Gravity in Curved Phase-Spaces and Two-Times Physics

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Abstract

The generalized (vacuum) field equations corresponding to gravity on curved $2d$-dimensional tangent bundle/phase spaces associated to the geometry of the (co)tangent bundle $TM_{d-1,1}(T^*M_{d-1,1})$ of a d-dim spacetime $M_{d-1,1}$ is investigated following the strict formalism of Lagrange-Finsler and Hamilton-Cartan geometry. It is found that there is *no* mathematical equivalence with Einstein's vacuum field equations in spacetimes of $2d$-dimensions, with *two* times, after a $d + d$ Kaluza-Klein-like decomposition of the $2d$-dim scalar curvature $\mathbf{R}$ is performed and involving the introduction of a nonlinear connection $A^a_\mu(x^\mu, y^b)$. The physical applications of the 4-dim phase space metric solutions found in this work, corresponding to the cotangent space of a 2-dim spacetime, deserve further investigation. The physics of *two* times may relevant in the solution to the problem of time in Quantum Gravity. Finding nontrivial solutions of the generalized gravitational field equations corresponding to the 8-dim cotangent bundle (phase space) of the 4-dim spacetime remains a challenging task.

1 Introduction : Born's Reciprocal Relativity in Phase Space

Born's reciprocal ("dual") relativity [1] was proposed long ago based on the idea that coordinates and momenta should be unified on the same footing, and consequently, if there is a limiting speed (temporal derivative of the position coordinates) in Nature there should be a maximal force as well, since force is the temporal derivative of the momentum. A *maximal* speed limit (speed of light) must be accompanied with a *maximal* proper force (which is also compatible with a *maximal* and *minimal* length duality). The generalized velocity and acceleration boosts (rotations) transformations of the $8D$ Phase space, where $X^i, T, E, P^i; i = 1,2,3$ are *all* boosted (rotated) into each-other, were given by [2] based on the group $U(1,3)$ and which is the Born version of the Lorentz group $SO(1,3)$.

The $U(1,3) = SU(1,3) \otimes U(1)$ group transformations leave invariant the symplectic 2-form $\Omega = - \, dt \wedge dp^0 + \delta_{ij} dx^i \wedge dp^j; i,j = 1,2,3$ and also the following Born-Green line interval in the $8D$ phase-space (in natural units $\hbar = c = 1$)

$$(d\sigma)^2 = (dt)^2 - (dx)^2 - (dy)^2 - (dz)^2 + \frac{1}{b^2} \left((dE)^2 - (dp_x)^2 - (dp_y)^2 - (dp_z)^2 \right) \tag{1.1}$$

the rotations, velocity and force (acceleration) boosts leaving invariant the symplectic 2-form and the line interval in the $8D$ phase-space are rather elaborate, see [2] for details. These transformations can be simplified drastically when the velocity and force (acceleration) boosts are both parallel to the x-direction and leave the transverse directions y, z, p_y, p_z intact. There is now a subgroup $U(1,1) = SU(1,1) \otimes U(1) \subset U(1,3)$ which leaves invariant the following line interval

$$(d\omega)^2 = (dT)^2 - (dX)^2 + \frac{(dE)^2 - (dP)^2}{b^2} =$$

$$(d\tau)^2 \left(1 + \frac{(dE/d\tau)^2 - (dP/d\tau)^2}{b^2} \right) = (d\tau)^2 \left(1 - \frac{F^2}{F^2_{max}} \right) \tag{1.2}$$

where one has factored out the proper time infinitesimal $(d\tau)^2 = dT^2 - dX^2$ in (2.2). The proper force interval $(dE/d\tau)^2 - (dP/d\tau)^2 = -F^2 < 0$ is "spacelike" when the proper velocity interval $(dT/d\tau)^2 - (dX/d\tau)^2 > 0$ is timelike. The analog of the Lorentz relativistic factor in eq-(2.2) involves the ratios of two proper *forces*.

If (in natural units $\hbar = c = 1$) one sets the maximal proper-force to be given by $b \equiv m_P A_{max}$, where $m_P = (1/L_P)$ is the Planck mass and $A_{max} = (1/L_p)$, then $b = (1/L_P)^2$ may also be interpreted as the maximal string tension. The units of b would be of $(mass)^2$. In the most general case there are four scales of time, energy, momentum and length that can be constructed from the three constants $b, c, \hbar$ as follows

$$\lambda_t = \sqrt{\frac{\hbar}{bc}}; \quad \lambda_l = \sqrt{\frac{\hbar \ c}{b}}; \quad \lambda_p = \sqrt{\frac{\hbar \ b}{c}}; \quad \lambda_e = \sqrt{\hbar \ b \ c} \tag{1.3}$$

The gravitational constant can be written as $G = \alpha_G \ c^4/b$ where α_G is a dimensionless parameter to be determined experimentally. If $\alpha_G = 1$, then the four scales (2.3) coincide with the *Planck* time, length, momentum and energy, respectively.

The $U(1,1)$ group transformation laws of the phase-space coordinates X, T, P, E which leave the interval (2.2) invariant are [2]

$$T' = T \ cosh\xi + \left(\frac{\xi_v \ X}{c^2} + \frac{\xi_a \ P}{b^2}\right) \frac{sinh\xi}{\xi} \tag{1.4a}$$

$$E' = E \cosh\xi + (-\xi_a X + \xi_v P) \frac{\sinh\xi}{\xi} \qquad (1.4b)$$

$$X' = X \cosh\xi + (\xi_v T - \frac{\xi_a E}{b^2}) \frac{\sinh\xi}{\xi} \qquad (1.4c)$$

$$P' = P \cosh\xi + (\frac{\xi_v E}{c^2} + \xi_a T) \frac{\sinh\xi}{\xi} \qquad (1.4d)$$

ξ_v is the velocity-boost rapidity parameter and the ξ_a is the force (acceleration) boost rapidity parameter of the primed-reference frame. These parameters are defined respectively in terms of the velocity $v = dX/dT$ and force $f = dP/dT$ (related to acceleration) as

$$\tanh(\frac{\xi_v}{c}) = \frac{v}{c}; \quad \tanh(\frac{\xi_a}{b}) = \frac{f}{F_{max}} \qquad (1.5)$$

It is straightforwad to verify that the transformations (1.4) leave invariant the phase space interval $c^2(dT)^2 - (dX)^2 + ((dE)^2 - c^2(dP)^2)/b^2$ but *do not* leave separately invariant the proper time interval $(d\tau)^2 = dT^2 - dX^2$, nor the interval in energy-momentum space $\frac{1}{b^2}[(dE)^2 - c^2(dP)^2]$. Only the *combination*

$$(d\sigma)^2 = (d\tau)^2 \left(1 - \frac{F^2}{F^2_{max}} \right) \qquad (1.6)$$

is truly left invariant under force (acceleration) boosts (1.4).

The physics of a limiting value of the proper acceleration in spacetime [4] has been studied by Brandt [3] from the perspective of the tangent bundle geometry. Generalized $8D$ gravitational equations reduce to ordinary Einstein-Riemannian gravitational equations in the *infinite* acceleration limit. A pedagogical monograph on Finsler geometry can be found in [9] and [10] where, in particular, Clifford/spinor structures were defined with respect to nonlinear connections associated with certain nonholonomic modifications of Riemann–Cartan gravity.

We explored in [5] some novel consequences of Born's reciprocal Relativity theory in flat phase-space and generalized the theory to the curved spacetime scenario. We provided, in particular, *six* specific results resulting from Born's reciprocal Relativity and which are *not* present in Special Relativity. These are : momentum-dependent time delay in the emission and detection of photons; energy-dependent notion of locality; superluminal behavior; relative rotation of photon trajectories due to the aberration of light; invariance of areas-cells in phase-space and modified dispersion relations. A different approach to the physics of curved momentum space has been undertaken by [22], [23], [24].

The purpose of this work is to analyze the geometry of the *curved* $8D$ (co) tangent bundle within the context of the geometry of Lagrange-Finsler (tangent space) and Hamilton-Cartan spaces (phase spaces), instead of viewing Born's reciprocal complex gravitational theory [6] as an $8D$ local gauge theory of the Quaplectic group [2] that is given by the semi-direct product of the pseudo-unitary group $U(1,3)$ with the Weyl-Heisenberg group involving four coordinates and momenta. We shall work with real metrics, instead of complex metrics having symmetric real components and antisymmetric imaginary ones [6]. We must emphasize that Reciprocity symmetry and $U(1,3)$ invariance will *not* be invoked in this work.

2 Two-Times Physics and Phase Space Metrics

Through the years it has become evident that the $(2,2)$-signature is not only mathematically interesting [13] (see also Refs. [14], [15]) but also physically. In fact, the $(2,2)$ signature emerges in several physical contexts, including self-dual gravity *a la* Plebanski (see Ref. [16] and references therein), consistent $N=2$ superstring theory as discussed by Ooguri and Vafa [17], and the $N=(2,1)$ heterotic string [19].

