and at the same time M-matrices defining the rows of the unitary U-matrix between zero energy states.

6.1 Gromov-Witten invariants

Gromov-Witten invariants [14] are rational numbers $GW^{X,A}_{g,n}$, which in a loose sense count the number of pseudo-holomorphic curves of genus g and n marked points and homology equivalence class A in symplectic space X meeting n surfaces of X with given homology equivalence classes. These invariants can distinguish between different symplectic manifolds. Since also the proposed generalized homology groups would define symplectic invariants if the realization of braided Galois groups as symplectic flows works, the attempt to understand the relation of Gromov-Witten invariants of TGD is well-motivated.

Let X be a symplectic manifold with almost complex structure J (the transition functions are not holomorphic) and C be an algebraic variety in X of genus g and with complex structure j having n marked points $x_1,...x_n$, which are points of X. Pseudo-homolomorphic maps of C to X are by definition maps, whose Jacobian map commutes with the multiplication of the tangent space vectors with the antisymmetric tensor representing imaginary unit $J \circ df = df \circ j$. If the symplectic manifold allows Kähler structure, one can say that pseudohomolomorphic maps commute with the multiplication by imaginary unit so that tangent plane of complex 2-manifold is mapped to a complex tangent plane of X.

The moduli space $M_{g,n}(X)$ of the pseudoholomorphic maps is finite-dimensional. One considers also its subspaces $M_{g,n}(X,A)$ of $M_{g,n}(X)$, where A represents a fixed homology equivalence class A for the image of C in X. The so called evaluation map from $M_{g,n}(X,A)$ to $M_{g,n}(X)) \times X^n$ defined by $(C, x_1, x_2, ...x_n, f) \rightarrow (st(C, x_1, x_2, ...x_n); f(x_1), ..., f(x_n))$. Here $st(C, x_1, x_2, ...x_n)$ denotes so called stabilization of $(C, x_1, ...x_n)$ defined in the following manner. A smooth component of Riemann surface is said to be stable if the number of automorphisms (conformal transformations) leaving the marked and nodal (double) points invariant is finite. Stabilization is obtained by dropping away the unstable components from the domain of C.

The image of the fundamental class of the moduli space $M_{g,n}(X)$ defines a homology class in $M_{g,n}(X)) \times X^n$. Since the homology groups of $M_{g,n}(X)) \times X^n$ are by Künneth theorem expressible as convolutions of homology groups of $M_{g,n}(X))$ and n copies of X, this homology class can be expressed as a sum

$$\sum_{\beta,\alpha_i} GW^{X,A}_{g,n} \beta \times \alpha_1 ... \times \alpha_n \quad .$$

The coefficients which in the general case are rational define Gromov-Witten invariants. One can rouhgly say that these rational numbers count the number of surfaces C intersecting the n homology classes α_i of X.

6.2 Gromov-Witten invariants and topological string theory of type A

Gromow-Witten invariants appear in topological string theory of type A [35] for which the scattering amplitudes depend on Kähler structure of X only. The target space X of this theory is 6-dimensional symplectic manifold. X can correspond to 6-dimensional Calabi-Yau manifold. Twistor space is one particular example of this kind of manifold and one can indeed relate twistor amplitudes to those of topological string theory in twistor space.

Type A topological string theory contains both fundamental string orbits, which are 2-surfaces wrapping over 2-real-D homorphic curves in X and D2 branes, whose 3-D "orbits" in X wrap over Lagrangian manifolds having by definition a vanishing induced symplectic form. There are also strings connecting the branes. C corresponds now to the world sheet of string with n marked points representing emitted particles. Gromov-Witten invariants are defined as integrals over the moduli spaces $M_{g,n}(X)$ and provide a rigorous definition for path integral and the free energy at given genus g is the generating function for Gromov-Witten invariants.

Witten introduced the formulation of the topological string theories in terms of topological sigma models [34]. The formulation involves the analog of BRST symmetry encountered in gauge fixing meaning that one replaces target space with super-space by assigning to target space-coordinates anticommutating partners which do not however represent genuine fermionic degrees of freedom. One also replaces string world sheet with a super-manifold $\mathcal{N} = (2,2)$ SUSY and spinors are world sheet spinors and Lorentz transformations act on string world sheet. Topological string models are characterized by continuous R-symmetries and the mixing of rotational and R-symmetries takes place. The R-symmetry associated with 2-D world sheet Lorentz transformation compensates for the spin rotation so that one indeed obtains a BRST charge Q (for elementary introduction to BRST symmetry see [7]), which is scalar and the condition $Q^2 = 0$ is satisfied identically so that cohomology is obtained.

6.3 Gromov-Witten invariants and WCW spinor fields in zero mode degrees of freedom

One can ask whether Gromow-Witten invariants of something more general could emerge naturally in TGD framework. A natural guess is that these invariants or their generalizations emerge in the construction of WCW spinor fields in zero mode degrees of freedom, which do not contribute to the line element of WCW Kähler metric.

6.3.1 Comparison of the basic geometric frameworks

The basic geometric frameworks are sufficiently similar to encourage the idea that Gromov-Witten type invariants might make sense in TGD framework.

