The structure of Fourier series

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Fourier series is constructed basing on the idea to model the elementary oscillation \((-1, +1)\) by the exponential function with negative base, viz. \((-1)^n\).

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We inspect the dependence, say, on time of a bounded quantity \(f\) expanding it into the sum of periodic processes \(w_k\).

First of all, consider a discrete process \(f_j\), rendered by the vector comprised of \(n\) values, \(f = (f_0, ..., f_j, ..., f_k, ..., f_{n-1})\), and, accordingly, expand it into the sum of vectors \(w_k = (w_{0k}, ..., w_{jk}, ..., w_{kk}, ..., w_{n-1,k})\):

\[
f = \sum_{k=0}^{n-1} c_k w_k = c_0 \begin{pmatrix} w_{00} \\ \vdots \\ w_{0j} \\ \vdots \\ w_{0k} \\ \vdots \\ w_{0n-10} \end{pmatrix} + \cdots + c_j \begin{pmatrix} w_{j0} \\ \vdots \\ w_{jj} \\ \vdots \\ w_{jk} \\ \vdots \\ w_{jn-1j} \end{pmatrix} + \cdots + c_k \begin{pmatrix} w_{k0} \\ \vdots \\ w_{kk} \\ \vdots \\ w_{kn} \\ \vdots \\ w_{kn-1k} \end{pmatrix} + \cdots + c_{n-1} \begin{pmatrix} w_{n-10} \\ \vdots \\ w_{n-1j} \\ \vdots \\ w_{n-1k} \\ \vdots \\ w_{n-1n-1} \end{pmatrix}.
\] (1)

We seek for the coefficients \(c_k\) of the expansion (1).

The simplest discrete periodic process is described by the vector \(w = (1, q, q^2, ..., q^j, ..., q^k, ..., q^{n-1})\), where \(q = -1\).

Process (2) is the oscillation \(w_j = (-1)^j\) between 1 and \(-1\) depending on integer argument \(j\) and has the period 2.

We will generalize this configuration from \(q = -1\) to

\[
q_k = (-1)^{n/k}.
\] (3)

Then process \(w_k\),

\[
w_k = (1, q_k, q_k^2, ..., q_k^j, ..., q_k^k, ..., q_k^{n-1}),
\] (4)

will have period \(n/k\): starting from \(w_{0k} = 1\) through \(j = n/k\) points there will be again \(w_{jk} = q_k^j = 1\), and at the intermediate value \(j = n/(2k)\) the function is \(w_{jk} = q_k^j = -1\). At \(k = 0\) the period of the oscillation is infinite, i.e. the quantity is constant. At \(k = n/2\) the process is identical to (2). When \(k > n/2\) the period of the oscillation will be less than 2. In the whole, the frequency of the processes varies in the range \(0 \leq k/n < 1\).

The set of vectors \(w_k\) is orthogonal in the sense that

\[
w_j^* \cdot w_k = \sum_{m=0}^{n-1} w_{jm}^* w_{mk} = \sum_{m=0}^{n-1} q_j m q_k m = \sum_{m=0}^{n-1} (-1)^{2(k-j)m} = n \delta_{jk}
\] (5)

where \(\delta_{jk}\) is the Kronecker delta. Equality (5) follows from the formula of geometric progression

\[
1 + p + p^2 + \cdots + p^{n-1} = \frac{(1 - p^n)}{(1 - p)}
\] (6)

with the denominator \(p = (-1)^{(2(k-j))/n}\).

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The property of orthogonality of vectors $\mathbf{w}_k$ is convenient by that multiplying the expansion (1) by vector $\mathbf{w}_k^*$, there can be immediately, making use of (5), found coefficient $c_k$:

$$\mathbf{w}_k^* \cdot \mathbf{f} = \sum_{m=0}^{n-1} w_{mk}^* f_m = nc_k.$$  

(7)

In the result we obtain

$$f_j = \sum_{k=0}^{n-1} c_k (-1)^{n \frac{k}{j}},$$  

(8)

$$c_k = \frac{1}{n} \sum_{m=0}^{n-1} f_m (-1)^{m \frac{2k}{n}}.$$  

(9)

The fractional power of $-1$, as in (8) and (9), can be reduced to $\sqrt{-1}$ and rendered in the exponential form with a positive base.