In [18] it was shown how a $\mathcal{N}=2$ Supersymmetric Wess-Zumino-Novikov-Witten model valued in the area-preserving (super) diffeomorphisms group is Self Dual Supergravity in $2+2$ and $3+1$ dimensions depending on the signatures of the base manifold and target space. The interplay among $\mathcal{W}_\infty$ gravity, $\mathcal{N}=2$ Strings, self dual membranes, $SU(\infty)$ Toda lattices and $SU^*(\infty)$ Yang-Mills instantons in $2+2$ dimensions can be found also [18] .

More recently, using the requirement of $SL(2,R)$ and Lorentz symmetries it has been proved by [21] that the $2+2$-target spacetime of a 0-brane is an exceptional signature, where a special kind of 0-brane called quatl shows that the $2+2$-target spacetime can be understood either as the $2+2$-world-volume or as a $1+1$ matrix-brane. Another recent motivation of physical interest is that the $2+2$-signature emerges in the discovery of hidden symmetries of the Nambu-Goto action found by [20]. In fact, Duff was able to rewrite the Nambu-Goto action in a $2+2$-target spacetime in terms of Cayley's hyper-determinant, revealing apparently new hidden symmetries of such an action. Duff's observation has been linked with the matrix-brane idea [21]. Black-hole-like solutions (spacetimes with singularities) of Einstein field equations in $3+1$ and $2+2$-dimensions were studied in [8].

Bars [12] has proposed a gauge symmetry in phase space. One of the consequences of this gauge symmetry is a new formulation of physics in spacetime. Instead of one time there must be *two* times, while phenomena described by one-time physics in $3+1$ dimensions appear as various shadows of the same phenomena that occur in $4+2$ dimensions with one extra space and one extra time dimensions (more generally, $d+2$). Problems of ghosts and causality are resolved automatically by the $Sp(2,R)$ gauge symmetry in phase space.

Instead of working with $d+2$-dim spaces with two times [12] we shall propose a different picture and view the two temporal directions as those corresponding to the time *and* energy variables of phase space, whereas the remaining phase space variables correspond to the spatial coordinates and momenta. Following this premise, firstly, we shall search for candidate 4-dim phase space metrics of signature $(2,2)$ which are associated with metrics in 4-dim curved spacetimes of signature $(2,2)$ and that are solutions to the ordinary vacuum Einstein field equations. In the next section, we shall formulate the rigorous construction of the geometry of the cotangent bundle (phase space) of spacetime within the context of the geometry of Hamilton-Cartan and Lagrange-Finsler spaces [9], [10], [11].

Let us begin with 4-dim spacetimes of signature $(2,2)$. Given a 4-dim spacetime of signature $+,+,-,-$ (two times, t_1, t_2), a parametrization of the t_2, x, y coordinates in terms of the variables

$$r \geq 0; \quad -\infty \leq \xi \leq +\infty; \quad 0 \leq \theta \leq 2\pi; \quad -\infty \leq t_2 \leq +\infty \tag{2.1}$$

given by

$$t_2 = r \ sinh \ \xi; \quad x = r \ cosh \ \xi \ cos \ \theta; \quad y = r \ cosh \ \xi \ sin \ \theta. \tag{2.2}$$

where r is the variable *throat* size of the 2-dim hyperboloid $\mathcal{H}^2$ embedded in 3-dim and that can be defined analytically in terms of t_2, x, y as

$$-(t_2)^2 \ + x^2 \ + y^2 \ = \ r^2. \tag{2.3}$$

shows that the flat spacetime metric

$$(ds)^2 \ = \ - (dt_1)^2 - (dt_2)^2 + (dx)^2 + (dy)^2 \tag{2.4}$$

can be recast in terms of the new coordinates as

$$ds^2 = -(dt_1)^2 + (dr)^2 + r^2 [\ cosh^2 \ \xi \ (d\theta)^2 \ - \ (d\xi)^2 \ ]. \tag{2.5}$$

Notice that we have a two times $(-,+,+,-)$ signature in eq-(2.5), as one should. The topology corresponding to metric in eq-(2.5) is $\mathcal{R} \times \mathcal{R}^* \times \mathcal{H}^2$. $\mathcal{R}^*$ is the half-interval $[0, \infty]$ representing the values of the radial coordinate. One temporal variable t_1 is characterized by the real line $\mathcal{R}$ and whose values range from $(-\infty, +\infty)$, and the other temporal variable t_2 is one of the 3 coordinates (t_2, x, y) which parametrize the two-dim hyperboloid $\mathcal{H}^2$ described by eq-(2.3).

A curved spacetime version of the metric of eq-(2.5) is

$$ds^2 = -e^{\mu(r)}(dt_1)^2 \ + \ e^{\nu(r)}(dr)^2 \ + \ (R(r))^2 [\ cosh^2 \ \xi \ (d\theta)^2 \ - \ (d\xi)^2 \ ]. \tag{2.6}$$

The metric in eq-(2.6) whose signature is $(2,2) = (-,+,+,-)$ is the hyperbolic version of the Schwarzschild metric. In general, due to radial reparametrization invariance, one can replace the radial variable $r \to R(r)$ for another variable called the area-radius, since Einstein's equations do not determine the form of the radial function $R(r)$ as explained in the appendix. We still must determine what are the functional forms of $\mu(r)$ and $\nu(r)$. In the appendix we find the solutions to Einstein's vacuum field equations in D-dimensions for metrics associated with a $D-2$-dim homogeneous space of constant positive (negative

) scalar curvature. In particular when $D = 4$ and the two-dim homogeneous space $\mathcal{H}^2$ has a constant positive scalar curvature, like two-dim de Sitter space, the metric components, in natural units $G = \hbar = c = 1$, are given by

$$(ds)^2 \ = \ -(1-\frac{2M}{r})\ (dt)^2 \ + \ (1-\frac{2M}{r})^{-1}\ (dr)^2 \ + \ r^2\ (\ cosh^2\xi\ (d\theta)^2 \ - \ (d\xi)^2\ ) \tag{2.7}$$

The 2-dim hyperboloid homogeneous space whose metric is defined by $(ds)^2 = r^2(cosh^2\xi\ (d\theta)^2 - (d\xi)^2)$ coincides with the metric of a 2-dim de Sitter space of constant positive scalar curvature and whose throat-size is r. Anti de Sitter space has a constant negative scalar curvature. There is a physical singularity at $r = 0$, the location of the point mass source, when the hyperboloid $\mathcal{H}^2$ degenerates to a *cone* since the throat size r has been pinched to zero.

By postulating that the momentum coordinates E, P are curled-up along the 2-dim de Sitter space allows to set the correspondence $\frac{E}{\lambda_e} \leftrightarrow \xi$; $\frac{P}{\lambda_p} \leftrightarrow \theta$, while the spacetime coordinates correspondence is $X \leftrightarrow r$; $T \leftrightarrow t$. λ_p, λ_e are the scales of momentum and energy that can be constructed from the three constants $b, c, \hbar$ as described in eq-(1.3). A nontrivial metric associated to a curved $4D$ phase-space and which is a putative solution to the vacuum field (generalized gravitational) equations corresponding to the cotangent bundle T^*M of a two-dim spacetime M is

$$(ds)^2 \ = \ -(1-\frac{2M}{X})\ (dT)^2 + (1-\frac{2M}{X})^{-1}\ (dX)^2 + X^2\ [\ cosh^2(\frac{E}{\lambda_e})\ \frac{(dP)^2}{\lambda_p^2} - \frac{(dE)^2}{\lambda_e^2}\ ] \tag{2.8}$$

One has the correct $(2,2)$ signature $-,+,+,-$ and the domain of values of the X, T, E, P variables are respectively

$$X \geq 0; \quad -\infty \leq T \leq +\infty; \quad -\infty \leq \frac{E}{\lambda_e} \leq +\infty; \quad 0 \leq \frac{P}{\lambda_p} \leq 2\pi \tag{2.9}$$