1. In the standard formulation of TGD the 6-dimensional symplectic manifold is replaced with the metrically 6-dimensional manifold $\delta M^4_\pm \times CP_2$ having degenerate symplectic and Kähler structure and reducing effectively (metrically) to the symplectic manifold $S^2 \times CP_2$. Partonic 2-surfaces at the light-like boundaries of CD identifiable as wormhole throats define the counterparts of fundamental string like object of topological string theory of type A. The n marked points of Gromov-Witten theory could correspond to the ends of braid strands carrying purely bosonic quantum numbers characterized by the attached $\delta M^4_\pm \times CP_2$ Hamiltonians with well defined angular momentum and color quantum numbers. One must distinguish these braid strands from the braid strands carrying fermion quantum numbers.

2. There are also differences. One assigns 3-D surfaces to the boundaries of CD and partonic 2-surfaces at CD are connected with are interpreted as strings so that partonic 2-surfaces have also brane like character. One can identify 3-D surfaces for which induced Kähler forms of CP_2 and $\delta M^4_\pm$ vanish (any surface with 1-D projection to $\delta M^4_\pm$ and 2-D CP_2 projection with Lagrangian manifold would define counterpart of brane) but it is not natural to raise these objects to a special role.

3. The imbedding maps of the partonic 2-surface to $\delta M^4_\pm \times CP_2$ should be pseudo-holomorphic in some sense. One can express the light-cone boundary as $\delta M^4_\pm = S^2 \times R_+$, where S^2 corresponds to a sphere associated with a given choice of a rest system and R_+ the radial light-like line from the tip of the light-cone with radial coordinate r. Lorentz boosts parametrize the choices of the spheres. Lorentz boosts however also affect the second tip of CD and so that they act on the moduli space of CDs.

4. One should understand what pseudo-holomorphy means. Since the moduli space of pseudo-holomorphic surfaces is finite-dimensional, only a very restricted set of partonic 2-surfaces satisfies pseudo-holomorphy condition. The induced metric of the partonic 2-surface defines a unique complex structure. Pseudo-holomorphy states that Jacobian takes the complex tangent place of partonic 2-surface to a comlex plane of the tangent space of $\delta M^4_\pm \times CP_2$. Pseudo-holomorphy is implied by holomorphy stating that both CP_2 coordinates and S^2 coordinates as functions of

the complex coordinate of the partonic 2-surface are holomorphic functions implying that the induced metric as the standard $ds^2 = g_{z\overline{z}}dzd\overline{z}$. Pseudo-holomorphy is also implied if one can express as a variety using functions which are holomorphic functions of $\delta M^4_\pm$ and CP_2 complex coordinates and analytic functions of the radial coordinate r. These surfaces are characterized by the homology-equivalence classes of their projections in $\delta M^4_\pm$ (3-D Euclidian space with puncture at origin) and in CP_2. Both are characterized by integer. These surfaces obviously define a subset of partonic 2-surfaces and one can actually assign to the string-like objects as cartesian products of string world sheets satisfying minimal surface equations and of 2-D complex sub-manifolds of CP_2.

5. The counter argument is that pseudo-holomorphy condition allows only finite-dimensional moduli space whereas the space of partonic 2-surfaces is infinite-dimensional. Could the finite-measurement resolution imply an effective reduction of the space of partonic 2-surfaces to this moduli space take place? Finite measurement resolution could be understood also as a kind of gauge invariance when realized in terms of inclusion of hyper-finite factors of type II_1 (HFFs) with the action of sub-factor having no effect on its observable properties. Pseudo-holomorphy would serve as a gauge fixing condition.

   I have proposed that quantum TGD is analogous to a physical analog of Turing machine in the sense that the inclusions of HFFs could allow to emulate any QFT with almost gauge group assignable to the included algebran [6]. The representation of these gauge groups as subgroups of symplectic transformations leaving the marked points of the partonic 2-surfaces invariant gives hopes of realizing this idea mathematically. Symplectic groups are indeed completely exceptional because of their representative power [33] and used already in classical mechanics and field theory to represent symmetries. An interesting question is whether the symplectic group associated with $\delta M^4_\pm \times CP_2$ could be universal in the sense that any gauge group of this kind allowsf faithful homomorphism to this group.

6.3.2 Could the analog of type A topological string theory make sense in TGD framework

The observations of previous paragraphs motivate the question whether an analog of type A topological string theory could emerge in the construction of the zero mode dependence of WCW spinor fields. The basic problem is to understand how the WCW spinor fields depend on symplectic invariants defining zero modes (non-quantum fluctuting degrees of freedom).

1. The encouraging symptom is that the n-point functions of both A and B type topological string theories are non-trivial only in dimension $D = 6$, which is the metric dimension of $\delta M^4_\pm \times CP_2$. Since the n-point functions of type A topological string theory depend only on the Kähler structure associated now by CP_2 and $\delta M^4_\pm$ Kähler forms they could code for the physics associated with the zero modes representing non-quantum fluctuating degrees of freedom. Since type B topological string theory requires vanishing of the first Chern class implying Calabi-Yau property, this theory is not possible in the standard formulation of TGD.