Theorem:

$$e^{\sqrt{-1} \varphi} = \cos \varphi + \sqrt{-1} \sin \varphi.$$  

(10)

Proof (for definition of the Euler’s number $e$ see Appendix).

Indeed, on the one side we have:

$$e^{\sqrt{-1} \varphi_1} e^{\sqrt{-1} \varphi_2} = e^{\sqrt{-1}(\varphi_1 + \varphi_2)}.$$

On the other, by the trigonometry:

$$(\cos \varphi_1 + \sqrt{-1} \sin \varphi_1)(\cos \varphi_2 + \sqrt{-1} \sin \varphi_2) = \cos(\varphi_1 + \varphi_2) + \sqrt{-1} \sin(\varphi_1 + \varphi_2).$$

Besides, differentiating the left-hand side of (10) we obtain

$$\frac{d}{d\varphi} e^{\sqrt{-1} \varphi} = \sqrt{-1} e^{\sqrt{-1} \varphi}.$$  

While the differential of the right-hand side (10) is

$$\frac{d}{d\varphi} (\cos \varphi + \sqrt{-1} \sin \varphi) = -\sin \varphi + \sqrt{-1} \cos \varphi = \sqrt{-1}(\cos \varphi + \sqrt{-1} \sin \varphi).$$

This is sufficient in order to substantiate equality (10).

Consider the change of $(-1)^z$ in dependence on $x$:

$$(-1)^0 = 1, \quad (-1)^{1/2} = \sqrt{-1}, \quad (-1)^1 = -1, \quad (-1)^{3/2} = -\sqrt{-1}, \quad (-1)^2 = 1.$$  

According to (10), we have $\exp(\sqrt{-1}\pi) = -1$ and, consequently, $\exp(\sqrt{-1}\pi x)$ changes with $x$ as

$$e^{\sqrt{-1}\pi \cdot 0} = 1, \quad e^{\sqrt{-1}\pi \cdot 1/2} = \sqrt{-1}, \quad e^{\sqrt{-1}\pi \cdot 1} = -1, \quad e^{\sqrt{-1}\pi \cdot 3/2} = -\sqrt{-1}, \quad e^{\sqrt{-1}\pi \cdot 2} = 1.$$  

Thus we have shown that

$$(-1)^z = e^{\sqrt{-1}\pi x}.$$  

(11)

Now we have the convenient form (11), (10) which visually demonstrates the oscillation as rotation of a unit vector in coordinates $(1, \sqrt{-1})$. Fig.1 may illustrate the distribution on the plane in these coordinates of components of a vector (4) with the denominator (3).
Using (11) in (8) and (9), we find the standard form of the Fourier expansion

$$f_j = \sum_{k=0}^{n-1} c_k e^{i \frac{2\pi k}{n} j},$$

$$c_k = \frac{1}{n} \sum_{m=0}^{n-1} f_m e^{-\frac{2\pi k}{n} m}$$

where the well-known designation for the imaginary unit $\sqrt{-1} = i$ is assumed.

Next we proceed to the continuous presentation supposing

$$t = j \Delta t, \quad T = n \Delta t = \text{const}, \quad \Delta t \to 0.$$  

Using (14) in (12), (13):

$$f(t) = \sum_{k=0}^{n-1} c_k e^{i \frac{2\pi k}{n} j \Delta t} \to \sum_{k=0}^{\infty} c_k e^{i \frac{2\pi k}{T} t},$$

$$c_k = \frac{1}{n \Delta t} \sum_{m=0}^{n-1} f_m e^{-\frac{2\pi k}{n} m \Delta t} \to \frac{1}{T} \int_0^T f(t) e^{-\frac{2\pi k}{T} t} \, dt.$$  

In (15) and (16) $T$ is the time duration of the process, and $T/k$ period of the $k$-th harmonics.

