Mean value theorems for Local fractional integrals on fractal space

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Abstract: In this paper, by some properties of Local fractional integral,we establish the generalized Mean value theorems for Local Fractional Integral.

Keywords: fractal space, Local fractional integral, local fractional Mean value theorems

MSC2010: 28A80,

1 Introduction

local fractional calculus (also called Fractal calculus) has played an important role in not only mathematics but also in physics and engineers [1-15]. Local fractional integral of $f(x)$ [6-7,9] was written in the form

$$ {}_aI_b^{(\alpha)}f(x) = \frac{1}{\Gamma(1+\alpha)}\int_a^b f(t)(dt)^\alpha = \frac{1}{\Gamma(1+\alpha)}\lim_{\Delta t\to 0}\sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^\alpha, $$

with $\Delta t_j = t_{j+1} - t_j$ and $\Delta t = \max\{\Delta t_1, \Delta t_2, \ldots, \Delta t_j, \ldots\}$, where for $j = 1, 2, \ldots, N-1 t_0 = a$ and $t_N = b$, $[t_j, t_{j+1}]$ is a partition of the interval $[a, b]$.

The purpose of this paper is to establish some Mean value theorems for Local fractional integrals on fractal space. We generalize the results of [1].

2 Preliminaries

In this section, we give some properties of Local fractional integral, that will be used later in this paper.

Theorem 2.1 *[1]Constant function $f(x) = c$ is Local fractional integrable from a to b and*

$$ {}_aI_b^{(\alpha)}f(x) = \frac{c(b-a)^\alpha}{\Gamma(1+\alpha)}. $$

Theorem 2.2 *Local fractional monotone functions on $[a, b]$ are integrable.*

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Theorem 2.3 *[1] Every local fractional continuous function on $[a, b]$ is integrable.*

Theorem 2.4 *If $f(x)$ is a local fractional bounded function that is integrable on $[a, b]$. Then $f(x)$ is integrable on every subinterval $[c, d]$ of $[a, b]$.*

Theorem 2.5 *[1] If $f(x)$ and $g(x)$ are local fractional integrable functions on $[a, b]$ and $c \in \mathrm{R}$. Then*

(1) $cf(x)$ is local fractional integrable and ${}_aI_b^{(\alpha)} cf(x) = c{}_aI_b^{(\alpha)} f(x)$;

(2) $f(x) \pm g(x)$ is local fractional integrable and ${}_aI_b^{(\alpha)}[f(x) \pm g(x)] = {}_aI_b^{(\alpha)} f(x) \pm {}_aI_b^{(\alpha)} g(x)$.

Theorem 2.6 *If $f(x)$ and $g(x)$ are local fractional integrable on $[a, b]$, then so is their product $f(x)g(x)$.*

Theorem 2.7 *[1] Let $f(x)$ be a function defined on $[a, b]$ and $a < c < b$.If $f(x)$ is local fractional integrable from a to c and from c to b, then $f(x)$ is local fractional integrable from a to b and*

$$ {}_aI_b^{(\alpha)} f(x) = {}_aI_c^{(\alpha)} f(x) + {}_cI_b^{(\alpha)} f(x). $$

Theorem 2.8 *[1] If $f(x)$ and $g(x)$ are local fractional integrable on $[a, b]$ and $f(x) \geq g(x)$ for all $x \in [a, b]$, then*

$$ {}_aI_b^{(\alpha)} f(x) \geq {}_aI_b^{(\alpha)} g(x). $$

Theorem 2.9 *[1] Let $f(x)$ be local fractional integrable on $[a, b]$, then so is $|f(x)|$ and*

$$ |{}_aI_b^{(\alpha)} f(x)| \leq {}_aI_b^{(\alpha)} |f(x)|. $$

3 Mean value theorems for Local fractional integrals

Theorem 3.1 *(First Mean Value Theorem). If $f(x)$ and $g(x)$ are local fractional bounded and integrable functions on $[a, b]$, and let $g(x)$ be nonnegative (or nonpositive) on $[a, b]$. Set $m = \inf\{f(x) : x \in [a, b]\}$ and $M = \sup\{f(x) : x \in [a, b]\}$. Then there exists a point ξ in (a, b) such that*