   The emergence of the topological string theory of type A seems to be in conflict with what twistorialization suggests. Witten suggested in his classic article [8] boosting the twistor revolution, that the Fourier transforms of the sattering amplitudes from momentum space to twistor space scattering amplitudes for perturbative $\mathcal{N} = 4$ SUSY could be interpreted in terms of D-instanton expansion of topological string theory of type B defined in twistor space CP_3. Twistorial considerations however led to a proposal [17] that TGD allows formulation also in terms of 6-dimensional surfaces in $CP_3 \times CP_3$, which are sphere bundles. CP_3 is a Calabi-Yau manifold and the natural question is whether the analog of topological string theory of type B might emerge in this formulation. The counterpart of the mirror symmetry relating A and B type models for different Calabi-Yau models would relate the two formulations of quantum TGD.

2. The earlier proposal [2] about a symplectic QFT defined as a generalization of conformal QFT coding for these degrees of freedom assigns to the partonic 2-surface collections of marked points defining a its division to 2-polygons carrying Kähler magnetic flux together the signed area

defined by R^3_+ symplectic form (essentially solid angle assignable to partonic 2-surface or its portion with respect to the tip of light-cone). A given assignment of marked points defines symplectic fusion algebra and these algebras integrate to an operad with a product defined by the product of fusion algebras.

One can identify the marked points as the end points of both space-like and time-like braids but it is not natural to assign them fermionic quantum numbers except those of covariantly constant right-handed neutrino spinor with the points of symplectic triangulation. This is well-motivated since symplectic algebra extends to super-symplectic algebra with covariantly constant right handed neutrino spinor defining the super-symmetry. One can assign the values of Hamiltonians of $\delta M^4_\pm \times CP_2$ to the marked points belonging to the irreducible representations of rotation group and color group such that the total quantum numbers vanish by the symplectic invariance. n-point functions would be correlation functions for Hamiltonians. In a well-defined sense one would have color and angular momentum confinement in WCW degrees of freedom.

The vanishing of net quantum numbers need not hold true for single connected partonic 2-surface. Also it could hold true only for a collection of partonic 2-surfaces associated with same 3-surface at either end of CD. The most general condition would be that the total color and spin numbers of positive and negative energy parts of the state sum up to zero in symplectic degrees of freedom.

3. The generating function for Gromov-Witten invariants is defined for a connected pseudo-holomorphic 2-surface with a fixed genus g as such is not general enough if one allows partonic 2-surfaces with several components. The generalization would provide information about the preferred extremal of Kähler action and about the topology of space-time surface. The generalization of the Gromov-Witten partition function in zero modes would define as its inverse the normalization factor for zero energy state identifiable as M-matrix defined as a positive diagonal square root of density matrix multiplied by S-matrix for which initial partons possess fixed genus and which contains superposition over braids with arbitrary number of strands. The intuition from ordinary thermodynamics suggests that this partitition function is in a reasonable approximation expressible as convolution for n-points functions for individual partonic 2-surfaces allowing the set of marked points to carry net $\delta M^4_\pm$ angular momentum and color quantum numbers.

6.3.3 Description of super-symmetries in TGD framework

It is interesting to see whether the formulation of super-symmetries in the framework of topological sigma models [34] has any reasonable relation to TGD where the notion of super-space does not look natural as a fundamental notion although it might be very useful as a formal tool in the formulation of SUSY QFT limit [8] and even quantum TGD itself.

1. Almost topological QFT property means that Kähler action for the preferred extremals reduces to Chern-Simons action assuming the weak form of electric magnetic duality. In the fermionic sector one must use modified gamma matrices defined as contractions of the canonical momentum densities for Kähler action (Kähler-Chern-Simon action) with imbedding space gamma matrices in the counterpart of Dirac action in the interior of space-time sheet and at 3-D wormhole throats. The modified gamma matrices define effective metric quadratic in canonical momentum densities which is typically highly degenerate. It contains information about the induced metric. Therefore one cannot expect that topological sigma model approach could work as such in TGD framework.

2. In TGD framework supersymmetries are generated by right-handed covariantly constant neutrinos and antineutrinos with both spin directions. These spinors are imbedding space spinors rather than world sheet spinors but one can say that the induction of the spinor structure makes them world sheet spinors. Since the momentum of the spinors is vanishing, one can assign all possible spin directions to the neutrinos.

3. Covariantly constant right-handed neutrino and antineutrino can have all possible spin directions and for fixed choice of quantization axes two spin directions are possible. Therefore one could say that rotation group acts as non-Abelian group of R-symmetries. TGD formulation need not be based on sigma model so that it is not all clear whether a twisted Lorenz transformations

are needed. If so, the most obvious guess is that space-time rotations are accompanied by R-symmetry rotation of right-handed neutrino spinors compensating the ordinary rotation it as in the case of topological sigma model originally introduced by Witten.

It is interesting to look the situation also from the point of view of the breaking of SUSY for supergravity defined in dimension 8 by using the table listing super-gravities in various dimensions [1].

1. One can assign to the causal diamond a fixed direction as a WCW correlate for the fixing of spin quantization axis and this direction corresponds to a particular modulus. The preferred time directiond defined by the line connecting the tips of CD and this direction define a plane of non-physical polarizations having in number theoretical approach as a preferred hypercomplex plane of hyper-octonions [16]. Hence it would seem that by the symmetry breaking by the choice of quantization axes allows only two spin directions the right handed neutrino and antineutrino and that different choices of the quantization axes correspond to different values for the moduli space of CDs.