A different identification for the phase space variables can be found from the following correspondence

$$-(t_2)^2 \ + \ x^2 \ + \ y^2 \ = \ r^2 \ \leftrightarrow -e^2 \ + \ x^2 \ + \ p^2 \ = \ r^2 \ = \ X^2 \tag{2.10}$$

so that

$$T \ = \ t_1 \ = \ t; \quad e \ = \ t_2 \ = \ r\ sinh\xi \ = \ X\ sinh(\frac{E}{\lambda_e}) \tag{2.11}$$

$$x \ = \ r\ cosh\xi\ cos\theta \ = \ X\ cosh(\frac{E}{\lambda_e})\ cos(\frac{P}{\lambda_p})$$

$$p \ = \ r\ cosh\xi\ sin\theta \ = \ X\ cosh(\frac{E}{\lambda_e})\ sin(\frac{P}{\lambda_p}) \tag{2.12}$$

$$\frac{P}{\lambda_p} = \theta = arctan(\frac{p}{x}) \Rightarrow d\theta = \frac{1}{1+(p/x)^2} d(\frac{p}{x}) \qquad (2.13)$$

$$(dX)^2 = (dr)^2 = [-e^2 + x^2 + p^2]^{-1} (-ede + xdx + pdp)^2; \qquad (2.14)$$

$$r^2 (d\xi)^2 = X^2 \frac{(dE)^2}{\lambda_e^2} =$$

$$\frac{1}{x^2+p^2} [(-e^2+x^2+p^2)(de)^2 + \frac{e^2}{-e^2+x^2+p^2} (-ede+xdx+pdp)^2] -$$

$$\frac{1}{x^2+p^2} [2e\ de\ (-ede + xdx + pdp)] \qquad (2.15)$$

etc .... leading to a very complicated expression for the metric (2.8) rewritten in terms of the t, x, e, p variables. In the *flat* phase limit, $M = 0$, the metric (2.8) can be rewritten in the standard pseudo-Euclidean form

$$(ds)^2 = -(dt)^2 - (de)^2 + (dx)^2 + (dp)^2 \leftrightarrow -(dt_1)^2 - (dt_2)^2 + (dx)^2 + (dy)^2 \qquad (2.16)$$

as expected.

We have shown that a particular $4D$ metric of topology $R \times S^1 \times R \times S^1$ which solves the $4D$ vacuum Einstein field equations in spaces of signature $(2,2)$ is of the form [8]

$$ds^2 = -(\lambda_l/\rho)\ (dt_1)^2 + (\rho/\lambda_l)\ (d\rho)^2 + \rho^2\ (\ (d\theta)^2 - \frac{(dt_2)^2}{\lambda_t^2}\ ) =$$

$$-(\lambda_l/\rho)\ (dt_1)^2 + (\rho/\lambda_l)\ (d\rho)^2 + \rho^2\ (\ (d\theta)^2 - (d\phi)^2\ ) \qquad (2.17)$$

where $\phi \equiv t_2/\lambda_t$ and λ_t, λ_l are constants. By assigning in eq-(2.17) the phase space variables

$$t_1 = T; \quad \rho = X; \quad \theta = \frac{P}{\lambda_p}; \quad t_2 = E; \quad \phi = \frac{E}{\lambda_e}; \qquad (2.18)$$

the 4-dim phase space metric of signature $(2,2)$ becomes

$$(ds)^2 = -\frac{(dT)^2}{(X/\lambda_l)} + \frac{X}{\lambda_l} (dX)^2 - X^2 \frac{(dE)^2}{\lambda_e^2} + X^2 \frac{(dP)^2}{\lambda_p^2} \qquad (2.19)$$

where $\lambda_t, \lambda_e, \lambda_p, \lambda_l$ are the four scales of time, energy, momentum and length that can be constructed from the three constants $b, c, \hbar$ as described in eq-(1.3). In natural units $\hbar = c = G = b = 1$ one has that $\lambda_t = \lambda_e = \lambda_p = \lambda_l = 1$.

Since the cotangent space of the circle S^1 is a cylinder $R \times S^1$, the $4D$ space of topology $R \times S^1 \times R \times S^1$ can be interpreted as the cotangent bundle (phase-space) of the 2-dim torus $S^1 \times S^1$. The latter torus is also the Shilov boundary of the $4D$ phase space. Another approach to gravity in curved phase spaces from the perspective of the geometry of bounded homogenous complex domains has been undertaken by [26]. Phase space is interpreted as the upper tubular regions of these complex domains, like the Siegel upper half plane (obtained from a conformal map of the Poincare disk), and spacetime corresponds to their Shilov boundary. Applications of curved phase spaces in Quantum Field Theories can be found in [27].

Physically, eq-(2.18) assigns the energy-momentum E, P variables to be confined along the two-torus $S^1 \times S^1$, and the space-time T, X variables are the fibers of the cotangent space of the two-torus. Therefore, the $4D$ phase-space is the cotangent bundle of the two-torus whose topology is $R \times S^1 \times R \times S^1$. By Born's reciprocity one may reverse the assignment so that the space-time T, X variables are confined along the two-torus $S^1 \times S^1$, while the energy-momentum E, P variables are the fibers of the cotangent space of the torus, setting aside for the moment the caveat behind having closed-timelike variables. One can bypass this problem by going to the covering space of the circle, like it is done with the temporal variable in AdS spacetime.

As shown in the Appendix, another metric solution in $D = 4$ when the two-dim homogeneous space has a constant *negative* scalar curvature, like two-dim de Anti de Sitter space AdS_2, in natural units $G = \hbar = c = 1$, is given by

$$(ds)^2 \ = \ (1+\frac{2M}{r}) \ (dt)^2 \ - \ (1+\frac{2M}{r})^{-1} \ (dr)^2 \ + \ r^2 \ (\ - \ cosh^2\xi \ (d\theta)^2 \ + \ (d\xi)^2 \ ) \tag{2.20}$$

in this case one has a $signature \ flip$ from the signature of the metric in eq-(2.7). By exchanging $r \leftrightarrow t$, like it occurs with the Kantowski-Sachs cosmological solution, eq-(2.20) becomes

$$(ds)^2 \ = \ (1+\frac{2M}{t}) \ (dr)^2 \ - \ (1+\frac{2M}{t})^{-1} \ (dt)^2 \ + \ t^2 \ (\ - \ cosh^2\xi \ (d\theta)^2 \ + \ (d\xi)^2 \ ) \tag{2.21}$$

If the energy-momentum coordinates E, P are curled-up along the 2-dim Anti de Sitter space, it allows to set the correspondence $\frac{E}{\lambda_e} \leftrightarrow \theta$; $\frac{P}{\lambda_p} \leftrightarrow \xi$, and $X \leftrightarrow r$; $T \leftrightarrow t$ so that eq-(2.21) becomes

$$(ds)^2 \ = \ (1+\frac{2M}{T}) \ (dX)^2 - (1+\frac{2M}{T})^{-1} \ (dT)^2 + T^2 \ [- \ cosh^2(\frac{P}{\lambda_p}) \ \frac{(dE)^2}{\lambda_e^2} \ + \ \frac{(dP)^2}{\lambda_p^2} \ ] \tag{2.22}$$

which is another nontrivial metric associated to a curved 4-dim phase-space and which is a putative solution to the vacuum field (generalized gravitational) equations corresponding to the cotangent bundle T^*M of a two-dim spacetime M. One has the correct $(2,2)$ signature $+,-,-,+$ in eq-(2.22) : $(dX)^2, (dP)^2$ appear with a positive sign, whereas $(dT)^2, (dE)^2$ appear with a negative sign. The domain of values of the X, T, P, E variables in eq-(2.22) are respectively

$$X \geq 0; \quad - \infty \leq T \leq +\infty; \quad - \infty \leq \frac{P}{\lambda_p} \leq +\infty; \quad 0 \leq \frac{E}{\lambda_e} \leq 2\pi \tag{2.23}$$

To sum up, we have displayed three nontrivial metrics in phase space given by eqs-(2.8, 2.19, 2.22) with split signature $(2,2)$ and which are obtained from solutions to the 4-dim Einstein field equations in spacetimes of signature $(2,2)$ by identifying one of the temporal variables with the energy and one of the spatial variables with the momentum. In the next section we shall study the geometry of the (co) tangent space within the framework of Lagrange-Finsler geometry and Hamilton-Cartan spaces and explore the field equations in order to verify the validity of the proposed metric solutions $g_{IJ}(x^i, p_i)$ in phase spaces, where the indices span $i, j = 1, 2, 3, ...d$ and $I, J = 1, 2, 3, ..., 2d$.

3 Gravity in Curved Phase Spaces

3.1 A Kaluza-Klein like approach

Some time ago, it was shown by [7] that a Kaluza-Klein formalism of Einstein's theory, based on the $(2+2)$-fibration of a generic 4-dimensional spacetime, describes General Relativity as a Yang-Mills gauge theory on the 2-dimensional base manifold, where the local gauge symmetry is the group of the diffeomorphisms of the 2-dimensional fibre manifold. They found the Schwarzschild solution by solving the field equations after a very laborious procedure. Their formalism was valid for any $d+n$ decomposition of the D-dim spacetime $D = d+n$.