Replacing in (15) and (16) $f(t)$ by $f(t + t_0)$ we obtain formulae for expansion of the function at any finite interval $(t_0, t_0 + T)$.

Notice that we may extend the geometric progression (6) at the same length $n$ into the region of negative powers:

$$p^{-n} + p^{-n+1} + \ldots + p^{-1} + 1 + p + p^2 + \ldots + p^{n-1} = (p^{-n} - p^n)/(1 - p).$$

In this event, for the extended vectors $w_k$ there holds the orthogonality with $q_k = (-1)^{k/n}$, so that we will have instead of (5)

$$w_j^* \cdot w_k = \sum_{m=-n}^{n-1} q_j^{-m} q_k^m = \sum_{m=-n}^{n-1} (-1)^{m} \frac{(k-j)m}{n} = 2n \delta_{jk}.$$
The symmetrical expansion has a more regular character: the periods of the discrete processes in question never become less than 2. At a given \( k \) the period equals to \( 2n/k \), i.e. two times more long than it is at the same \( k \) in the one-sided distribution (4). At \( k = -n \) the period equals to 2, then with the increase of \( k \) it grows up to infinity at \( k = 0 \), and further drops gradually to almost 2 at \( k = n - 1 \). So that the frequency varies in the range \(-1/2 \leq k/(2n) < 1/2\). In accord with (18), we recast relations (15) and (16) putting \( 2n \) in place of \( n \):

\[
f(t) = \sum_{k=-\infty}^{\infty} c_k e^{i\pi k \frac{t}{2T}},
\]

\[
c_k = \frac{1}{2T} \int_{-T}^{T} f(t) e^{-i\pi k \frac{t}{T}} dt
\]

where the function is taken at a finite interval \((-T, +T)\). If \( f(t) \) is defined on the interval \((t_1, t_2)\), then we have in (19), (20) \( T = (t_2 - t_1)/2 \), and \( f(t) \) should be substituted by \( f(t + (t_1 + t_2)/2) \).

Notice that if the function \( f \) is real, then (20) entails \( c_{-k} = c_k^* \), where * is the sign of complex conjugation. We will represent the expansion of the real function in the real form. From (19), using Euler formula (10):

\[
f(t) = c_0 + \sum_{k=1}^{\infty} \left( c_k e^{i\pi k \frac{t}{T}} + c_{-k} e^{-i\pi k \frac{t}{T}} \right) = c_0 + \sum_{k=1}^{\infty} \left[ (c_k + c_k^*) \cos \pi k \frac{t}{T} + i(c_k - c_k^*) \sin \pi k \frac{t}{T} \right].
\]

Coefficients \( c_0, c_k + c_k^*, i(c_k - c_k^*) \) of this expansion are real and can be determined from (20) as

\[
c_k + c_k^* = \frac{1}{T} \int_{-T}^{T} f(t) \cos \pi k \frac{t}{T} dt,
\]

\[
i(c_k - c_k^*) = \frac{1}{T} \int_{-T}^{T} f(t) \sin \pi k \frac{t}{T} dt
\]

where \( c_0^* = c_0 \).

Thus, at a finite interval a function can be expanded into the Fourier series.

Let the time interval be infinite: \( T \to \infty \). We will redefine variables:

\[
\frac{\pi k}{T} = \omega, \quad \frac{Tc_k}{\pi} \to c_\omega.
\]

Using (24) in (19) yields

\[
f(t) = \sum_{k=-\infty}^{\infty} c_\omega \frac{\pi}{T} e^{i\omega t} \to \int_{-\infty}^{\infty} c_\omega e^{i\omega t} d\omega
\]

since varying \( k \) by one \( \omega \) changes by \( \pi/T \), i.e. \( \delta \omega = \pi/T \). Using (24) in (20):

\[
c_\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt.
\]

So, we deal with the Fourier integral on the entire number axis.