2. Since imbedding space spinors are involved, the sugra counterpart of TGD is $\mathcal{N} = 2$ super gravity in dimension 8 for which super charges are Dirac spinors and their hermitian conjugates with $U(2)$ acting as R-symmetries. Note that the supersymmetry does not require Majorana spinors unlike $N = 1$ supersymmetry does in string model and fixes the target space dimension to $D = 10$ or $D = 11$. Just like $D = 11$ of M-theory is the unique maximal dimension if one requires fundamental Majorana spinors (for which there is no empirical support), $D = 8$ of TGD is the unique maximal dimension if one allows only Dirac spinors.

3. In dimensional reduction to $D = 6$, which is the metric dimension of the boundary of δCD a breaking of $\mathcal{N} = 8$ sugra $\mathcal{N} = (2, 2)$ sugra occurs, and one obtains decomposition into pseudo-real representations with supercharges in representations (4,0) and (0,4) of $R = Sp(2) \times Sp(2)$ ($Sp(2) = Sl(2, R)$ corresponds to 2-D symplectic transformations identifiable also as Lorentz group SO(1,2)). (4,0) and (0,4) could correspond to left and right handed neutrinos with both directions of helicities and thus potentially massive. CP_2 geometry breaks this supersymmetry.

4. The reduction to the level of right handed neutrinos requires a further symmetry breaking and $D = 5$ sugra indeed contains supercharges Q and their conjugates in 4-D pseudoreal representation of $R = Sp(4)$. Note that this group corresponds to 2×2 quaternionic matrices. A possible interpretation would be as a reduction in CP_2 degrees freedom to $U(2) \times U(1)$ invariant sphere.

5. The R-symmetries mixing neutrinos and antineutrinos are pysically questionable so that a breaking of R-symmetry to $Sp(2)\times Sp(2)$ to $SU(2)\times SU(2)$ or even $SU(2)$ should take place. A further reduction to homologically non-trivial geodesic sphere of CP_2 might reduce the action of $CP_2(2)$ holonomies to those generated by electric charge and weak isospin and thus leaving right-handed neutrinos invariant. Fixing the quantization axis of spin would reduce R-symmetry to $U(1)$. The inverse imaged of this geodesic sphere is identified as string world sheet [9].

6.3.4 What could bee the roles braided Galois homology and Gromov-Witten type homology in TGD framework?

In the proposed framework the view about construction of WCW spinor fields would be roughly following.

1. One can distinguish between WCW "orbital" degrees of freedom and fermionic degrees of freedom and in the case of WCW degrees of freedom also between zero modes and quantum fluctuating degrees of freedom. Zero modes correspond essentially to the non-local symplectic invariants assignable to the projections of the $\delta M^4_\pm$ and CP_2 Kähler forms to the space-time surface. Quantum fluctuating degrees of freedom correspond to the symplectic algebra in the basis defined by Hamiltonians belonging to the irreps of rotation group and color group.

2. At the level of partonic 2-surfaces finite measurement resolution leads to discretization in terms of braid ends and symplectic triangulation. At the level of WCW discretization replaces symplectic group with its discrete subgroup. This discrete subgroup must result as a coset space defined by

the subgroup of symplectic group acting as Galois group in the set of braid points and its normal subgroup leaving them invariant. The group algebra of this discrete subgroup of symplectic group would have interpretation in terms of braided Galois cohomology. This picture provides an elegant realization for finite measurement resolutions and there is also a conne. ction with the realization of finite measurement resolution using categorification [39], [2].

3. The generating function for Gromow Witten invariants would define an excellent candidate for the part of WCW spinor field defining on zero modes only. The generalization of Gromov-Witten invariants to n-point functions defined by Hamiltonians of $\delta M^4_\pm \times CP_2$ are symplectic invariants if net $\delta M^4_\pm \times CP_2$ quantum numbers vanish. The most general definition assumes that the vanishing of quantum numbers occurs only for zero energy states having disjoint unions of partonic 2-surfaces at the boundaries of CDs as geometric correlate.

4. The proposed generalized homology theory involving braided Galois group and symplectic group of $\delta M^4_\pm \times CP_2$ would realize the "almost" in TGD as almost topological QFT in finite measurement resolution replacing symplectic group with its discretized version. This algebra would relate to the quantum fluctuating degrees of freedom. The braids would carry only fermion number and there would be no Hamiltonians attached with them. The braided Galois homology could define in the more general situation invariants of symplectic isotopies.

5. One should also add four-momenta and twistors to this picture. The separation of dynamical fermionic and sup-symplectic degrees of freedom suggesets that the Fourier transforms for amplitudes containing the fermionic braid end points as arguments define twistorial amplitudes. The representations of light-like momenta using twistors would lead to a generalization of the twistor formalism. At zero momentum limit one would obtain symplectic QFT with states characterized by collections of Hamiltonians and their super-counterparts.

7 A connection between cognition, number theory, algebraic geometry, topology, and quantum physics

I have had some discussions with Stephen King and Hitoshi Kitada in a closed discussion group about the idea that the duality between Boolean algebras and Stone spaces could be important for the understanding of consciousness, at least cognition. In this vision Boolean algebras would represent conscious mind and Stone spaces would represent the matter: space-time would emerge.

