Could the notion of hyperdeterminant be useful in TGD framework?

M. Pitkänen
Email: matpitka@luukku.com.
http://tgd.wippiespace.com/public_html/.

September 17, 2010

1 Introduction

The vanishing of ordinary determinant tells that a group of linear equations possesses non-trivial solutions. Hyperdeterminant [1] generalizes this notion to a situation in which one has homogenous multilinear equations. The notion has applications to the description of quantum entanglement and has stimulated interest in physics blogs [2, 3]. Hyperdeterminant applies to hyper-matrices with n matrix indices defined for an n-fold tensor power of vector space - or more generally - for a tensor product of vector spaces with varying dimensions. Hyper determinant is an n-linear function of the arguments in the tensor factors with the property that all partial derivatives of the hyper determinant vanish at the point, which corresponds to a non-trivial solution of the equation. A simple example is potential function of n arguments linear in each argument.

1.1 About the definition of hyperdeterminant

Hyperdeterminant was discovered by Cayley for a tensor power of 2-dimensional vector space V_2 (n-linear case for n-fold tensor power of 2-dimensional linear space) and he gave an explicit formula for the hyperdeterminant in this case. For $n = 3$ the definition is following.

$$A^1_{i_3j_3} = \frac{1}{2}\epsilon^{i_1j_1}\epsilon^{i_2j_2}\epsilon^{i_3j_3}A_{i_1i_2i_3}A_{j_1j_2j_3} \quad .$$

In more general case one must take tensor product of $k = 2$ hyper-matrices and perform the contractions of indices belonging to the two groups in by using n 2-D permutations symbols.

$$det(A) = \frac{1}{2^n}(\prod_{a=1}^{n}\epsilon^{i^a_kj^a_k})A_{i^a_1i^a_2...i^a_n}A_{j^a_1j^a_2...j^a_n} \quad .$$

The first guess is that the definition for V_k, $k > 2$ is essentially identical: one takes the tensor product of k hyper-matrices and performs the contractions using k-dimensional permutation symbols.

Under some conditions one can define hyperdeterminant also when one has a tensor product of linear spaces with different dimensions. The condition is that the largest vector space dimension in the product does not exceed the sum of other dimensions.

1.2 Could hyperdeterminant be useful in the description of criticality of Kähler action?

Why the notion of hyperdeterminant- or rather its infinite-dimensional generalization- might be interesting in TGD framework relates to the quantum criticality of TGD stating that TGD Universe involves a fractal hierarchy of criticalities: phase transitions inside phase transitions inside... At classical level the lowest order criticality means that the extremal of Kähler action possesses non-trivial second variations for which the action is not affected. The system is critical. In QFT context one speaks about zero modes. The vanishing of the so called Gaussian (of functional) determinant associated with second variations is the condition for the existence of critical deformations. In QFT context this situation corresponds to the presence of zero modes.

The simplest physical model for a critical system is cusp catastrophe defined by a potential function $V(x)$ which is fourth order polynomial. At the edges of cusp two extrema of potential function stable and unstable extrema co-incide and the rank of the matrix defined by the potential function vanishes. This means vanishing of its determinant. At the tip of the cusp the also the third derivative vanishes of potential function vanishes. This situation is however not describable in terms of hyperdeterminant since it is genuinely non-linear rather than only multilinear.

In a complete analogy, one can consider also the vanishing of n:th variations in TGD framework as higher order criticality so that the vanishing of hyperdeterminant might serve as a criterion for the higher order critical point and occurrence of phase transition.

1. The field equations are formally multilinear equations for variables which correspond to imbedding space coordinates at different space-time points. The generic form of the variational equations is

$$\int \frac{\delta^n S}{\delta h^{k_1}(x_1)\delta h^{k_2}(x_2)...\delta h^{k_n}(x_n)}\delta h^{k_2}(x_2)...\delta h^{k_n}(x_n)\prod_{i=2}^{n} d^4x_k = 0 \quad .$$

   Here the partial derivatives are replaced with functional derivatives. On basis of the formula one has formally an n-linear situation. This is however an illusion in the generic case. For a local action the equations reduce to local partial differential equations involving higher order derivatives and field equations involve products of field variables and their various partial derivatives at single point so that one has a genuinely non-linear situation in absence of special symmetries.

