Abstract:

Two points on Poincaré disk and exterior disk which have the same equidistant curve coordinates have the relation of the inversion on a circle which divides both regions. An isometry is realized between exterior disk and lower half-plane.

1. Exterior disk

1.1. Another

An interior point of Poincaré disk is expressed as an intersection of two equidistant curves, namely two circles. An intersection of them is outside the disk too like Figure 1. The coordinates of the two intersections are, modifying a result of the reference [1],

\begin{align*}
x &= \frac{f(1, 2)x_a + f(0, 1)x_a}{2f(0, 2)x_a^2 + f(2, 0)y_b^2} \\
y &= \frac{f(2, 1)y_b + f(1, 0)y_b}{2f(0, 2)x_a^2 + f(2, 0)y_b^2}
\end{align*}

The inside and the outside of the disk adopt the minuses and the pluses respectively in the double signs of the expressions. We try to extend equidistant curve coordinates to the outside of it by use of an intersection outside it.

Assuming \( z_1 = x_a' \), \( z_2 = x_a \), \( z_3 = R \), \( z_4 = -R \)

\( \frac{z_1 - z_3}{z_2 - z_3} \frac{z_2 - z_4}{z_1 - z_4} = \frac{x_a' - R}{x_a - R} \frac{x_a + R}{x_a' + R} \)

Here, setting \( x_a' \to \infty \)

\( \frac{z_1 - z_3}{z_2 - z_3} \frac{z_2 - z_4}{z_1 - z_4} = \frac{z_2 - z_4}{z_1 - z_4} = \frac{x_a + R}{x_a - R} \)

is obtained. The equidistant curve coordinate by use of this cross ratio is

\( \frac{x^*}{x_a} = k \log \frac{x_a + R}{x_a - R} \)

From this
\[ x_a = R \coth \{x^*/(2k)\} \]

The value \( x^* \) is 0 at infinity and \( \infty \) at the sphere point \( R \). Similarly

\[ y^* = k \log \frac{y_b + R}{y_b - R} \]

\[ y_b = R \coth \{y^*/(2k)\} \]

Substituting \( x_a, y_b \) into \( x, y \) of which double signs are the pluses

\[ x = R \frac{\sinh \frac{x^*}{k}}{\sqrt{1 + \sinh^2 \frac{x^*}{k} + \sinh^2 \frac{y^*}{k} - 1}}, \quad y = R \frac{\sinh \frac{y^*}{k}}{\sqrt{1 + \sinh^2 \frac{x^*}{k} + \sinh^2 \frac{y^*}{k} - 1}} \]

We express a Poincaré metric in equidistant curve coordinates.

\[
\frac{dx^2 + dy^2}{(x^2 + y^2 - R^2)^2} = \frac{k^2 \left( \sqrt{1 + \sinh^2 \frac{x^*}{k} + \sinh^2 \frac{y^*}{k} - 1} \right)^2}{R^2} \]

Therefore

\[ ds^2 = \frac{4k^2 R^2 (dx^2 + dy^2)}{(x^2 + y^2 - R^2)^2} = \frac{\cosh \frac{x^*}{k} \cosh \frac{y^*}{k} (dx^2 + dy^2) - 2 \cosh \frac{x^*}{k} \cosh \frac{y^*}{k} \sinh \frac{x^*}{k} \sinh \frac{y^*}{k} dx^* dy^*}{1 + \sinh^2 \frac{x^*}{k} + \sinh^2 \frac{y^*}{k}} \]

By the following replacements

\[
\frac{\cosh \frac{x^*}{k} \cosh \frac{y^*}{k}}{\sqrt{1 + \sinh^2 \frac{x^*}{k} + \sinh^2 \frac{y^*}{k}}} dx^* \equiv ds_1, \quad \frac{\cosh \frac{x^*}{k} \cosh \frac{y^*}{k}}{\sqrt{1 + \sinh^2 \frac{x^*}{k} + \sinh^2 \frac{y^*}{k}}} dy^* \equiv ds_2
\]
\[ ds_1^2 + ds_2^2 - 2ds_1ds_2 \tanh \frac{x^*}{k} \tanh \frac{y^*}{k} \]