I am personally somewhat skeptic because I see consciousness and matter as totally different levels of existence. Consciousness (and information) is about something, matter just is. Consciousness involves always a change as we no from basic laws about perception. There is of course also the experience of free will and the associated non-determinism. Boolean algebra is a model for logic, not for conscious logical reasoning. There are also many other aspects of consciousness making it very difficult to take this kind of duality seriously.

I am also skeptic about the emergence of space-time say in the extremely foggy form as it used in entropic gravity arguments. Recent day physics poses really strong constraints on our view about space-time and one must take them very seriously.

This does not however mean that Stone spaces could not serve as geometrical correlates for Boolean consciousness. In fact, p-adic integers can be seen as a Stone space naturally assignable to Boolean algebra with infinite number of bits.

7.1 Innocent questions

I ende up with the innocent questions, as I was asked to act as some kind of mathematical consultant and explain what Stone spaces actually are and whether they could have a connection to p-adic numbers. Anyone can of course go to Wikipedia and read the article Stone's representation theorem for Boolean algebras. For a layman this article does not however tell too much.

Intuitively the content of the representation theorem looks rather obvious, at least at the first sight. As a matter fact, the connection looks so obvious that physicists often identify the Boolean algebra and its geometric representation without even realizing that two different things are in question. The subsets of given space- say Euclidian 3-space- with union and intersection as basic algebraic operations and inclusion of sets as ordering relation defined a Boolean algebra for the purposes of physicist. One can assign to each point of space a bit. The points for which the value of bit equals to one define the subset. Union of subsets corresponds to logical OR and intersection to AND. Logical implication B→ A corresponds to A contains B.

When one goes to details problems begin to appear. One would like to have some non-trivial form of continuity.

1. For instance, if the sets are form open sets in real topology their complements representing negations of statements are closed, not open. This breaks the symmetry between statement and it negation unless the topology is such that closed sets are open. Stone's view about Boolean algebra assumes this. This would lead to discrete topology for which all sets would be open sets and one would lose connection with physics where continuity and differential structure are in key role.

2. Could one dare to disagree with Stone and allow both closed and open sets of E^3 in real topology and thus give up clopen assumption? Or could one tolerate the asymmetry between statements and their negations and give some special meaning for open or closet sets- say as kind of axiomatic statements holding true automatically. If so, one an also consider algebraic varieties of lower dimension as collections of bits which are equal to one. In Zariski topology used in algebraic geometry these sets are closed. Again the complements would be open. Could one regard the lower dimensional varieties as identically true statements so that the set of identically true statements would be rather scarce as compared to falsities? If one tolerates some quantum TGD, one could ask whether the 4-D quaternionic/associative varieties defining classical space-times and thus classical physics could be identified as the axiomatic truths. Associativity would be the basic truth inducing the identically true collections of bits.

7.2 Stone theorem and Stone spaces

For reasons which should be clear it is perhaps a good idea to consider in more detail what Stone duality says. Stone theorem states that Boolean algebras are dual with their Stone spaces. Logic and certain kind of geometry are dual. More precisely, any Boolean algebra is isomorphic to closed open subsets of some Stone space and vice versa. Stone theorem respects category theory. The homomorphisms between Boolean algebras A and B corresponds to homomorphism between Stone spaces $S(B)$ and $S(A)$: one has contravariant functor between categories of Boolean algebras and Stone spaces. In the following set theoretic realization of Boolean algebra provides the intuitive guidelines but one can of course forget the set theoretic picture altogether and consider just abstract Boolean algebra.

1. Stone space is defined as the space of homomorphisms from Boolean algebra to 2-element Boolean algebra. More general spaces are spaces of homomorphisms between two Boolean algebras. The analogy in the category of linear spaces would be the space of linear maps between two linear spaces. Homomorphism is in this case truth preserving map: $h(A\ AND\ B) = h(a)\ AND\ h(B)$, $h(\ OR\ B) = h(a)\ OR\ h(B)$ and so on.

2. For any Boolean algebra Stone space is compact, totally disconnected Hausdorff space. Conversely, for any topological space, the subsets, which are both closed and open define Boolean algebra. Note that for a real line this would give 2-element Boolean algebra. Set is closed and open simultaneously only if its boundary is empty and in p-adic context there are no boundaries. Therefore for p-adic numbers closed sets are open and the sets of p-adic numbers with p-adic norm above some lower bound and having some set of fixed pinary digits, define closed-open subsets.

3. Stone space dual to the Boolean algebra does not conform with the physicist's ideas about space-time. Stone space is a compact totally disconnected Hausdorff space. Disconnected space is representable as a union of two or more disjoint open sets. For totally disconnected space this is true for every subset. Path connectedness is stronger notion than connected and says that two points of the space can be always connected by a curve defined as a mapping of *real* unit interval to the space. Our physical space-time seems to be however connected in this sense.