As shown by [7] the line element

$$ds^2 \ = \ g_{ab} \ dy^a \ dy^b \ + \ (g_{\mu\nu} + g_{ab} \ A^a_\mu \ A^b_\nu) \ dx^\mu \ dx^\nu \ + \ 2 g_{ab} \ A^b_\mu \ dx^\mu \ dy^a. \tag{3.1}$$

in light cone coordinates

$$u \ = \ \frac{1}{\sqrt{2}} \ (t + r), \quad v \ = \ \frac{1}{\sqrt{2}} \ (t - r). \tag{3.2}$$

$$A^a_u \ = \ \frac{1}{\sqrt{2}} \ (A^a_t \ + \ A^a_r), \quad A^a_v \ = \ \frac{1}{\sqrt{2}} \ (A^a_t \ - \ A^a_r). \tag{3.3}$$

after using the Polyakov ansatz

$$g_{\mu\nu} = \begin{pmatrix} -2 \ h(t,r) & -1 \\ -1 & 0 \end{pmatrix}, \tag{3.4}$$

becomes

$$ds^2 \ = \ g_{ab} \ dy^a \ dy^b \ - 2du \ dv \ - 2h(u) \ du^2 \ +$$

$$g_{ab} \ (A^a_u du + A^b_v dv) \ (A^b_u du + A^b_v dv) \ + 2 g_{ab} \ (A^a_u du + A^b_v dv) \ dy^a. \tag{3.5a}$$

Upon setting $g_{ab} = e^\sigma \ \rho_{ab}$ such that $det \ \rho_{ab} = 1$ and after a very laborious calculation Yoon [7] arrived finally at the expression for the scalar curvature

$$\mathcal{R} \;=\; g^{\mu\nu}\;(R^{\alpha}_{\mu\alpha\nu} \;+\; R^{b}_{\mu b\nu}) \;+\; g^{ab}(R^{c}_{acb} \;+\; R^{\mu}_{a\mu b}) \qquad (3.5b)$$

in the light-cone coordinates given by

$$\mathcal{R} \;=\; -\frac{1}{2}\; e^{2\sigma}\;\rho_{ab}\;F^{a}_{+-}\;F^{b}_{+-} \;+\; e^{\sigma}\;\mathcal{R}_2 \;+\; e^{\sigma}\;D_{+}\sigma\;D_{-}\sigma\;-$$

$$\frac{1}{2}\;e^{\sigma}\;\rho^{ab}\;\rho^{cd}\;(D_{+}\rho_{ab})\;(D_{-}\rho_{cd}) \;+\; \frac{1}{2}\;e^{\sigma}\;\rho^{ab}\;\rho^{cd}\;(D_{+}\rho_{ac})\;(D_{-}\rho_{bd}) \;+$$

$$2h_{++}\;e^{\sigma}\;[\;D^{2}_{-}\sigma \;+\frac{1}{2}(D_{-}\sigma)^{2} \;+\frac{1}{4}\;\rho^{ab}\;\rho^{cd}\;(D_{-}\rho_{ac})\;(D_{-}\rho_{bd})\;] \qquad (3.6)$$

plus surface terms. The Lie-bracket is

$$[\;A_{\mu}\;,\;g_{ab}\;] \;=\; (\partial_a\;A^{c}_{\mu}(x^{\mu},y^{a}))\;g_{bc}(x^{\mu},y^{a}) \;+\; (\partial_b\;A^{c}_{\mu}(x^{\mu},y^{a}))\;g_{ac}(x^{\mu},y^{a}) \;+$$

$$A^{c}_{\mu}(x^{\mu},y^{a})\;\partial_c\;g_{ab}(x^{\mu},y^{a}). \qquad (3.7)$$

the Yang-Mills-like field strength is

$$F^{a}_{\mu\nu} \;=\; \partial_{\mu}A^{a}_{\nu} \;-\partial_{\nu}A^{a}_{\mu} \;-\; [A_{\mu},A_{\nu}]^{a} \;=$$

$$\partial_{\mu}A^{a}_{\nu} \;-\partial_{\nu}A^{a}_{\mu} \;-\; A^{c}_{\mu}\partial_{c}A^{a}_{\nu} \;+A^{c}_{\nu}\partial_{c}A^{a}_{\mu}. \qquad (3.8)$$

The covariant derivative of a tensor *density* ρ_{ab} with weight 1 is

$$D_{\mu}\;\rho_{ab} \;=\; \partial_{\mu}\;\rho_{ab} \;-[\;A_{\mu}\;,\;\rho\;]_{ab} \;+(\partial_{c}A^{c}_{\mu})\;\rho_{ab} \;=$$

$$\partial_{\mu}\;\rho_{ab} \;-A^{c}_{\mu}\partial_{c}\;\rho_{ab} \;-(\partial_{a}A^{c}_{\mu})\;\rho_{cb} \;-(\partial_{b}A^{c}_{\mu})\;\rho_{ac} \;+(\partial_{c}A^{c}_{\mu})\;\rho_{ab}. \qquad (3.9)$$

the covariant derivative on the scalar density $\Omega = e^{\sigma}$ of weight -1 is

$$D_{\mu}\Omega \;=\; \partial_{\mu}\Omega \;-A^{a}_{\mu}\partial_{a}\Omega \;-(\partial_{a}A^{a}_{\mu})\Omega \;\Rightarrow. \qquad (3.10)$$

$$D_{\mu}\sigma \;=\; \partial_{\mu}\sigma \;-A^{a}_{\mu}\partial_{a}\sigma \;-(\partial_{a}A^{a}_{\mu}). \qquad (3.11)$$

after factoring the e^{σ} terms.

The authors [7] were able to solve the equations of motion associated with the Einstein-Hilbert action

$$S \;=\; \int\;du\;dv\;d^{2}y\;\mathcal{R}. \qquad (3.12)$$

by varying the Einstein-Hilbert action *before* imposing the gauge fixing conditions (3.4) and $det\;\rho_{ab} = 1$ giving a total of 10 equations for the 10 fields $\sigma, h_{++}, h_{--}, A^{a}_{+}, A^{a}_{-}, \rho_{ab}$ with $a,b = 1,2$. These 10 fields match the same number of idependent components of the metric $g_{\mu\nu}$ in $4D$. After a very laborious procedure the authors found solutions for the "vacuum" field configurations $A^{a}_{+} = 0, A^{a}_{-} = 0$ given by

$$ds^{2} \;=\; 2\;du\;dv \;-(1-\frac{2GM}{u})\;dv^{2} \;+u^{2}\;d\Omega^{2}. \qquad (3.13)$$

$$ds^2 \;=\; -2\; du\; dv\; - (1 - \frac{2GM}{v})\; du^2\; + v^2\; d\Omega^2. \qquad (3.14)$$

which have the *same functional form* as the Schwarzschild-Hilbert solution in the retarded and advanced temporal Eddington-Finkelstein coordinates, $u = t - r_*, v = t + r_*$

$$ds^2 \;=\; 2\; dr\; dv\; - (1 - \frac{2GM}{r})\; dv^2\; + r^2\; d\Omega^2. \qquad (3.15)$$

$$ds^2 \;=\; -2\; du\; dr\; - (1 - \frac{2GM}{r})\; du^2\; + r^2\; d\Omega^2. \qquad (3.16)$$

with the subtle technicality that r_* appearing in the definitions $u = t - r_*, v = t + r_*$ in eqs-(2.15, 2.16) is the *tortoise* radial coordinate $r_*(r)$ given by

$$\int dr_* \;=\; \int \frac{dr}{1 - 2GM/r} \;\Rightarrow\; r_* \;=\; r + 2GM\; ln\; |\frac{r}{2GM} - 1|. \qquad (3.17a)$$

It is well known [25] that one can introduce afterwards the Fronsdal-Kruskal-Szekeres coordinates in terms of the Eddington-Finkelstein coordinates and which allow an analytical extension of the Schwarzschild-Hilbert solution into the interior region of the black hole beyond the horizon.

The solutions in eqs-(3.13-3.16) reflect the spherical symmetry where $r^2(d\Omega)^2$ is the standard metric of the two-dim sphere S^2. In the split signature case $(2,2)$, one must replace the $r^2(d\Omega)^2$ metric interval by the one corresponding to an internal two-dim hyperboloid $\mathcal{H}^2$, a two-dim de Sitter space dS_2, and given by $r^2(cosh^2\xi(d\theta)^2 - (d\xi)^2)$ as indicated by eq-(2.7).

Secondly, in order to avoid confusion with the notation, let us label t', r' for the variables of the $1+1$ dim spacetime M_{1+1} used by [7], such that the equivalence of eq-(3.14) with eq-(3.16) is established by setting the correspondence

$$t' - r' \leftrightarrow r; \quad t' + r' \leftrightarrow t - r_* \;=\; t - (\; r + 2GM\; ln\; |\frac{r}{2GM} - 1|\;) \qquad (3.17b)$$

similar equivalence of eq-(3.13) with eq-(3.15) is established by setting the correspondence

$$t' + r' \leftrightarrow r; \quad t' - r' \leftrightarrow t + r_* \;=\; t + (\; r + 2GM\; ln\; |\frac{r}{2GM} - 1|\;) \qquad (3.17c)$$

To sum up, the equivalence among eqs-(3.13, 3.14) with eqs-(3.15,3.16) corroborates once more that the metric (2.7) is a solution of the $4D$ Einstein field equations, as well as being a solution to the field equations after a Kaluza-Klein-like $2+2$ decomposition of the $4D$ space is performed, in the split signature $(2,2)$ case.