$$ {}_aI_b^{(\alpha)} f(x)g(x) = f(\xi){}_aI_b^{(\alpha)} g(x). \tag{3.1} $$

Proof. We have

$$ m \leq f(x) \leq M, \text{for all} x \in [a, b]. \tag{3.2} $$

Suppose $g(x) \geq 0$. Multiplying (3.2) by $g(x)$ we obtain

$$ mg(x) \leq f(x)g(x) \leq Mg(x) \quad \text{for all} \quad x \in [a, b]. $$

Besides, each of the functions $mg(x)$, $Mg(x)$, and $f(x)g(x)$ is local fractional integrable from a to b by Theorem 2.5 and Theorem 2.6. Hence, we get from these inequalities, by using Theorem 2.8,

$$m_a I_b^{(\alpha)} g(x) \leq {}_a I_b^{(\alpha)} f(x)g(x) \leq M_a I_b^{(\alpha)} g(x). \tag{3.3}$$

If ${}_a I_b^{(\alpha)} g(x) = 0$, it follows from (3.3) that ${}_a I_b^{(\alpha)} f(x)g(x) = 0$, and therefore equality (3.1) is obvious; if ${}_a I_b^{(\alpha)} g(x) > 0$, then (3.3) implies

$$m \leq \frac{{}_a I_b^{(\alpha)} f(x)g(x)}{{}_a I_b^{(\alpha)} g(x)} \leq M.$$

there exists a point ξ in (a, b) such that

$$m \leq f(\xi) \leq M,$$

which obtains the desired result (3.1).

In particular, for $g(x) = 1$, we have from Theorem 3.1 the following result.

Corollary 3.1 *Let $f(x)$ be an local fractional integrable function on $[a, b]$ and let m and M be the infimum and supremum, respectively, of $f(x)$ on $[a, b]$. Then there exists a point ξ in (a, b) such that*

$${}_a I_b^{(\alpha)} f(x) = f(\xi) \frac{(b-a)^\alpha}{\Gamma(1+\alpha)}.$$

Remark. Conditions of Corollary 3.1. is weaker than those of Theorem 2.23 in [1].

In what follows we will make use of the following fact, known as Abel's lemma.

Lemma 3.2 *Let the numbers p_i for $1 \leq i \leq n$ satisfy the inequalities $p_1 \geq p_2 \geq \ldots \geq p_n$ and the numbers $S_k = \sum_{i=1}^{k} q_i$ for $1 \leq k \leq n$ satisfy the inequalities $m \leq S_k \leq M$ for all values of k, where q_i, m, and M are some numbers. Then $mp_1 \leq \sum_{i=1}^{n} p_i q_i \leq M p_1$.*

Theorem 3.3 *(Second Mean Value Theorem I). If $f(x)$ is a local fractional bounded function that is integrable on $[a, b]$. Let further m_F and M_F be the infimum and supremum, respectively, of the function $F(x) = \frac{1}{\Gamma(1+\alpha)} \int_a^x f(t)(dt)^\alpha$ on $[a, b]$. Then:*

(i) If $g(x)$ is nonincreasing with $g(x) \geq 0$ on $[a, b]$, then there is some point ξ in (a, b) such that $m_F \leq f(\xi) \leq M_F$ and

$${}_a I_b^{(\alpha)} f(x)g(x) = g(a)F(\xi). \tag{3.4}$$

(ii) If a function $g(x)$ is any local fractional monotone function on $[a, b]$, then there is some point ξ in (a, b) such that $m_F \leq F(\xi) \leq M_F$ and

$${}_a I_b^{(\alpha)} f(x)g(x) = [g(a) - g(b)]F(\xi) + g(b)_a I_b^{(\alpha)} f(x) \quad . \tag{3.5}$$

Proof. To prove part (i) of the theorem, suppose that $g(x)$ is nonincreasing and that $g(x) \geq 0$ for all $x \in [a, b]$. Consider an arbitrary $\varepsilon > 0$. Since $f(x)$ and $f(x)g(x)$ are integrable on $[a, b]$, we can choose, by definition of Local fractional integrals, a partition $a = x_0 < x_1 < \ldots x_{n-1} < x_n = b$ such that

$$\frac{1}{\Gamma(1+\alpha)} \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1})^{\alpha} < \varepsilon^{\alpha} \quad , \tag{3.6}$$