4. The points of the Stone space $S(B)$ can be identified ultrafilters. Ultrafilter defines homomorphism of B to 2-element of Boolean algebra Boolean algebra. Set theoretic realization allows to understand what this means. Ultrafilter is a set of subsets with the property that intersections belong to it and if set belongs to it also sets containing it belong to it: this corresponds to the fact that set inclusion $A \supset B$ corresponds to logical implication. Either set or its complement belongs to the ultrafilter (either statement or its negation is true). Empty set does not. Ultrafilter obviously corresponds to a collection of statements which are simultaneously true without contradictions. The sets of ultrafilter correspond to the statements interpreted as collections of bits for which each bit equals to 1.

5. The subsets of B containing a fixed point b of Boolean algebra define an ultrafilter and imbedding of b to the Stone space by assigning to it this particular principal ultrafilter. b represents a statement which is always true, kind of axiom for this principal ultrafilter and ultrafilter is the set of all statements consistent with b.

   Actually any finite set in the Boolean algebra consisting of a collection of fixed bits b_i defines an ultrafilter as the set all subsets of Boolean algebra containing this subset. Therefore the space of all ultra-filters is in one-one correspondence with the space of subsets of Boolean statements. This set corresponds to the set of statements consistent with the truthness of b_i analogous to axioms.

7.3 2-adic integers and 2-adic numbers as Stone spaces

I was surprised to find that p-adic numbers are regarded as a totally disconnected space. The intuitive notion of connected is that one can have a continuous curve connecting two points and this is certainly true for p-adic numbers with curve parameter which is p-adic number but not for curves with real parameter which became obvious when I started to work with p-adic numbers and invented the notion of p-adic fractal. In other words, p-adic integers form a continuum in p-adic but not in real sense. This example shows how careful one must be with definitions. In any case, to my opinion the notion of path based on p-adic parameter is much more natural in p-adic case. For given p-adic integers one can find p-adic integers arbitrary near to it since at the limit $n \to \infty$ the p-adic norm of p^n approaches zero. Note also that most p-adic integers are infinite as real integers.

Disconnectedness in real sense means that 2-adic integers define an excellent candidate for a Stone space and the inverse of the Stone theorem allows indeed to realize this expectation. Also 2-adic numbers define this kind of candidate since 2-adic numbers with norm smaller than 2^n for any n can be mapped to 2-adic integers. One would have union of Boolean algebras labelled by the 2-adic norm of the 2-adic number. p-Adic integers for a general prime p define obviously a generalization of Stone space making sense for effectively p-valued logic: the interpretation will be discussed below.

Consider now a Boolean algebra consisting of all possible infinitely long bit sequences. This algebra corresponds naturally to 2-adic integers. The generating Boolean statements correspond to sequences with single non-vanishing bit: by taking the unions of these points one obtains all sets. The natural topology is that for which the lowest bits are the most significant. 2-adic topology realizes this idea since n:th bit has norm 2^{-n}. 2-adic integers as an p-adic integers are as spaces totally disconnected.

That 2-adic integers and more generally, 2-adic variants of n-dimensional p-adic manifolds would define Stone bases assignable to Boolean algebras is consistent with the identification of p-adic space-time sheets as correlates of cognition. Each point of 2-adic space-time sheet would represent 8 bits as a point of 8-D imbedding space. In TGD framework WCW ("world of classical worlds") spinors correspond to Fock space for fermions and fermionic Fock space has natural identification as a Boolean algebra. Fermion present/not present in given mode would correspond to true/false. Spinors decompose to a tensor product of 2-spinors so that the labels for Boolean statements form a Boolean algebra too in this case. A possible interpretation is as statements about statements.

In TGD Universe life and thus cognition reside in the intersection of real and p-adic worlds. Therefore the intersections of real and p-adic partonic 2-surfaces represent the intersection of real and p-adic worlds, those Boolean statements which are expected to be accessible for conscious cognition. They correspond to rational numbers or possibly numbers in n algebraic estension of rationals. For rationals pinary expansion starts to repeat itself so that the number of bits is finite. This intersection is also always discrete and for finite real space-time regions finite so that the identification looks a very natural since our cognitive abilities seem to be rather limited. In TGD inspired physics magnetic bodies are the key players and have much larger size than the biological body so that their intersection with their p-adic counterparts can contain much more bits. This conforms with the interpretation that the evolution of cognition means the emergence of increasingly longer time scales. Dark matter hierarchy realized in terms of hierarchy of Planck constants realizes this.

7.4 What about p-adic integers with $p > 2$?

The natural generalization of Stone space would be to a geometric counterpart of p-adic logic which I discussed for some years ago. The representation of the statements of p-valued logic as sequences of pinary digits makes the correspondence trivial if one accepts the above represented arguments. The generalization of Stone space would consist of p-adic integers and imbedding of a p-valued analog of Boolean algebra would map the number with only n:th digit equal to $1, ..., p-1$ to corresponding p-adic number.

One should however understand what p-valued statements mean and why p-adic numbers near powers of 2 are important. What is clear that p-valued logic is too romantic to survive. At least our every-day cognition is firmly anchored to a reality where everything is experience to be true or false.

1. The most natural explanation for $p > 2$ adic logic is that all Boolean statements do not allow a physical representation and that this forces reduction of 2^n valued logic to $p < 2^n$-valued one. For instance, empty set in the set theoretical representation of Boolean logic has no physical representation. In the same manner, the state containing no fermions fails to represent anything physically. One can represent physically at most $2^n - 1$ one statements of n-bit Boolean algebra and one must be happy with $n - 1$ completely represented digits. The remaining statements containing at least one non-vanishing digit would have some meaning, perhaps the last digit allowed could serve as a kind of parity check.

2. If this is accepted then p-adic primes near to power 2^n of 2 but below it and larger than the previous power 2^{n-1} can be accepted and provide a natural topology for the Boolean statements grouping the binary digits to p-valued digit which represents the allowed statements in 2^n valued Boolean algebra. Bit sequence as a unit would be represented as a sequence of physically realizable bits. This would represent evolution of cognition in which simple yes or not statements are replaced with sequences of this kind of statements just as working computer programs are fused as modules to give larger computer programs. Note that also for computers similar evolution is taking place: the earliest processors used byte length 8 and now 32, 64 and maybe even 128 are used.

3. Mersenne primes $M_n = 2^n - 1$ would be ideal for logic purposes and they indeed play a key role in quantum TGD. Mersenne primes define p-adic length scales characterize many elementary particles and also hadron physics. There is also evidence for p-adically scaled up variants of hadron physics (also leptohadron physics allowed by the TGD based notion of color predicting colored excitations of leptons). LHC will certainly show whether M_{89} hadron physics at TeV energy scale is realized and whether also leptons might have scaled up variants.

4. For instance, M_{127} assignable to electron secondary p-adic time scale is .1 seconds, the fundamental time scale of sensory perception. Thus cognition in .1 second time scale single pinary statement would contain 126 digits as I have proposed in the model of memetic code. Memetic codons would correspond to 126 digit patterns with duration of .1 seconds giving 126 bits of information about percept.

If this picture is correct, the interpretation of p-adic space-time sheets- or rather their intersections with real ones- would represent space-time correlates for Boolean algebra represented at quantum level by fermionic many particle states. In quantum TGD one assigns with these intersections braids- or number theoretic braids- and this would give a connection with topological quantum field theories (TGD can be regarded as almost topological quantum field theory).

7.5 One more road to TGD

The following arguments suggests one more manner to end up with TGD by requiring that fermionic Fock states identified as a Boolean algebra have their Stone space as space-time correlate required by quantum classical correspondence. Second idea is that space-time surfaces define the collections of binary digits which can be equal to one: kind of eternal truths. In number theoretical vision associativity condition in some sense would define these divine truths. Standard model symmetries are a must- at least as their p-adic variants -and simple arguments forces the completion of discrete lattice counterpart of M^4 to a continuum.

1. If one wants Poincare symmetries at least in p-adic sense then a 4-D lattice in M^4 with $SL(2,Z)\times T^4$, where T^4 is discrete translation group is a natural choice. $SL(2,Z)$ acts in discrete Minkowski space T^4 which is lattice. Poincare invariance would be discretized. Angles and relative velocities would be discretized, etc..

2. The p-adic variant of this group is obtained by replacing Z and T^4 by their p-adic counterparts: in other words Z is replaced with the group Z_p of p-adic integers. This group is p-adically continuous group and acts continuously in T^4 defining a p-adic variant of Minkowski space consisting of all bit sequences consisting of 4-tuples of bits. Only in real sense one would have discreteness: note also that most points would be at infinity in real sense. Therefore it is possible to speak about analytic functions, differential calculus, and to write partial differential equations and to solve them. One can construct group representations and talk about angular momentum, spin and 4-momentum as labels of quantum states.

3. If one wants standard model symmetries p-adically one must replace T^4 with $T^4 \times CP_2$. CP_2 would be now discrete version of CP_2 obtained from discrete complex space C^3 by identifying points different by a scaling by complex integer. Discrete versions of color and electroweak groups would be obtained.

The next step is to ask what are the laws of physics. TGD fan would answer immediately: they are of course logical statements which can be true identified as subsets of $T^4 \times CP_2$ just as subset in Boolean algebra of sets corresponds to bits which are true.

1. The collections of 8-bit sequences consisting of only 1:s would define define 4-D surfaces in discrete $T^4 \times CP_2$. Number theoretic vision would suggest that they are quaternionic surfaces so that one associativity be the physical law at geometric level. The conjecture is that preferred extremals of Kähler action are associative surfaces using the definition of associativity as that assignable to a 4-plane defined by modified gamma matrices at given point of space-time surface.

2. Induced gauge field and metric make sense for p-adic integers. p-Adically the field equations for Kähler action make also sense. These p-adic surfaces would represent the analog of Boolean algebra. They would be however something more general than Stone assumes since they are not closed-open in the 8-D p-adic topology.

One however encounters a problem.

1. Although the field equations associated with Kähler action make sense, Kähler action itself does not exists as integral nor does the genuine minimization make sense since p-adically numbers are not well ordered and one cannot in general say which of two numbers is the larger one. This is a real problem and suggests that p-adic field equations are not enough and must be accompanied by real ones. Of course, also the metric properties of p-adic space-time are in complete conflict with what we believe about them.

2. One could argue that for preferred extremals the integral defining Kähler action is expressible as an integral of 4-form whose value could be well-defined since integrals of forms for closed algebraic surfaces make sense in p-adic cohomology theory pioneered by Grothendieck. The idea would be to use the definition of Kähler action making sense for preferred extremals as its definition in p-adic context. I have indeed proposed that space-time surfaces define representatives for homology with inspiration coming from TGD as almost topological QFT. This would give powerful constraints on the theory in accordance with the interpretation as a generalized Bohr orbit.