2. If one has multi-linearity, the tensor product is formally an infinite tensor power of 8-D (or actually 4-D by General Coordinate Invariance) linear tangent spaces of H associated with the space-time points. A less formal representation is in terms of some discrete basis for the deformations allowing also linear ordering of the basis functions. One might hope in some basis vanishing diagonal terms in all orders and multilinearity.

3. When one uses discretization, the equations stating the vanishing of the second variation couple nearest neighbour points given as infinite-D matrix with non-vanishing elements at diagonal and in a band along diagonal. For higher variations one obtains similar matrix along a diagonal of infinite cube and the width of the band increases by two units as n increases by 1 unit. One might perhaps say that the range of long range correlations increases as n increases. The vanishing of the elements at the diagonal- not necessarily in this representation- is necessary in order to achieve multi-linear situation.

1.3 Could the field equations for higher variations be multilinear?

The question is whether for some highly symmetric actions- say Kähler action for preferred extremals- the notion of functional (or Gaussian) determinant could have a generalization to hyperdeterminant allowing to concisely express whether the solutions allow deformations for which the action is not affected.

1. In standard field theory framework this notion need not be of much use but in TGD framework, where Kähler action has infinite-dimensional vacuum degeneracy, the situation is quite different. Vacuum degeneracy means that every space-time surface with at most 2-D CP_2 projection which is so called Lagrangian manifold is vacuum extremal. Physically this correspond to Kähler gauge potential, which is pure gauge and implies spin glass degeneracy. This dynamical and local $U(1)$ symmetry of vacua is induced by symplectic transformations of CP_2 and has nothing to do with U(1) gauge invariance. For non-vacua it corresponds to isometries of "world of classical worlds". In particular, for M^4 imbedded in canonical manner to $M^4 \times CP_2$ fourth order variation is the first non-vanishing variation. The static mechanical analogy is potential function which is fourth order polynomial. Dynamical analogy is action for which both kinetic and potential terms are fourth order polynomials.

2. The vacuum degeneracy is responsible for much of new physics and mathematics related to TGD. Vacuum degeneracy and the consequent complete failure of canonical quantization and path integral approach forced the vision about physics as geometry of "World of Classical Worlds" (WCW) meaning a generalization of Einstein's geometrization of physics program. 4-D spin glass degeneracy is of the physical implications and among other things allows to have a failure of the standard form of classical determinism as a space-time correlate of quantum non-determinism. There are reasons to hope that also the hierarchy of Planck constants reduces to the 1-to-many correspondence between canonical momentum densities and time derivatives of imbedding space-coordinates. Quantum criticality and its classical counterparts is a further implication of the vacuum degeneracy and has provided a lot of insights to the world according to TGD. Therefore it would be nice if the generalization of the hyperdeterminant could provide new insights to quantum criticality.

1.4 Multilinearity, integrability, and cancellation of infinities

The multilinearity in the general sense would have a very interesting physical interpretation. One can consider the variations of both Kähler action and Kähler function defined as Kähler action for a preferred extremal.

1. Multilinearity would mean multi-linearization of field equations in some discrete basis for deformations-say the one defined by second variations. Dynamics would be only apparently non-linear. One might perhaps say that the theory is integrable- perhaps even in the usual sense. The basic idea behind quantum criticality is indeed the existence of infinite number of conserved currents assignable to the second variations hoped to give rise to an integrable theory. In fact, the possibility -or more or less the fact - that also higher variations can vanish for more restricted configurations would imply further conserved currents.

2. Second implication would be the vanishing of local divergences. These divergences result in QFT from purely local interaction terms with degree higher than two. Even mass insertion which is second order produces divergences. If diagonal terms are absent from Kähler function, also these divergences are absent in the functional integral. The main idea behind the notion of Kähler function is that it is a non-local functional of 3-surface although Kähler action is a local functional of space-time sheet serving as the analog of Bohr orbit through 3-surface. As one varies the 3-surface, one obtains a 3-surface (light-like wormhole throats with degenerate four-metric) which is also an extremal of Chern-Simons action satisfying weak form of electric magnetic duality.