And

$$\left| \frac{1}{\Gamma(1+\alpha)} \sum_{i=1}^{n} f(x_{i-1}) g(x_{i-1})(x_i - x_{i-1})^{\alpha} - \frac{1}{\Gamma(1+\alpha)} \int_a^b f(x)g(x)(dx)^{\alpha} \right| < \varepsilon^{\alpha}, \tag{3.7}$$

where m_i and M_i are the infimum and supremum, respectively, of $f(x)$ on $[x_{i-1}, x_i)$. Since $g(x_{i-1}) \geq 0$, we get from $m \leq f(x_{i-1}) \leq M$ that

$$\begin{aligned} &\frac{1}{\Gamma(1+\alpha)} \sum_{i=1}^{n} m_i g(x_{i-1})(x_i - x_{i-1})^{\alpha} \\ &\leq \frac{1}{\Gamma(1+\alpha)} \sum_{i=1}^{n} f(x_{i-1}) g(x_{i-1})(x_i - x_{i-1})^{\alpha} \leq \frac{1}{\Gamma(1+\alpha)} \sum_{i=1}^{n} M_i g(x_{i-1})(x_i - x_{i-1})^{\alpha}, \end{aligned} \tag{3.8}$$

holds. Next, by Corollary 3.1, there exist numbers ξ_i for $1 \leq i \leq n$ such that $m_i \leq f(\xi_{i-1}) \leq M_i$ and

$$\frac{1}{\Gamma(1+\alpha)} \int_{x_{i-1}}^{x_i} f(x)(dx)^{\alpha} = f(\xi_i) \frac{(x_i - x_{i-1})^{\alpha}}{\Gamma(1+\alpha)}.$$

Consider the numbers

$$S_k = \sum_{i=1}^{k} f(\xi_i) \frac{(x_i - x_{i-1})^{\alpha}}{\Gamma(1+\alpha)} = \frac{1}{\Gamma(1+\alpha)} \int_a^{x_k} f(x)(dx)^{\alpha}.$$

for $1 \leq k \leq n$. Obviously, $m_F \leq S_k \leq M_F$, where m_F and M_F are the infimum and supremum, respectively, of $F(x)$ on $[a, b]$. Put

$$p_i = g(x_{i-1}) \quad \text{and} \quad q_i = f(\xi_i) \frac{(x_i - x_{i-1})^{\alpha}}{\Gamma(1+\alpha)}.$$

for $1 \leq i \leq n$. Since $g(x)$ is nonincreasing and $g(x) \geq 0$, we have

$$p_1 \geq p_2 \geq \ldots \geq p_n.$$

The numbers p_i ,S_i, and q_i satisfy the conditions of Lemma 3.2. Therefore

$$m_F g(a) \leq \sum_{i=1}^{n} g(x_{i-1}) f(\xi_i) \frac{(x_i - x_{i-1})^{\alpha}}{\Gamma(1+\alpha)} \leq M_F g(a) \quad . \tag{3.9}$$

On the other hand,

$$\sum_{i=1}^{n} m_i g(x_{i-1}) \frac{(x_i - x_{i-1})^{\alpha}}{\Gamma(1+\alpha)} \leq \sum_{i=1}^{n} g(x_{i-1}) f(\xi_i) \frac{(x_i - x_{i-1})^{\alpha}}{\Gamma(1+\alpha)} \leq \sum_{i=1}^{n} M_i g(x_{i-1}) \frac{(x_i - x_{i-1})^{\alpha}}{\Gamma(1+\alpha)}. \tag{3.10}$$

From (3.8) and (3.10) we have, taking into account the monotonicity of $g(x)$and (3.6),

$$\begin{aligned}
&\left|\frac{1}{\Gamma(1+\alpha)}\sum_{i=1}^{n} g(x_{i-1})[f(x_{i-1})-f(\xi_i)](x_i-x_{i-1})^\alpha\right| \\
&\le \frac{1}{\Gamma(1+\alpha)}\sum_{i=1}^{n}(M_i-m_i)g(x_{i-1})(x_i-x_{i-1})^\alpha \\
&\le \frac{g(a)}{\Gamma(1+\alpha)}\sum_{i=1}^{n}(M_i-m_i)(x_i-x_{i-1})^\alpha \le g(a)\varepsilon
\end{aligned} \tag{3.11}$$

From this and (3.7) it follows that

$$\left|\frac{1}{\Gamma(1+\alpha)}\int_a^b f(x)g(x)(dx)^\alpha - \frac{1}{\Gamma(1+\alpha)}\sum_{i=1}^{n} g(x_{i-1})f(\xi_i)(x_i-x_{i-1})^\alpha\right| < \varepsilon^\alpha + g(a)\varepsilon^\alpha.$$

Hence, using (3.9), we obtain

$$-\varepsilon^\alpha - g(a)\varepsilon^\alpha + m_F g(a) < \frac{1}{\Gamma(1+\alpha)}\int_a^b f(x)g(x)(dx)^\alpha < \varepsilon^\alpha + g(a)\varepsilon^\alpha + M_F g(a).$$

Since $\varepsilon > 0$ is arbitrary, we get

$$m_F g(a) \le \frac{1}{\Gamma(1+\alpha)}\int_a^b f(x)g(x)(dx)^\alpha \le M_F g(a). \tag{3.12}$$

If $g(a) = 0$, it follows from (3.11) that $\int_a^b f(x)g(x)(dx)^\alpha = 0$, and therefore equality (3.4) becomes obvious; if $g(a) > 0$, then (3.11) implies

$$m_F \le \frac{{}_aI_b^{(\alpha)} f(x)g(x)}{g(a)} \le M_F.$$

there exists a point ξ in (a, b) such that

$$m_F \le F(\xi) = \frac{{}_aI_b^{(\alpha)} f(x)g(x)}{g(a)} \le M_F.$$

which yields the desired result (3.4).

Let now $g(x)$ be an arbitrary nonincreasing function on $[a, b]$. Then the function h defined by $h(t) = g(t) - g(b)$ is nonincreasing and $h(t) \ge 0$ on $[a, b]$. therefore, applying formula (3.4) to the function $h(t)$, we can write

$$\begin{aligned}
&{}_aI_b^{(\alpha)} f(x)[g(x)-g(b)] \\
&= \frac{1}{\Gamma(1+\alpha)}\int_a^b f(x)[g(x)-g(b)](dx)^\alpha = [g(a)-g(b)]F(\xi)
\end{aligned}.$$

which obtains the formula (3.5) of part (ii) for nonincreasing functions $g(x)$. If $g(x)$ is nondecreasing, then the function $g_1(x) = -g(x)$ is nonincreasing, and applying the obtained result to $g_1(x)$, we have the same result for nondecreasing functions $g(x)$ as well. Thus, part (ii) is proved for all monotone functions $g(x)$.

The following theorem can be proved in a similar way as Theorem 3.3.

Theorem 3.4 *(Second Mean Value Theorem II). If $f(x)$ be a local fractional bounded function that is integrable on $[a,b]$. Let further m_G and M_G be the infimum and supremum, respectively,of the function $G(x) = \frac{1}{\Gamma(1+\alpha)}\int_x^b f(t)(dt)^\alpha$ on $[a,b]$. Then:*

(i) If a function $g(x)$ is nonincreasing with $g(x) \geq 0$ on $[a,b]$, then there is some point ξ in (a,b) such that $m_G \leq G(\xi) \leq M_G$ and

$$_aI_b^{(\alpha)}f(x)g(x) = g(b)G(\xi).$$

(ii) If $g(x)$ is any local fractional monotone function on $[a,b]$, then there is some point ξ in (a,b) such that $m_G \leq G(\xi) \leq M_G$ and

$$_aI_b^{(\alpha)}f(x)g(x) = [g(b)-g(a)]G(\xi) + g(a)_aI_b^{(\alpha)}f(x).$$

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