**Figure 1**

1.2. Inversion

Points of the inside and the outside of the disk are expressed in the following expressions by use of equidistant curve coordinates.

\[
x = R \frac{\sinh \frac{x^*}{k}}{\sqrt{1 + \sinh^2 \frac{x^*}{k} + \sinh^2 \frac{y^*}{k} \pm 1}}, \quad y = R \frac{\sinh \frac{y^*}{k}}{\sqrt{1 + \sinh^2 \frac{x^*}{k} + \sinh^2 \frac{y^*}{k} \pm 1}}
\]

The inside and the outside of the disk adopt the pluses and the minuses respectively in the double signs of the expressions. We consider two points \( P, P' \) of the inside and the outside of it of which \( x^*, y^* \) are the same. Because their ratios of \( x : y \) are the same, they are on a line which passes through origin. Multiplying two squares of a distance from origin

\[
OP^2 \times OP'^2 = (x^2 + y^2)(x'^2 + y'^2)
= \frac{R^2(\sinh^2 \frac{x^*}{k} + \sinh^2 \frac{y^*}{k})}{(\sqrt{1 + \sinh^2 \frac{x^*}{k} + \sinh^2 \frac{y^*}{k} + 1})^2}
= \frac{R^2(\sinh^2 \frac{x^*}{k} + \sinh^2 \frac{y^*}{k})}{(\sqrt{1 + \sinh^2 \frac{x^*}{k} + \sinh^2 \frac{y^*}{k} - 1})^2}
= R^4
\]

That is to say, the two points are in the relation of the inversion on a circle of which radius is \( R \) and which separates the inside and the outside. The outside is called exterior disk in the sense of copy of the disk because of the relation.

2. Lower half plane

A point \( w = x + iy \) of the inside of the disk is mapped to a point \( z = u + iv \) of upper half plane by the following linear transformation.
Because a point of the outside of it has the value $R^2 - x^2 - y^2 < 0$, it is mapped to a point of lower half plane. We adopt the complex conjugate of $z$ if we hope that it is mapped to upper half plane.

$$z = \frac{iR - w}{R + w} = \frac{2R^2 y + iR(R^2 - x^2 - y^2)}{(x + R)^2 + y^2}$$

$$\bar{z} = \frac{2R^2 y - iR(R^2 - x^2 - y^2)}{(x + R)^2 + y^2}$$

References:
[2] Morio Kikuchi, "On coordinate systems by use of spheres (the 1st, . . . , the 4th)"
(2009)(in Japanese)
\[ z_1 = x'_a, \ z_2 = x_a, \ z_3 = R, \ z_4 = -R \quad \text{として} \]

\[
\frac{z_1 - z_3 z_2 - z_4}{z_2 - z_3 z_1 - z_4} = \frac{x'_a - R x_a + R}{x_a - R x'_a + R}
\]

ここで、\( x'_a \to \infty \) すれば

\[
\frac{z_1 - z_3 z_2 - z_4}{z_2 - z_3 z_1 - z_4} = \frac{x_a + R}{x_a - R}
\]

となります。この複比を用いた等距離線座標は

\[ x^* = k \log \frac{x_a + R}{x_a - R} \]

となり、これを \( x_a \) について求めれば

\[ x_a = R \coth \{ x^*/(2k) \} \]

となります。\( x^* \) は無限遠方で 0、球面 \( R \) で無限大となります。同様に

\[ y^* = k \log \frac{y_b + R}{y_b - R} \]

\[ y_b = R \coth \{ y^*/(2k) \} \]

\( x_a, y_b \) を複号を + とした \( x, y \) に代入して

\[
x = R \frac{\sinh \frac{x^*}{k}}{\sqrt{1 + \sinh^2 \frac{x^*}{k} + \sinh^2 \frac{y^*}{k} - 1}}, \quad y = R \frac{\sinh \frac{y^*}{k}}{\sqrt{1 + \sinh^2 \frac{x^*}{k} + \sinh^2 \frac{y^*}{k} - 1}}
\]