3. The weak of electric magnetic duality together with the Beltrami property for conserved currents associated with isometries and for Kähler current and corresponding instanton current imply that the Coulomb term in Kähler action vanishes and it reduces to Chern-Simons term at 3-D light-like wormhole throats plus Lagrange multiplier term taking care that the weak electric magnetic duality is satisfied. This contributes a constraint force to field equations so that the theory does not reduce to topological QFT but to what could be called almost topological QFT.

4. Chern-Simons term is a local functional of 3-surface and one argue that the dangerous locality creeps in via the electric-magnetic duality after all. By using the so called Darboux coordates (P_i, Q_i) for CP_2 Chern-Simons action reduces to a third order polynomial proportional to $\epsilon^{ijk} P_i dP_j dQ_k$ so that one indeed has multilinearity rather than non-linearity. The Lagrangian multiplier term however breaks strict locality and also contributes to higher functional derivatives of Kähler function and is potentially dangerous. It contains information about the preferred extremals via the normal derivatives associated with the Kähler electric field in normal direction and it higher derivatives.

5. One has however good hopes about multilinearity of higher variations Kähler function and of Kähler action for preferred extremals on basis of general arguments related to the symmetric space property of WCW. As a matter fact, effective two-dimensionality seems to guarantee genuine non-locality. Recall that effective two-dimensionality is implied by the strong form of General Coordinate Invariance stating that the basic geometric objects can be taken to be either light-like 3-surfaces or space-like 3-surfaces at the ends of space-time surface at boundaries

of causal diamond. This implies that partonic 2-surfaces defining the intersections of these surfaces plus their 4-D tangent space-data code for physics. By effective 2-dimensionality Chern-Simons action is a non-local functional of data about partonic 2-surface and its tangent space. Hence the n:th variation of 3-surface and space-time surface reduces to a non-local functional of n:th variation of the partonic 2-surface and its tangent space data. This is just what genuine multilinearity means.

1.5 Hyperdeterminant and entanglement

A highly interesting application of hyperdeterminants is to the description of quantum entanglement- in particular to the entanglement of n qubits in quantum computation. For pure states the matrix describing entanglement between two systems has minimum rank for pure states and thus vanishing determinant. Hyper-matrix and hyperdeterminant emerge naturally when one speaks about entanglement between n quantum systems. The vanishing of hyperdeterminant means that the state is not maximally non-pure.

For the called hyper-finite factor defined by second quantized induced spinor fields one has very formally infinite tensor product of 8-D H-spinor space. By induced spinor equation the dimension effectively reduces to four. Similar formal $8 \rightarrow 4$ reduction occurs by General Coordinate Invariance for the n:th variations. Quantum classical correspondence states that many-fermion states have correlates at the level of space-time geometry. The very naive question inspired also by supersymmetry is whether the vanishing of n-particle hyperdeterminant for the fermionic entanglement has as a space-time correlate n:th order criticality. If so, one could say that the non-locality with all its beautiful consequences is forced by quantum classical correspondence!

1.6 Could multilinear Higgs potentials be interesting?

It seems that hyperdeterminant has quite limited applications to finite-dimensional case. The simplest situation corresponds to a potential function $V(x_1, .., x_n)$. In this case one obtains also partial derivatives up to n:th order for single variable and one has genuine non-linearity rather than multilinearity. This spoils the possibility to apply the notion of hyperdeterminant to tell whether critical deformations are possible unless the potential function is multilinear function of its arguments. An interesting idea is that Higgs potential of this form. In this case the extrema allow scalings of the coordinates x_i. In 3-D case 3-linear function of 6 coordinates coming as doublets (x_i, y_i) , $i = 1, 2, 3$ and characterized by a matrix $A_{i_1 i_2 i_3}$, where i_k is two-valued index, would provide an example of this kind of Higgs potential.

References

[1] *Hyperdeterminant*, `http://en.wikipedia.org/wiki/Hyperdeterminant`.

[2] *Duff, string theory, entanglement and hyperdeterminants.* `http://blog.vixra.org/2010/09/02/duff-string-theory-entanglement-and-hyperdeterminants/`.

[3] *M-theory lesson 350.* `http://pseudomonad.blogspot.com/2010/09/m-theory-lesson-350.html`.