3. This argument together with what we know about the topology of space-time on basis of everyday experience however more or less forces the conclusion that also real variant of $M^4 \times CP_2$ is there and defines the proper variational principle. The finite points (on real sense) of $T^4 \times CP_2$ (in discrete sense) would represent points common to real and p-adic worlds and the identification in terms of braid points makes sense if one accepts holography and restricts the consideration to partonic 2-surfaces at boundaries of causal diamond. These discrete common points would represent the intersection of cognition and matter and living systems and provide a representation for Boolean cognition.

4. Finite measurement resolution enters into the picture naturally. The proper time distance between the tips would be quantized in multiples of CP_2 length. There would be several choices for the discretized imbedding space corresponding to different distance between lattice points: the interpretation is in terms of finite measurement resolution.

It should be added that discretized variant of Minkowski space and its p-adic variant emerge in TGD also in different manner in zero energy ontology.

1. The discrete space $SL(2,Z) \times T^4$ would have also interpretation as acting in the moduli space for causal diamonds identified as intersections of future and past directed light-cones. T^4 would represent lattice for possible positions of the lower tip of CD and and $SL(2,Z)$ leaving lower tip invariant would act on hyperboloid defined by the position of the upper tip obtained by discrete Lorentz transformations. This leads to cosmological predictions (quantization of red shifts). CP_2 length defines a fundamental time scale and the number theoretically motivated assumption is that the proper time distances between the tips of CDs come as integer multiples of this distance.

2. The stronger condition explaining p-adic length scale hypothesis would be that only octaves of the basic scale are allowed. This option is not consistent with zero energy ontology. The reason is that for more general hypothesis the M-matrices of the theory for Kac-Moody type algebra with finite-dimensional Lie algebra replaced with an infinite-dimensional algebra representing hermitian square roots of density matrices and powers of the phase factor replaced with powers of S-matrix. All integer powers must be allowed to obtain generalized Kac-Moody structure, not only those which are powers of 2 and correspond naturally to integer valued proper time distance between the tips of CD. Zero energy states would define the symmetry Lie-algebra of S-matrix with generalized Yangian structure.

3. p-Adic length scale hypothesis would be an outcome of physics and it would not be surprising that primes near power of two are favored because they are optimal for Boolean cognition.

The outcome is TGD. Reader can of course imagine alternatives but remember the potential difficulties due to the fact that minimization in p-adic sense does not make sense and action defined as integral does not exist p-adically. Also the standard model symmetries and quantum classical correspondence are to my opinion "must":s.

7.6 A connection between cognition and algebraic geometry

Stone space is synonym for profinite space. The Galois groups associated with algebraic extensions of number fields represent an extremely general class of profinite group [26]. Every profinite group appears in Galois theory of some field K. The most most interesting ones are inverse limits of $Gal(F_1/K)$ where F_1 varies over all intermediate fields. Profinite groups appear also as fundamental groups in algebraic geometry. In algebraic topology fundamental groups are in general not profinite. Profiniteness means that p-adic representations are especially natural for profinite groups.

There is a fascinating connection between infinite primes and algebraic geometry discussed above leads to the proposal that Galois groups - or rather their projective variants- can be represented as braid groups acting on 2-dimensional surfaces. These findings suggest a deep connection between space-time correlates of Boolean cognition, number theory, algebraic geometry, and quantum physics and TGD based vision about representations of Galois groups as groups lifted to braiding groups acting on the intersection of real and p-adci variants of partonic 2-surface conforms with this.

Fermat theorem serves as a good illustration between the connection between cognitive representations and algebraic geometry. A very general problem of algebraic geometry is to find rational points of an algebraic surface. These can be identified as common rational points of the real and p-adic variant of the surface. The interpretation in terms of consciousness theory would be as points defining cognitive representation as rational points common to real partonic 2-surface and and its p-adic variants. The mapping to polynomials given by their representation in terms of infinite primes to braids of braids of braids.... at partonic 2-surfaces would provide the mapping of n-dimensional problem to 2-dimensional one.

One considers the question whether there are integer solutions to the equation $x^n + y^n + z^n = 1$. This equation defines 2-surfaces in both real and p-adic spaces. In p-adic context it is easy to construct solutions but they usually represent infinite integers in real sense. Only if the expansion in powers of p contains finite number of powers of p, one obtains real solution as finite integers.

The question is whether there are any real solutions at all. If they exist they correspond to the intersections of the real and p-adic variants of these surfaces. In other words p-adic surface contains cognitively representable points. For $n > 2$ Fermat's theorem says that only single point $x = y = z = 0$ exists so that only single p-adic multi-bit sequence $(0, 0, 0, ...)$ would be cognitively representable.

This relates directly to our mathematical cognition. Linear and quadratic equations we can solve and in these cases the number in the intersection of p-adic and real surfaces is indeed very large. We learn the recipes already in school! For $n > 2$ difficulties begin and there are no general recipes and it requires mathematician to discover the special cases: a direct reflection of the fact that the number of intersection points for real and p-adic surfaces involved contains very few points.

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