ポアンカレ計量を等距離線座標で表します。

\[
dx^2 + dy^2 = \left( \frac{\partial x}{\partial x^*} dx^* + \frac{\partial x}{\partial y^*} dy^* \right)^2 + \left( \frac{\partial y}{\partial x^*} dx^* + \frac{\partial y}{\partial y^*} dy^* \right)^2
\]

\[
= \frac{R^2}{k^2 \left( \sqrt{1 + \sinh^2 \frac{x^*}{k} + \sinh^2 \frac{y^*}{k} - 1} \right)^2}
\]

\[
\times \cosh^2 \frac{x^*}{k} \cosh^2 \frac{y^*}{k} (dx^2 + dy^2) - 2 \cosh \frac{x^*}{k} \cosh \frac{y^*}{k} \sinh \frac{x^*}{k} \sinh \frac{y^*}{k} dx^* dy^*
\]

\[
\frac{1 + \sinh^2 \frac{x^*}{k} + \sinh^2 \frac{y^*}{k}}{1 + \sinh^2 \frac{x^*}{k} + \sinh^2 \frac{y^*}{k}}
\]
したがって

\[
ds^2 = \frac{4k^2R^2(dx^2 + dy^2)}{(x^2 + y^2 - R^2)^2} = \frac{\cosh^2 \frac{x}{k} \cosh^2 \frac{y}{k} (dx^* + dy^2) - 2 \cosh \frac{x}{k} \cosh \frac{y}{k} \sinh \frac{x}{k} \sinh \frac{y}{k} dx^* dy^*}{1 + \sinh^2 \frac{x}{k} + \sinh^2 \frac{y}{k}}
\]

上式において

\[
\cosh \frac{x}{k} \cosh \frac{y}{k} dx^* \equiv ds_1, \quad \cosh \frac{x}{k} \cosh \frac{y}{k} dy^* \equiv ds_2
\]

として

\[
= ds_1^2 + ds_2^2 - 2ds_1 ds_2 \tanh \frac{x}{k} \tanh \frac{y}{k}
\]

図 1

1.2. 反転
円板の内外の点は等距離線座標を用いて

\[
x = R \frac{\sinh \frac{x}{k}}{\sqrt{1 + \sinh^2 \frac{x}{k} + \sinh^2 \frac{y}{k} \pm 1}}, \quad y = R \frac{\sinh \frac{y}{k}}{\sqrt{1 + \sinh^2 \frac{x}{k} + \sinh^2 \frac{y}{k} \pm 1}}
\]

のように表されます。複号は、円板の内が +、円板の外が - です。\(x^*, y^*\) が同じである内外の2点 \(P, P'\) について考察します。\(x: y\) が同じなので2点は直角座標の原点を通る直線上にあります。直角座標の原点からの距離の二乗を乗じてみると
\[ \mathbf{OP}^2 \times \mathbf{OP}'^2 = (x^2 + y^2)(x'^2 + y'^2) \]
\[ = \frac{R^2(\sinh^2 \frac{x^2}{k} + \sinh^2 \frac{y^2}{k})}{(\sqrt{1 + \sinh^2 \frac{x^2}{k} + \sinh^2 \frac{y^2}{k}} + 1)^2} \frac{R^2(\sinh^2 \frac{x'^2}{k} + \sinh^2 \frac{y'^2}{k})}{(\sqrt{1 + \sinh^2 \frac{x'^2}{k} + \sinh^2 \frac{y'^2}{k}} - 1)^2} \]
\[ = R^4 \]

すなわち、2点は内外を分ける半径 \( R \) の円に関して反転の関係にあります。このような関係があるので、円板の外を円板のコピーという意味で外円板と称します。

2. 下半平面
円板内の点 \( w = x + iy \) は一次変換

\[ z = iR \frac{R - w}{R + w} = \frac{2R^2y + iR(R^2 - x^2 - y^2)}{(x + R)^2 + y^2} \]

によって上半平面上の点 \( z = u + iv \) に写されます。円板外の点は \( R^2 - x^2 - y^2 < 0 \) となるので下半平面上の点に移されます。円板外の点を上半平面上の点に写すには複素共役を採用します。

\[ \bar{z} = \frac{2R^2y - iR(R^2 - x^2 - y^2)}{(x + R)^2 + y^2} \]

参考文献: