The Exact Solution of The Pioneer Anomaly According to The General Theory of Relativity and The Hubble's Law

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Abstract

Radio metric data from Pioneer 10/11 indicate an apparent anomalous, constant, acceleration acting on the spacecraft with a magnitude $\sim 8 \times 10^{-10}\, m/s^2$, directed towards the Sun[1,2]. Turyshev [7] examined the constancy and direction of the Pioneer anomaly, and concluded that the data a temporally decaying anomalous acceleration $-2 \times 10^{-11}\, \frac{m}{s^2.yr}$ with an over 10% improvement in the residuals compared to a constant acceleration model. Anderson, who is retired from NASA's Jet Propulsion Laboratory (JPL), is that study's first author. He finds, so "it's either new physics or old physics we haven't discovered yet." New physics could be a variation on Newton's laws, whereas an example of as-yet-to-be-discovered old physics would be a cloud of dark matter trapped around the sun[12].

In this paper I introduce the exact solution for the Pioneer anomaly depending on the general theory of relativity and the Hubble's law. According to my solution, there are two terms of decelerations that controls the Pioneer anomaly. The first is produced by moving the Pioneer spacecraft through the gravitational field of the Sun, which causes the velocity of the spacecraft to be decreased according to the Schwarzschild Geometry of freely infalling particle. This deceleration is responsible for varying behaviour of the Pioneer anomaly in Turyshev [7]. The second term is produced by the Hubble's law which is constant and equals to the Hubble's constant multiplied by the speed of light in vacuum.

Theory

In the Schwarzschild geometry a radially infalling particle falling from infinity with vanishing initial velocity moves on a path described by[5,6]

$$\left(1-\frac{2m}{r}\right)\ \frac{dt}{d\tau}=1\ \ and\ \ \left(\frac{dr}{d\tau}\right)^2=\frac{2m}{r} \tag{1}$$

C (speed of light) is taken to be 1 in the equations above, and $m=\dfrac{GM}{C^2}$. Thus, from these equations, we finally find the measured (observed) speed of the particle falling freely in the gravitational field for the reference frame of the earth observer is V' given as

$$V'=\frac{dr/d\tau}{dt/d\tau}=\frac{dr}{dt}=\sqrt{\frac{2GM}{r}}\ \left(1-\frac{2GM}{C^2r}\right) \tag{2}$$

Thus we can express the equation above as

$$V'=V\left(1-\frac{2GM}{C^2r}\right) \tag{3}$$

V is the proper speed of the object which is equal to $\dfrac{dr}{d\tau}$ (the derivative of distance with respect to the proper time τ , which is according to eq. (1), $\dfrac{dr}{d\tau}=\sqrt{\dfrac{2GM}{r}}$).

For the reference frame of the earth observer, and according to its time, the observed speed of the freely falling object under the gravitational field is $V'=\dfrac{dr}{dt}$ given according to eq. (3).

We get from eq. (3) that the observed speed of the freely falling object will be decreased for the reference frame of the earth observer compared to the proper speed, and thus, the electromagnetic wave that is transmitted from the object will produce a slight red shift on top of the larger blue shift for the earth observer if the direction of the velocity is toward the earth, and the earth observer will think there is some force that is pulling the object back ward. If the direction of the velocity is in the opposite direction to the earth, in this case there must produce a slight blue shift on top of the lager red shift, and then the earth observer will think that there is some force that is pulling the object back ward, similar as what happened in the case of Pioneer 10/11

The Pioneer Anomaly

Now if a particle is located at a distance r from a big mass M. If we give this particle an energy to move with escape velocity $\frac{dr}{d\tau} = V = \sqrt{\frac{2GM}{r}}$. That means according to the Schwarzschild Geometry of freely infalling particle, we reverse the motion, and if we try to measure the observed velocity of this particle at any distance r from the mass M, we shall find it equals as in eq. (3). Thus, if we consider the Pioneer velocity as in eq. (3), then we get

$$V' = \left(1 - \frac{2GM}{C^2 r}\right) V$$

V' is the observed Pioneer velocity for the reference frame of the earth observer which is equal to $\frac{dr}{dt}$ (derivative of distance with respect to earth time), V is the proper Pioneer velocity which is equal to $\frac{dr}{d\tau}$ (derivative of distance with respect to proper time), G is the gravitational constant, M is the Sun mass, C is the speed of light in vacuum, and r is the distance between the spacecraft and the Sun.

Thus from eq. (3), in the case of Pioneer 10/11, and since they are going away from the Sun, we can conclude that their observed velocities should be less than their proper velocities for the earth observer. Therefore that must produce a slight blue shift on top of the larger red shift for the earth observer [1].

The JPL analysis of unmodeled accelerations used the JPL's Orbit Determination Program (ODP) [1,2]. Over the years the data continually indicated that the largest systematic error in the acceleration residuals is a constant bias of $a_p \approx (8 \pm 3) \times 10^{-10}\, m/s^2$ directed toward the Sun to within the beam-width of the Pioneers' antennae [1].

According to the special relativity theory, the equation that describes the clock motion of the Pioneer spacecraft comparing with a clock motion on the earth for an observer on the earth is given according to the equation

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - \frac{V^2}{C^2}}} \qquad (4)$$

$\Delta t'$ is the reading of Pioneer's clock for the earth observer

Δt is the reading of the earth clock for the earth observer

We get from eq. (4) that the Pioneer's clock motion will be slower than the earth clock motion for the earth observer. That is referring to the time dilation in relativity. But since the observed velocity of the Pioneer spacecraft is less than the proper velocity for the earth observer according to eq. (3), where

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - \frac{\left(1 - \frac{2GM}{C^2 r}\right)^2 V^2}{C^2}}} \tag{5}$$

The slowing rate of the clock of the Pioneer spacecraft according to eq. (5) is less than according to eq. (4) for the earth observer. Thus, the earth observer will think that there is some force that is pulling the spacecraft back toward the Sun, that is because of the effect of the gravitational field of the Sun. Also according to eq. (5) there is a slight blue shift on top of the larger red shift.

Now, according to eq. (4), for low velocities comparing to the speed of light, the difference between the predicted frequency and the reference frequency ν_0 as the result of the red shift is $\Delta \nu_{\mathrm{mod}el}$ given as

$$\frac{\Delta \nu_{\mathrm{mod}\,el}}{\nu_0} = \frac{V}{C} \tag{6}$$

But the observed frequency difference $\Delta \nu_{obs}$ that is given according to eq. (5)

$$\frac{\Delta \nu_{obs}}{\nu_0} = \frac{\left(1 - \frac{2GM}{C^2 r}\right) V}{C} \tag{7}$$

Thus from eqs. (6) and (7) we get

$$\left[\Delta \nu_{obs} - \Delta \nu_{\mathrm{mod}\,el}\right] / \nu_0 = -\left(\frac{2GM}{C^2 r}\right) \frac{V}{C} \tag{8}$$

From eq. (8) we get the observed difference frequency is less than the predicted. That means there is a slight blue shift. According to the Pioneer team calculations, the observed, two-way anomalous effect by a DSN antenna can be expressed to first order in V/C as [1]

$$\left[\Delta \nu_{obs} - \Delta \nu_{\mathrm{mod}el}\right]_{DSN} / \nu_0 = -\frac{2\ a'_p\, t}{C} \tag{9}$$

By DSN convention [1], $\left[\Delta \nu_{obs} - \Delta \nu_{\mathrm{mod}el}\right]_{usual} = -\left[\Delta \nu_{obs} - \Delta \nu_{\mathrm{mod}el}\right]_{DSN}$

Thus from that and from eq. (8) we get

$$-\left(\frac{2GM}{C^2 r}\right)\frac{V}{C} = \frac{2a'_p\, t}{C}$$

Thus the spacecraft acceleration a'_p, where $t = \frac{r}{C}$

$$a'_p = -\frac{GM}{r^2}\frac{V}{C} \tag{10}$$

By substituting in eq. (10) $V = \frac{dr}{d\tau} = \sqrt{\frac{2GM}{r}}$ is the proper escape velocity of the spacecraft at distance r from the Sun, we get

$$a'_p = -\left(\left(\frac{GM}{r}\right)^{3/2}\frac{2^{1/2}}{r\ C}\right) \tag{11}$$

The distance r between the spacecraft and the Sun for the earth observer is give as

$$\vec{r}_{sp} = \vec{r}_{se} + \vec{r}_{ep}$$

$\vec{r}_{sp}$ is the distance between the spacecraft and Sun for the earth observer.

$\vec{r}_{se}$ is the distance between the Sun and the earth for the earth observer.

$\vec{r}_{ep}$ is the distance between the spacecraft and the earth for the earth observer.

And since $\vec{r}_{se}$ is small compared to $\vec{r}_{ep}$, thus we can consider

$$\vec{r}_{sp} \approx \vec{r}_{ep}$$

Now by considering, $G = 6.67 \times 10^{-11} m^3 / kg.s^2$, $M = 1.99 \times 10^{30} kg$ are respectively gravitational constant and mass of the Sun. NASA data [3] show that in the very middle part (1983-1990) of the whole observation period of Pioneer 10 its radial distance from the Sun changes from $r \cong 28.8\ AU = 4.31 \times 10^{12}\ m$ to $r \cong 48.1\ AU = 7.2 \times 10^{12}\ m$. Thus by computing a'_{p10} according to eq. (11), we get $a'_{p10} = -1.87 \times 10^{-10} m/s^2$ and $a'_{p10} = -0.52 \times 10^{-10} m/s^2$.

Analogous computations for Pioneer 11, as checking point, show the following. Full time of observation of Pioneer 11 is shorter so observational period is taken from 1984 to 1989, with observational data from the same source [3]. Radial distances for beginning and end of the period are $r \cong 15.1\ AU = 2.26 \times 10^{12}\ m, and\ r = 25.2\ AU = 3.77\ \times 10^{12}\ m$. By using eq. (11) to compute a'_{p11}, we get, $a'_{p11} = -9.5 \times 10^{-10} m/s^2$ and $a'_{p11} = -2.62 \times 10^{-10}\ m/s^2$.

Hubble's Law

We can derive the value of the Hubble's constant according to our previous discussion as;

Suppose the universe as a sphere of radius r with constant mass density ρ anywhere. The measured force per unit mass that is exerted on an object falling freely from the edge of the universe for the earth observer is given as [7]

$$f = \frac{dV'}{dr}V' = \left(\frac{d}{dr}\left(\frac{dr}{dt}\right)\right)\frac{dr}{dt} \tag{12}$$

Thus from eq. (3) we get

$$f = -\frac{GM}{r^2} + \frac{8G^2M^2}{C^2r^3} - \frac{12G^3M^3}{C^4r^4} \tag{13}$$

In the case of a sphere of radius r and constant mass density ρ, we get $M = \frac{4}{3}\pi\ r^3\rho$. Thus by substituting the value of M in eq. (13), we get

$$f = -\frac{4\pi G\rho r}{3} + \frac{128\pi^2G^2\rho^2r^3}{9C^2} - \frac{768\pi^3G^3\rho^3r^5}{27C^4} \tag{14}$$

Equation (14) represents the measured force per unit mass (acceleration) that is exerted on an object falling freely under the effect of gravity for the reference frame located inside the sphere (for example the earth). Thus for an observer located on the earth surface, observing an object falling freely from the edge of the universe, there are three terms control the observed acceleration of this object relative to this observer. The first term which is the Newton's force acceleration and the third term which has the same sign of Newton's acceleration, which is added to it. The second term is always to the opposite direction of the velocity which makes the object to decelerate. Thus if an object falling freely toward the center of the sphere, in this case the earth observer will observe that there is some force that is pulling the object backward, and this must produce a slight red shift on top of the larger blue shift. Now we can solve eq. (14) for f=0 to find the radius at which the repulsive force which is pulling the object upward and it is causing the red shift, to be equal to the gravitational force which is pulling the object downward toward the center of the sphere for the reference frame of the earth observer.

We call to r at f=0 as r_H (Hubble distance). Thus from eq. (14) we have

$$r_H = \pm\sqrt{\frac{3}{8\pi G\rho}}\ C$$
$$or$$
$$r_H = \pm\sqrt{\frac{1}{8\pi G\rho}}\ C \qquad (15)$$
$$or$$
$$r_H = 0$$

Thus from eq.(15) we get the Hubble's constant H, where $H = C / r_H$, thus

$$H = \sqrt{\frac{8\pi G\rho}{3}} \ \ or \ \ H = \sqrt{8\pi G\rho} \qquad (16)$$

From the previous discussion we found that, since Hubble took his measurements from the stars and galaxies that are very far from us, and since these stars and galaxies are affected by the gravitational force of the universe which defined by the mass density of the universe. Furthermore, as we have seen how the gravitational field of the Sun affected on the speed of the Pioneer spacecraft to be decreased. Thus the gravitational field which produced by the mass density of the universe must affect on the velocity of any object to be decreased. The amount of the decreased velocity of any object that is falling freely under the effect of the mass density of the universe which is called the recessional velocity is given according the Hubble's relation

$$v = Hr \qquad (17)$$

v is the recessional velocity, typically expressed in km/s.

r is the proper distance between the object and the observer of the earth.

By dividing eq. (18) by t=r/C, that will lead us to the object accelerates with constant acceleration equals to

$$a_H = HC \qquad (18)$$

and the direction of this acceleration is upward since the direction of the gravitational force is downward. Now, if we reverse the motion, instead of an object falling toward us, we propose an object moving with velocity V going far away from us, in this case, a constant acceleration HC will affect on this object in the opposite direction of the velocity, and the earth observer will think there is a mysterious force that is pulling the object backward, same as what happened in

Pioneer 10/11, where scientists observed at first the deceleration of Pioneer 10/11 was very close to HC. We get from eqs.(16) and (17), the recessional velocity at $r = r_H$ is equal to C.

From that we conclude, the recessional velocity is produced by decreasing the velocity of any object under the effect of the gravitational field, same as the light speed will decrease when passing through a medium of refractive index greater than 1.

Now let's back to the Pioneer anomaly. We found from the previous discussions there are two terms of decelerations that affected on the Pioneer 10/11 anomaly. Since Pioneer 10/11 are going far away from the earth, then Pioneer 10/11 deceleration a_p is given as

$$a_p = -HC - \left(\left(\frac{GM}{r} \right)^{3/2} \frac{2^{1/2}}{r\ C} \right) \tag{19}$$

A recent 2011 estimate of the Hubble constant, which used a new infrared camera on the Hubble Space Telescope (HST) to measure the distance and redshift for a collection of astronomical objects, gives a value of *H*= 73.8 ± 2.4 (km/s)/Mpc or about $2.40 \times 10^{-10} m/s^2$ [10,11].

Thus, from eq. (17) we get

$$a_H = 7.20 \times 10^{-10} m/s^2$$

Thus by adding a_H to the measured Pioneer deceleration according to eq. (11) we get, for Pioneer 10 at distance $r = 28.8\ AU$ or after 11 years of lunch;
$a_{p10} = a_D + a'_{p10} = -7.20 \times 10^{-10} - 1.87 \times 10^{-10} = -9.07 \times 10^{-10} m/s^2$

This quantity is very agreed with the observed Pioneer 10 acceleration (at t=11 years of lunch), in fig. (1) taken from Turyshev [7].

And at distance $r = 48.1\ AU$ at t=18 years of lunch, we get
$a_{p10} = -7.20 \times 10^{-10} - 0.52 \times 10^{-10} = -7.72 \times 10^{-10} m/s^2$. This quantity is agreed with the observed Pioneer 10 acceleration (at t=18 years of lunch), in fig. (1) taken from Turyshev [7].

We see from that the Pioneer deceleration is decreased by increasing distance or time. We can compute $\dot{a}$ for Pioneer 10 as

$$\dot{a} = \frac{\Delta a}{\Delta t} = \frac{7.72 \times 10^{-10} - 9.07 \times 10^{-10}}{7.5\ years} = -1.80 \times 10^{-11}\ \text{m}/s^2/\text{yr} \tag{20}$$

7.5 years is the period of observation from 1983-1990, and as noted by Anderson [13]. We see from eq. (20) that the predicted $\dot{a}$ is agreed with the observed deceleration decaying in Turyshev[7] . Markwardt [8] obtained an improved fit of Pioneer 10 data when estimating a jerk of $\dot{a} = -1.8 \times 10^{-11} m/s^2$ /yr. Also Toth [9] obtained $\dot{a} = -2.1 \times 10^{-11} m/s^2$/yr which is agreed as in eq. (20).

For Pioneer 11, we see in Turyshev[7] and Anderson[1], the observed deceleration for Pioneer 11 at first was greater than the observed deceleration for Pioneer 10. That is agreed with our calculations according to eq. (11), where the Pioneer 11 was much closer to the Sun than Pioneer 10. At the time that Pioneer 11 approaches to the distances equal to the distances of Pioneer 10 from the Sun. both of data are very close to each other.

Fig. (2) illustrates the predicted Pioneer 10 anomaly according to my solution versus r -the distance from the Sun as in eq. (19).

Conclusion:

There are two terms of decelerations that controls the Pioneer anomaly. The first is produced by moving the Pioneer spacecraft through the gravitational field of the Sun, which causes the velocity of the spacecraft to be decreased according to the Schwarzschild Geometry of freely infalling particle. This deceleration is responsible for varying behaviour of the Pioneer anomaly in Turyshev [7]. The second term is produced by the Hubble law, which is constant and equals to the Hubble's constant multiplied by the speed of light in vacuum.

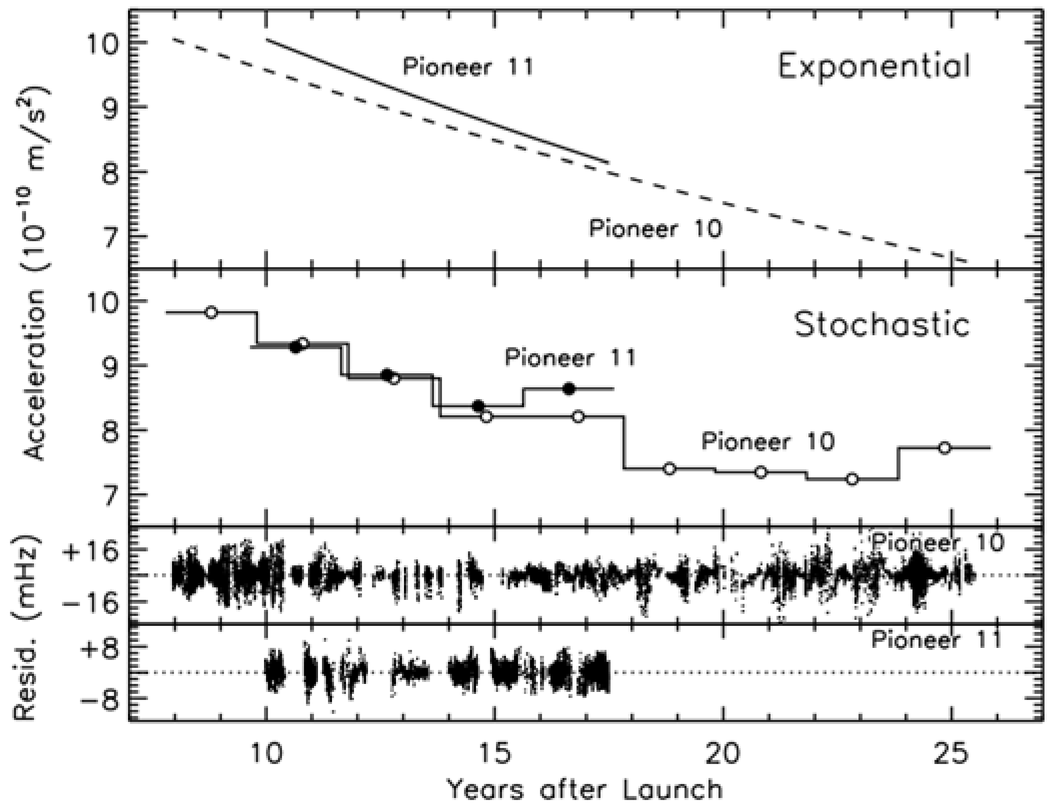

FIG. 1: Top panel: Estimates of the anomalous acceleration of Pioneer 10 (dashed line) and Pioneer 11 (solid line) using an exponential model. Second panel: Stochastic acceleration estimates for Pioneer 10 (open circles) and Pioneer 11 (filled circles), shown as step functions. Bottom two panels: Doppler residuals of the stochastic acceleration model. Note the difference in vertical scale for Pioneer 10 vs. Pioneer 11. Turyshev [7].

$a_p (\times 10^{-10} m/s^2)$, $H = 73.8 \pm 2.4$ (km/s)/Mpc, $a_p = HC + \left(\left(\frac{GM}{r} \right)^{3/2} \frac{2^{1/2}}{r\ C} \right)$

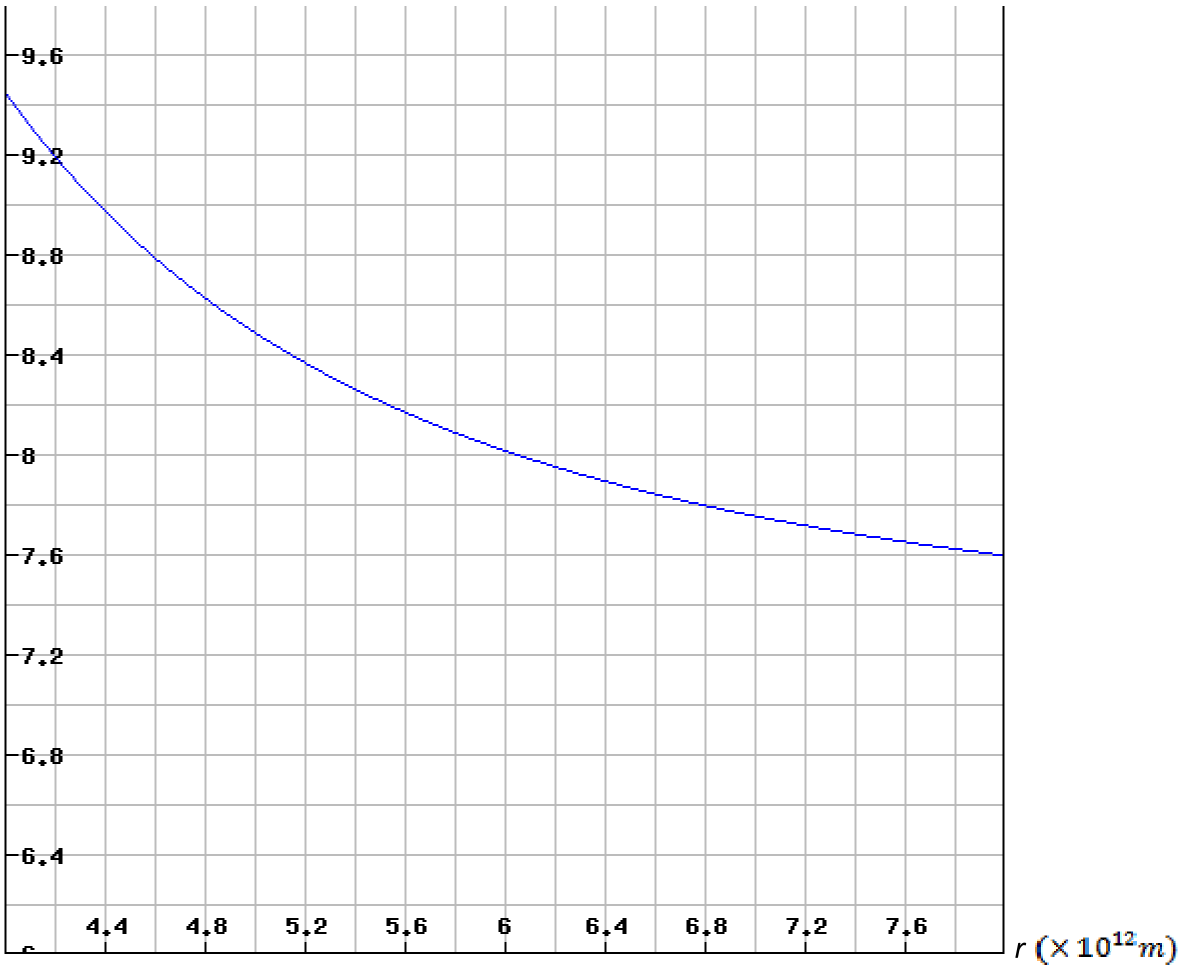

Fig. (2), the predicted Pioneer 10 anomaly versus distance from the Sun according to my solution.

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