There can be deduced a formula for the concise rendering and remembering of the Fourier expansion. First, refashion (25) as

\[
f(t) = \int_{-\infty}^{\infty} d\omega' c_\omega' e^{i\omega' t}.
\]
Substituting (27) into (26):

\[ c(\omega) = \int_{-\infty}^{\infty} d\omega' c(\omega') \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} dt e^{i(\omega' - \omega)t} \right]. \tag{28} \]

From (28) we have

\[ \delta(\omega' - \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{i(\omega' - \omega)t}. \tag{29} \]

Similarly to (29), there can be written \( \delta \)-function for \( t \):

\[ \delta(t' - t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{i\omega(t' - t)}. \tag{30} \]

Using the representation (30), we may easily obtain formula for the Fourier transform

\[ f(t) = \int_{-\infty}^{\infty} dt' f(t') \delta(t' - t) = \int_{-\infty}^{\infty} d\omega \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} dt' f(t') e^{-i\omega t'} \right] e^{i\omega t}. \tag{31} \]

Compare (31) with (25) and (26).

Let

\[ f(t) = \sum_k c_k e^{i\omega_k t} \tag{32} \]

i.e. we have a set of harmonic oscillators. Substituting (32) in (26):

\[ c(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_k c_k e^{i(\omega_k - \omega)t} dt = \sum_k c_k \delta(\omega - \omega_k) \tag{33} \]

where the definition (29) is used. In reality, (32) is blurred, and the discrete spectrum (33) degrades into the sum of Gauss components

\[ c(\omega) = \sum_k \frac{c_k}{\sqrt{2\pi}\sigma_k^2} \exp \left[ -\frac{(\omega - \omega_k)^2}{2\sigma_k^2} \right] \tag{34} \]

that is shown in Fig.2.

Figure 2: A realistic spectrum of the composite signal.
Appendix A: THE FIRST REMARKABLE LIMIT

Binomial $1 + 1/n$ raised to the power $n$ at $n \to \infty$ is bounded by excess and deficiency in the following way:

$$2^{1/2} < \left(1 + \frac{1}{n}\right)^n = 1 + n \frac{1}{n} + \frac{n(n-1)}{2!} \frac{1}{n^2} + ... \xrightarrow{n \to \infty} 1 + 1 + \frac{1}{2!} + ... < 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + ... = 1 + \frac{1}{1 - \frac{1}{2}} = 3.$$

Denoting

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$$

we are seeking for

$$\frac{d}{dx} e^x = \lim_{\Delta x \to 0} \frac{e^{x + \Delta x} - e^x}{\Delta x} = e^x \lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x}.$$ 

Supposing $\Delta x = 1/n$ gives

$$\lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x} = \lim_{n \to \infty} n \left(1 + \frac{1}{n}\right)^n \frac{1}{n} = 1.$$ 

Hence:

$$\frac{d}{dx} e^x = e^x.$$

Relationship (A1) can be generalized. We have

$$S_1 = \left(1 + \frac{m}{n}\right)^n = 1 + n \cdot \frac{m}{n} + \frac{n(n-1)}{2!} \left(\frac{m}{n}\right)^2 + ... \xrightarrow{n \to \infty} 1 + m + \frac{m^2}{2!} + \frac{m^3}{3!} + ...,$$

$$S_2 = \left(1 + \frac{1}{n}\right)^{nm} = 1 + mn \cdot \frac{1}{n} + \frac{mn(mn-1)}{2!} \left(\frac{1}{n}\right)^2 + ... \xrightarrow{n \to \infty} S_1.$$

Therefore:

$$e^m = \left[\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n\right]^m = \lim_{n \to \infty} \left(1 + \frac{m}{n}\right)^n.$$