Having written the explicit $2+2$ decomposition and above solutions in eqs-(3.15, 3.16), one may assign the internal variables, that parametrize the hyperbolic two-dim de Sitter space dS_2, to correspond to the energy E and momentum P, and the $1+1$ base spacetime variables to correspond to the X, T coordinates of the 4-dim phase space. In order to have consistent units one must have that $y^a \leftrightarrow p_a/\mathbf{b}; \partial/\partial y^a \leftrightarrow \mathbf{b}\ \partial/\partial p_a$ and $A_i^a \partial_{y^a} \leftrightarrow \mathbf{b} A_{ia} \partial_{p_a}$, with $i,j = 1,2; a = 1,2$. In natural units $\hbar = G = c = 1$ it yields $\mathbf{b} = 1$, as shown in eq-(1.3), such that it simplifies matters.

Concluding, the 4-dim curved phase-space metric (2.8) is a viable solution to the gravitational field equations following the Kaluza-Klein-like $2+2$ decomposition and after identifying the internal coordinates with the E, P variables. The other solutions in eqs-(2.19, 2.22) are harder to verify. A more *rigorous* method to find and verify solutions to the generalized gravitational field equations in tangent spaces/phase spaces requires to study directly the geometry of the (co)tangent bundle $T^M_{1+1}(T^*M_{1+1})$ of the two-dim spacetime M_{1+1}, rather than following the Kaluza-Klein-like method of [7] described above. We find that the Kaluza-Klein decomposition *is not equivalent* to the more rigorous methods of Finsler-Lagrange and Hamilton-Cartan spaces [9]. We proceed to show this in the next section. See also the related work by [3].

3.2 Geometry of the Tangent and Cotangent Bundle of Spacetime : Lagrange-Finsler and Hamilton-Cartan Spaces

In this section we shall present the essentials behind the geometry of the tangent and cotangent space. We will follow closely the description by authors [9], [10], [11]. The metric associated with the tangent space TM_d can be written in the in the following block diagonal form

$$(ds)^2 \ = \ g_{ij}(x^k, y^a)\ dx^i d\ x^j \ + \ h_{ab}(x^i, y^a)\ \delta y^a\ \delta y^b \qquad (3.18)$$

$(i,j,k = 1,2,3,....d;\ a,b,c = 1,2,3,....d)$ if instead of the standard coordinate-basis one introduces the anholonomic frames (non-coordinate basis) defined as

$$\delta_i \ = \ \partial_i - N_i^b(x,y)\ \partial_b \ = \ \partial/\partial x^i - N_i^b(x,y)\ \partial_b; \quad \partial_a \ = \ \frac{\partial}{dy^a} \qquad (3.19)$$

and its dual basis is

$$\delta^\alpha \equiv \delta u^\alpha = (\delta^i = dx^i, \ \ \delta^a = dy^a + N_k^a(x,y)\ dx^k) \qquad (3.20)$$

where the N–coefficients define a nonlinear connection, N–connection structure, see details in [9], [10], [11]. As a very particular case one recovers the ordinary linear connections if $N_i^a(x,y) = \Gamma^a_{bi}\,(x)\, y^b$.

The N–connection structures can be naturally defined on (pseudo) Riemannian spacetimes and one can relate them with some anholonomic frame fields (vielbeins) satisfying the relations $\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha = W^\gamma_{\alpha\beta}\delta_\gamma$, with nontrivial anholonomy coefficients

$$W^k_{ij} \ = \ 0; \ \ W^k_{aj} \ = \ 0; \ \ W^k_{ia} \ = \ 0; \ \ W^k_{ab} \ = \ 0; \ \ W^c_{ab} = 0$$

$$W^a_{ij} \ = \ -\Omega^a_{ij}; \ \ W^a_{bj} \ = \ -\partial_b N^a_j; \ \ W^b_{ia} \ = \ \partial_a N^b_j \tag{3.21}$$

where

$$\Omega^a_{ij} \ = \ \delta_j N^a_i \ - \ \delta_i N^a_j \tag{3.22}$$

is the nonlinear connection curvature (N–curvature). This is the same object as $F^a_{\mu\nu}$ described in the previous section when $N^a_j \leftrightarrow A^a_\mu$.

A metric of type given by eq-(3.18) with arbitrary coefficients $g_{ij}(x^k, y^a)$ and $h_{ab}(x^k, y^a)$ defined with respect to a N–elongated basis is called a distinguished metric. A linear connection $D_{\delta_\gamma}\delta_\beta = \Gamma^\alpha_{\ \beta\gamma}(x,y)\delta_\alpha$ associated to an operator of covariant derivation D is compatible with a metric $g_{\alpha\beta}$ and N–connection structure on a pseudo–Riemannian spacetime if $D_\alpha g_{\beta\gamma} = 0$. The linear distinguished connection is parametrized by irreducible (horizontal, vertical) h–v–components, $\Gamma^\alpha_{\ \beta\gamma} = \left(L^i_{\ jk}, L^a_{\ bk}, C^i_{\ jc}, C^a_{\ bc}\right)$ such that [9], [10], [11].

$$\begin{aligned}
L^i_{\ jk} \ &= \ \frac{1}{2} g^{in}\left(\delta_k g_{nj} + \delta_j g_{nk} - \delta_n g_{jk}\right) \\
L^a_{\ bk} \ &= \ \partial_b N^a_k \ + \ \frac{1}{2} h^{ac}\left(\delta_k h_{bc} - h_{dc}\partial_b N^d_k - h_{db}\partial_c N^d_k\right) \\
C^i_{\ jc} \ = \ \frac{1}{2} g^{ik}\partial_c g_{jk}; \quad C^a_{\ bc} \ &= \ \frac{1}{2} h^{ad}\left(\partial_c h_{db} + \partial_b h_{dc} - \partial_d h_{bc}\right).
\end{aligned} \tag{3.23}$$

This defines a canonical linear connection (distinguished by a N–connection) which is similar to the metric connection introduced by Christoffel symbols in the case of holonomic bases. The anholonomic coefficients $w^\gamma_{\ \alpha\beta}$ and N–elongated derivatives give nontrivial coefficients for the torsion tensor, $T(\delta_\gamma, \delta_\beta) = T^\alpha_{\ \beta\gamma}\delta_\alpha$. One arrives at

$$T^\alpha_{\ \beta\gamma} \ = \ \Gamma^\alpha_{\ \beta\gamma} \ - \ \Gamma^\alpha_{\ \gamma\beta} \ + \ w^\alpha_{\ \beta\gamma}, \tag{3.24}$$

and at the curvature tensor, $R(\delta_\tau, \delta_\gamma)\delta_\beta = R_{\beta\ \ \gamma\tau}^{\ \alpha}\delta_\alpha$

$$R_{\beta\ \ \gamma\tau}^{\ \alpha} \ = \ \delta_\tau\Gamma^\alpha_{\ \beta\gamma} \ - \ \delta_\gamma\Gamma^\alpha_{\ \beta\tau} \ + \ \Gamma^\sigma_{\ \beta\gamma}\Gamma^\alpha_{\ \sigma\tau} \ - \ \Gamma^\sigma_{\ \beta\tau}\Gamma^\alpha_{\ \sigma\gamma} \ + \ \Gamma^\alpha_{\ \beta\sigma}w^\sigma_{\ \gamma\tau} \tag{3.25}$$

One should note the key presence of the last term in (3.25) due to the non-vanishing anholonomic coefficients $w^\gamma_{\ \alpha\beta}$. One is not accustomed to see this term in ordinary textbooks. The torsion distinguished tensor has the following irreducible, nonvanishing, h–v–components, $T^\alpha_{\ \beta\gamma} = \left(T^i_{\ jk}, C^i_{\ ja}, S^a_{\ bc}, T^a_{\ ij}, T^a_{\ bi}\right)$ given by

$$\begin{aligned}
T^i_{jk} \ = \ L^i_{jk} \ - \ L^i_{kj}; \ \ T^i_{ja} \ &= \ C^i_{ja}; \ \ T^i_{aj} \ = \ - \ C^i_{ja} \\
T^i_{ja} \ = \ 0; \ \ T^a_{bc} \ &= \ S^a_{bc} \ = \ C^a_{bc} \ - \ C^a_{cb} \\
T^a_{ij} \ = \ -\Omega^a_{ij}; \ \ T^a_{bi} \ &= \ \partial_b N^a_i \ - \ L^a_{bi}; \ \ T^a_{ib} \ = \ - \ T^a_{bi}
\end{aligned} \tag{3.26}$$

and where $\Omega^a_{ij} = \delta_j N^a_i - \delta_i N^a_j$ can be interpreted as the "field strength" associated with the nonlinear connection N^a_i.

The curvature distinguished tensor has the following irreducible, non-vanishing, h–v–components $R_{\beta\ \ \gamma\tau}^{\ \alpha} = \left(R^i_{hjk}, R^a_{bjk}, P^i_{jka}, P^c_{bka}, S^i_{jbc}, S^a_{bcd}\right)$ given by [9], [10], [11]

$$R^i_{hjk} \ = \ \delta_k L^i_{hj} \ - \ \delta_j L^i_{hk} \ + \ L^m_{hj} L^i_{mk} \ - \ L^m_{hk} L^i_{mj} \ - \ C^i_{ha} \Omega^a_{jk}$$

$$R^a_{bjk} \ = \ \delta_k L^a_{bj} \ - \ \delta_j L^a_{bk} \ + \ L^c_{bj} L^a_{ck} \ - \ L^c_{bk} L^a_{cj} \ - \ C^a_{bc} \Omega^c_{jk}$$

$$P^i_{jka} \ = \ \partial_a L^i_{jk} \ + \ C^i_{jb} T^b_{ka} \ - \ \left(\delta_k C^i_{ja} \ + \ L^i_{lk} C^l_{ja} \ - \ L^l_{jk} C^i_{la} \ - \ L^c_{ak} C^i_{jc} \right)$$

$$P^c_{bka} \ = \ \partial_a L^c_{bk} \ + \ C^c_{bd} T^d_{ka} \ - \ \left(\delta_k C^c_{ba} \ + \ L^c_{dk} C^d_{ba} \ - \ L^d_{bk} C^c_{da} \ - \ L^d_{ak} C^c_{bd} \right)$$

$$S^i_{jbc} \ = \ \partial_c C^i_{jb} \ - \ \partial_b C^i_{jc} \ + \ C^h_{jb} C^i_{hc} \ - \ C^h_{jc} C^i_{hb}$$

$$S^a_{bcd} \ = \ \partial_d C^a_{bc} \ - \ \partial_c C^a_{bd} \ + \ C^e_{bc} C^a_{ed} \ - \ C^e_{bd} C^a_{ec} \tag{3.27}$$

Having reviewed the geometry of the tangent bundle TM we proceed with the cotangent bundle case T^*M (phase space). In the case of the cotangent space of a d-dim manifold T^*M_d the metric can be equivalently rewritten in the block diagonal form [9] as

$$(ds)^2 \ = \ g_{ij}(x^k, p_a) \ dx^i d \ x^j \ + \ h^{ab}(x^k, p_c) \ \delta p_a \ \delta p_b \tag{3.28}$$

$i, j, k = 1, 2, 3, ...d, \ a, b, c = 1, 2, 3, ...d,$ if instead of the standard coordinate basis one introduces the following anholonomic frames (non-coordinate basis)

$$\delta_i \ = \ \delta/\delta x^i \ = \ \partial_{x^i} \ + \ N_{ia} \ \partial^a \ = \ \partial_{x^i} \ + \ N_{ia} \ \partial_{p_a}; \ \ \partial^a \ \equiv \ \partial_{p_a} \ = \ \frac{\partial}{\partial p_a} \tag{3.29}$$

One should note the *key* position of the indices that allows us to distinguish between derivatives with respect to x^i and those with respect to p_a. The dual basis of $(\delta_i = \delta/\delta x^i; \partial^a = \partial/\partial p_a)$ is

$$dx^i, \ \ \delta p_a \ = \ dp_a \ - \ N_{ja} \ dx^j \tag{3.30}$$

where the N–coefficients define a nonlinear connection, N–connection structure. An N-linear connection D on T^*M can be uniquely represented in the adapted basis in the following form

$$D_{\delta_j}(\delta_i) \ = \ H^k_{ij} \ \delta_k; \ \ D_{\delta_j}(\partial^a) \ = \ - \ H^a_{bj} \ \partial^b; \tag{3.31}$$

$$D_{\partial^a}(\delta_i) \ = \ C^{ka}_i \ \delta_k; \ \ D_{\partial^a}(\partial^b) \ = \ - \ C^{ba}_c \ \partial^c \tag{3.32}$$

where $H^k_{ij}(x,p), H^a_{bj}(x,p), C^{ka}_i(x,p), C^{ba}_c(x,p)$ are the connection coefficients. For any N-linear connection D with the above coefficients the torsion 2-forms are

$$\Omega^i \;=\; \frac{1}{2} T^i_{jk}\; dx^j \wedge dx^k \;+\; C^{ia}_j\; dx^j \wedge \delta p_a$$

$$\Omega_a \;=\; \frac{1}{2} R_{jka}\; dx^j \wedge dx^k \;+\; P^b_{aj}\; dx^j \wedge \delta p_b \;+\; \frac{1}{2} S^{bc}_a\; \delta p_b \wedge \delta p_c \qquad (3.33)$$

and the curvature 2-forms are

$$\Omega^i_j \;=\; \frac{1}{2} R^i_{jkm}\; dx^k \wedge dx^m \;+\; P^{ia}_{jk}\; dx^k \wedge \delta p_a \;+\; \frac{1}{2} S^{iab}_j\; \delta p_a \wedge \delta p_b \qquad (3.34a)$$

$$\Omega^a_b \;=\; \frac{1}{2} R^a_{bkm}\; dx^k \wedge dx^m \;+\; P^{ac}_{bk}\; dx^k \wedge \delta p_c \;+\; \frac{1}{2} S^{acd}_b\; \delta p_c \wedge \delta p_d \qquad (3.34b)$$

where one must recall that the dual basis of $\delta_i = \delta/\delta x^i,\ \partial^a = \partial/\partial p_a$ is given by $dx^i,\ \delta p_a = dp_a - N_{ja} dx^j$.

The distinguished torsion tensors are of the form [9]

$$T^i_{jk} \;=\; H^i_{jk} \;-\; H^i_{kj};\;\; S^{ab}_c \;=\; C^{ab}_c \;-\; C^{ba}_c;\;\; P^a_{bj} \;=\; H^a_{bj} \;-\; \partial^a\; N_{jb}$$

$$R_{ija} \;=\; \frac{\delta N_{ja}}{\delta x^i} \;-\; \frac{\delta N_{ia}}{\delta x^j} \qquad (3.35)$$

the last tensor R_{ija} has a one to one correspondence with the field strength $F^a_{\mu\nu}$ of section **3.1**. The distinguished tensors of the curvature are of the form

$$R^i_{kjh} \;=\; \delta_h H^i_{kj} \;-\; \delta_j H^i_{kh} \;+\; H^l_{kj}\; H^i_{lh} \;-\; H^l_{kh}\; H^i_{lj} \;-\; C^{ia}_k\; R_{jha}$$

$$P^{ab}_{cj} \;=\; \partial^a\; H^b_{cj} \;+\; C^{ad}_c\; P^b_{dj} \;-\; \left(\delta_j\; C^{ab}_c \;+\; H^b_{dj}\; C^{da}_c \;+\; H^a_{dj}\; C^{bd}_c \;-\; H^d_{cj}\; C^{ab}_d\right)$$

$$P^{ak}_{ij} \;=\; \partial^a\; H^k_{ij} \;+\; C^{al}_i\; T^k_{lj} \;-\; \left(\delta_j\; C^{ak}_i \;+\; H^a_{bj}\; C^{bk}_i \;+\; H^k_{lj}\; C^{al}_i \;-\; H^l_{ij}\; C^{ak}_l\right)$$

$$S^{abc}_d \;=\; \partial^c\; C^{ab}_d \;-\; \partial^b\; C^{ac}_d \;+\; C^{eb}_d\; C^{ac}_e \;-\; C^{ec}_d\; C^{ab}_e;\;\; etc.......... \qquad (3.36)$$

where we have omitted the other components and once again we have for our notation $\partial^a = \partial/\partial p_a$ and $\delta/\delta x^i = \partial_{x^i} + N_{ia}\,\partial^a$. Equipped with these curvature tensors one can perform suitable contractions involving g_{ij}, h^{ij}, N_{ia} to obtain three curvature scalars of the $\mathcal{R}, \mathcal{P}, \mathcal{S}$ type

$$\mathcal{R} \;=\; \delta^j_i\; R^i_{kjl}\; g^{kl};\;\; \mathcal{S} \;=\; \delta^d_b\; S^{abc}_d\; h_{ac}$$

$$\mathcal{P}_1 = \delta^c_b \, P^{ab}_{cj} \, g^{jk} N_{ka}; \; \mathcal{P}_2 = \delta^c_a \, P^{ab}_{cj} \, N_{bk} \, g^{kj}; \; \mathcal{P}_3 = \delta^i_k \, P^{ak}_{ij} \, g^{jl} \, N_{la}; \; \mathcal{P}_4 = \delta^j_k \, P^{ak}_{ij} \, N_{al} \, g^{li} \tag{3.37}$$

and construct phase space Lagrangians involving a linear combination of the three curvature scalars

$$\mathcal{L}_{phase\ space} \;=\; a\,\mathcal{R} \;+\; b\,\mathcal{S} \;+\; \sum_{n=1}^{4} c_n\,\mathcal{P}_n \tag{3.38}$$

where a, b, c_n are real-valued numerical coefficients. Notice that the Kaluza-Klein-like decomposition of the scalar curvature in higher dimensions given in eq-(3.6) involves a very *specific* linear combination of the curvature tensor contractions given by

$$\mathbf{R} \;=\; G^{MN}\;R_{MN} \;=\; g^{\mu\nu}\;(R^{\alpha}_{\mu\alpha\nu} \;+\; R^{b}_{\mu b\nu}) \;+\; g^{ab}(R^{c}_{acb} \;+\; R^{\mu}_{a\mu b}) \tag{3.39}$$

There is a direct and exact correspondence/match between the components in eq-(3.37) and those in eq-(3.39) given by

$$\mathcal{R} \longleftrightarrow g^{\mu\nu}\;R^{\alpha}_{\mu\alpha\nu}; \;\; \mathcal{S} \longleftrightarrow g^{ab}\;R^{c}_{acb}; \tag{3.40a}$$

However, there is *no* correspondence/match between the $\mathcal{P}$ curvature scalars

$$c_1\;\mathcal{P}_1 \;+\; c_2\;\mathcal{P}_2 \;+\; c_3\;\mathcal{P}_3 \;+\; c_4\;\mathcal{P}_4 \tag{3.40b}$$

with the following scalar contractions stemming from the Kaluza-Klein decomposition (3.39) $g^{\mu\nu}R^{b}_{\mu b\nu} + g^{ab}R^{\mu}_{a\mu b}$. In the tangent bundle case, for instance, if one were to exchange the $b \leftrightarrow j$ indices in the expression for R^{a}_{bjk} in eq-(3.27), one obtains R^{a}_{jbk} that has the same index structure as $R^{c}_{\mu b\nu}$ whose contraction in the bc indices gives the desired term $R^{b}_{\mu b\nu}$ in eq-(3.39). However, there is a problem with this procedure since exchanging the $b \leftrightarrow j$ indices leads to an exchange $L^{a}_{bk} \to L^{a}_{jk}$ which is problematic because there is *no* connection coefficient of the form L^{a}_{jk} associated to the covariant derivation D.

Therefore, due to the fact that the scalar curvature contractions stemming from the Kaluza-Klein decomposition (3.39) $g^{\mu\nu}R^{b}_{\mu b\nu} + g^{ab}R^{\mu}_{a\mu b}$ do *not* have a proper match/correspondence with the curvature contractions appearing in the geometry of the (co) tangent space, the $2d$-dimensional phase space Lagrangian (3.38) corresponding to the cotangent bundle of the d-dim spacetime $\mathcal{L}_{phase\ space}$ does *not* have the same structure (same functional form up to total derivatives) as the Einstein-Hilbert Lagrangian $\mathbf{R}$ in $2(d-1)+2 = 2d$ dimensions (with two times), after a Kaluza-Klein-like $d+d$ decomposition of $\mathbf{R}$ described in section **3.1** is performed, and upon imposing the identifications $y^a \leftrightarrow p_a/\mathbf{b}$, $\partial/\partial y^a \leftrightarrow \mathbf{b}\;\partial/\partial p_a$; and $A^a_i \leftrightarrow N_{ia}$ with $i, j = 1, 2, 3, ...d; a, b = 1, 2; 3, ..., d$. Consequently, the field equations obtained from a variational principle of the phase space Lagrangian $\mathcal{L}$ in eq-(3.38) are *not* equivalent to the gravitational field equations in $2d$-dimensions corresponding to the $d+d$ decomposition of the Einstein-Hilbert Lagrangian $\mathbf{R}$.

The same result applies to the $2d$-dimensional tangent bundle case TM ; i.e. their field equations are not equivalent. Hence, the putative phase space metrics in eqs-(2.8,2.19, 2.22), with split signature $(2,2)$ are viable solutions to the field equations obtained from the Klauza-Klein-like decomposition of the 4-dim space, but are *not* solutions to the vacuum field equations in the 4-dim phase space associated with the cotangent bundle T^*M of the 2-dim spacetime of signature $(1,1)$.

The generalized (vacuum) field equations corresponding to gravity on the curved $2d$-dimensional tangent bundle/phase spaces associated to the geometry of the (co)tangent bundle $TM_{d-1,1}(T^*M_{d-1,1})$ of a d-dim spacetime $M_{d-1,1}$ with *one* time, and which are obtained from a direct variation of the tangent space/phase space actions with respect to the respective fields

$$g_{ij}(x^k,y^a),\ h_{ab}(x^k,y^a),\ N_i^a(x^k,y^a);\quad g_{ij}(x^k,p_k),\ h^{ab}(x^k,p_a),\ N_{ia}(x^k,p_a) \tag{3.41}$$

needs to be investigated further because there is *no* mathematical equivalence with the ordinary Einstein vacuum field equations in spacetimes of $2d$-dimensions with *two* times

$$\mathbf{R}_{MN}(X)\ -\ \frac{1}{2}\,\mathbf{g}_{MN}(X)\,\mathbf{R}(X)\ =0;\quad M,N=1,2,3,.......,2d \tag{3.42}$$

To display explicitly the $difficulty$ in solving the field equations in Lagrange-Finsler and Hamilton-Cartan spaces, we devote the next section to provide specific nontrivial solutions in a "simple" case.

3.3 Integrable solutions in the Tangent Bundle TM_2 case

To solve the most general solutions for $g_{ij}(x^k,y^a), h_{ab}(x^k,y^a), N_i^a(x^k,y^a)$ is a very difficult task even in the simpler case of actions involving linear terms in the scalar curvature. Einstein's equations in dimensions $2+2$ as a toy model of Einstein-Finsler gravity in the tangent bundle TM_2 over a 2-dimensional manifold M_2 were studied by [10], [11]. An *integrable* class of solutions was found which are $different$ from the solutions (3.13,3.14) provided by [7]. Solutions in dimensions $2+2+2.....;3+2+2+...$ have also been found by [11]. However we must stress that the equations and solutions found by [10], [11] in the tangent bundle TM_2 case $differ$ from the field equations one obtains from a variation of the most general action involving the 3 types of curvature scalars $\mathcal{R},\mathcal{P},\mathcal{S}$ because Vacaru sets $\mathcal{P}=0$.

The field equations used by Vacaru are

$$\mathcal{R}_{ij}\ -\ \frac{1}{2}\,(\ \mathcal{R}+\mathcal{S}\ )\ g_{ij}\ =0\ ;\quad \mathcal{S}_{ab}\ -\ \frac{1}{2}\,(\ \mathcal{R}+\mathcal{S}\ )\ h_{ab}\ =0 \tag{3.43a}$$

$$\mathcal{P}_{ia}\ =\ 0;\ \ \mathcal{P}_{ai}\ =\ 0;\ \ \mathcal{P}_{ia}\ \neq\ \mathcal{P}_{ai};\quad i,j,k=1,2;\ a,b,c=1,2 \tag{3.43b}$$

$\mathcal{R}_{ij}, \mathcal{S}_{ab}, \mathcal{P}_{ia}, \mathcal{P}_{ai}$ are obtained from the curvature contractions $R^k_{ikj}, S^c_{acb}, P^k_{ika}, P^b_{abi}$, respectively. A further contraction $g^{ij}\mathcal{R}_{ij} = \mathcal{R}$; $h^{ab}S_{ab} = \mathcal{S}$ yields the curvature scalars. Since Vacaru sets $\mathcal{P}_{ia}, \mathcal{P}_{ai}$ to *zero* this leads to $\mathcal{P}_1 = \mathcal{P}_2 = 0$.

A special family of solutions was found by [10] for the case

$$g_{ij} = g_{ij}(x^k); \quad h_{ab} = h_{ab}(x^k, y^1); \quad N^a_i = N^a_i(x^k, y^1) \tag{3.44}$$

i.e. when the solutions do *not* depend on the coordinate y^2. The solutions are determined by the generating functions $f(x^i, y^1)$, the integration functions ${}^0f(x^i), {}^0h(x^k), {}^0w_j(x^k), {}^0n_i(x^k)$ and the signature coefficients $\epsilon_1 = \pm 1, \epsilon_2 = \pm 1, \epsilon_3 = \pm 1, \epsilon_4 = \pm 1$. As explained by [10] one should chose and/or fix such functions following additional assumptions on symmetry of solutions, boundary conditions etc. For instance, the solutions to the wave equation in two dimensions are given in terms of two generating functions $\Psi = f(x + ct) + g(x - ct)$.

By defining $N^1_i = w_i, \; N^2_i = n_i, \; i = 1, 2$, the integrable solutions to the vacuum equations (3.43) found by Vacaru [10], [11] are given by

$$g_{11} \; = \; \epsilon_1 \; e^{\Psi(x^k)}; \;\; g_{22} \; = \; \epsilon_2 \; e^{\Psi(x^k)} \;\; for \;\; \epsilon_1 \; \partial^2_{x^1}\Psi \; + \; \epsilon_2 \; \partial^2_{x^2}\Psi \; = \; 0 \tag{3.45}$$

$$h_{11} \; = \; \epsilon_3 \; {}^0h(x^i) \; [\; \partial_{y^1} f(x^i, y^1) \;]^2; \;\; h_{22} \; = \; \epsilon_4 \; [\; f(x^i, y^1) \; - \; {}^0f(x^i) \;]^2 \tag{3.46}$$

$$w_j \; = \; {}^0w_j(x^k) \; exp \left(- \int_0^{y^1} dy^1 [\frac{2h_{11}\partial_{y^1}A}{\partial_{y^1}h_{11}}] \right) \int_0^{y^1} dy^1 \; [\frac{h_{11}B_j}{\partial_{y^1}h_{11}}] \; exp \left(- \int_0^{y^1} dy^1 [\frac{2h_{11}\partial_{y^1}A}{\partial_{y^1}h_{11}}] \right) \tag{3.47}$$

for $j = 1, 2$.

$$n_i \; = \; {}^0n_i(x^k) \; + \; \int dy^1 \; h_{11} \; K_i; \;\; i = 1, 2.$$

The expressions for A, B_k, K_i are given by

$$A \; = \; \frac{\partial_{y^1}h_{11}}{2h_{11}} \; + \; \frac{\partial_{y^1}h_{22}}{2h_{22}} \tag{3.48}$$

$$B_k \; = \; \left(\frac{\partial_{y^1}h_{22}}{2h_{22}} \right) \left(\frac{\partial_{x^k}g_{11}}{2g_{11}} \; - \; \frac{\partial_{x^k}g_{22}}{2g_{22}} \right) \; - \; \partial_{x^k}A; \;\; k = 1, 2. \tag{3.49}$$

$$K_1 \; = \; -\frac{1}{2} \left(\frac{\partial_{x^2}g_{11}}{g_{22}h_{11}} \; + \; \frac{\partial_{x^1}g_{22}}{g_{22}h_{22}} \right); \;\; K_2 \; = \; \frac{1}{2} \left(\frac{\partial_{x^1}g_{22}}{g_{11}h_{11}} \; - \; \frac{\partial_{x^2}g_{22}}{g_{22}h_{22}} \right) \tag{3.50}$$

Due to the *duality* between the tangent and cotangent bundles of spacetime, one expects analogous (not identical) field equations and solutions for the 4-dim phase space case (associated with the 2-dim space). In this particular case one could have solutions of the form $g_{ij}(X,T); h^{ab}(X,T,E); N_{ia}(X,T,E)$, reminiscent of the so-called rainbow metrics (energy dependent). Solutions of the form $g_{ij}(X,T); h^{ab}(X,T,P); N_{ia}(X,T,P)$ are equally valid, or solutions depending on X,T and the combinations like $E+P$, or $E-P$ but not both.

The real challenge is to find solutions to Einstein's equations associated with the 8-dim cotangent bundle (phase space) of the 4-dim spacetime; i.e. one has a $8 = 4+4$ decomposition with *two* temporal directions T,E and 6 spatial ones $X_1, X_2, X_3, P_1, P_2, P_3$. Furthermore, more general actions than the Einstein-Hilbert type, including torsion-squared and curvature-squared terms in phase space should be studied. To the author, one of the most physically interesting extensions of gravity in phase spaces (Born reciprocal gravity) is the study of higher order (co) tangent spaces (Jet bundles) which are related to higher order accelerations and naturally involve higher dimensions than $D = 8$. The principle of maximal speed (special relativity), maximal acceleration/force (Born reciprocal relativity) would extend to a generalized relativity of maximal higher order accelerations/forces. Physics in phase space uniting space-time-momentum-energy should reveal important clues to the true origins of *mass* that do not rely on the Higgs mechanism.

Finally, it is desirable to find if there are any connections among the latter theories (higher order accelerations/forces) and the higher spin theories of Vasiliev. The connection between Finsler geometry and the higher conformal spin theories behind $\mathcal{W}_\infty$ gravity/strings was suggested by Hull long ago [28]. Yoon also has discussed the emergence of $\mathcal{W}_\infty$ gravity Lagrangians from the Lagrangian of eq-(3.6). The physical applications of the phase space metrics found in section **2** deserve further investigation, as well as how the physics of *two* times may resolve the problem of time in Quantum Gravity. Finding non-trivial solutions of the generalized field equations corresponding to the 8-dim cotangent bundle (phase space) of the 4-dim spacetime is a challenging task of paramount importance.

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APPENDIX

Let us start with the line element

$$ds^2 = -e^{\mu(r)}(dt_1)^2 + e^{\nu(r)}(dr)^2 + R^2(r)\tilde{g}_{ij}d\xi^i d\xi^j. \qquad (A.1)$$

Here, the metric $\tilde{g}_{ij}$ corresponds to a homogeneous space and $i,j = 3,4,...,D-2$. The only nonvanishing Christoffel symbols are

$$\begin{array}{lll} \Gamma^1_{21} = \frac{1}{2}\mu', & \Gamma^2_{22} = \frac{1}{2}\nu', & \Gamma^2_{11} = \frac{1}{2}\mu' e^{\mu-\nu}, \\ \Gamma^2_{ij} = -e^{-\nu}RR'\tilde{g}_{ij}, & \Gamma^i_{2j} = \frac{R'}{R}\delta^i_j, & \Gamma^i_{jk} = \tilde{\Gamma}^i_{jk}, \end{array} \qquad (A.2)$$

and the only nonvanishing Riemann tensor are

$$\mathcal{R}^1_{212} = -\tfrac{1}{2}\mu'' - \tfrac{1}{4}\mu'^2 + \tfrac{1}{4}\nu'\mu', \qquad \mathcal{R}^1_{i1j} = -\tfrac{1}{2}\mu' e^{-\nu} RR'\tilde{g}_{ij},$$

$$\mathcal{R}^2_{121} = e^{\mu-\nu}(\tfrac{1}{2}\mu'' + \tfrac{1}{4}\mu'^2 - \tfrac{1}{4}\nu'\mu'), \qquad \mathcal{R}^2_{i2j} = e^{-\nu}(\tfrac{1}{2}\nu' RR' - RR'')\tilde{g}_{ij},$$

$$\mathcal{R}^i_{jkl} = \tilde{R}^i_{jkl} - R'^2 e^{-\nu}(\delta^i_k \tilde{g}_{jl} - \delta^i_l \tilde{g}_{jk}). \tag{A.3}$$

The field equations are

$$\mathcal{R}_{11} = e^{\mu-\nu}(\frac{1}{2}\mu'' + \frac{1}{4}\mu'^2 - \frac{1}{4}\mu'\nu' + \frac{(D-2)}{2}\mu'\frac{R'}{R}) = 0, \tag{A.4}$$

$$\mathcal{R}_{22} = -\frac{1}{2}\mu'' - \frac{1}{4}\mu'^2 + \frac{1}{4}\mu'\nu' + (D-2)(\frac{1}{2}\nu'\frac{R'}{R} - \frac{R''}{R}) = 0, \tag{A.5}$$

and

$$\mathcal{R}_{ij} = \frac{e^{-\nu}}{R^2}(\frac{1}{2}(\nu' - \mu')RR' - RR'' - (D-3)R'^2)\tilde{g}_{ij} + \frac{k}{R^2}(D-3)\tilde{g}_{ij} = 0, \tag{A.6}$$

where $k = \pm 1$, depending if $\tilde{g}_{ij}$ refers to positive or negative curvature. From the combination $e^{-\mu+\nu}R_{11} + R_{22} = 0$ we get

$$\mu' + \nu' = \frac{2R''}{R'}. \tag{A.7}$$

The solution of this equation is

$$\mu + \nu = \ln R'^2 + a, \tag{A.8}$$

where a is a constant.

Substituting (A.7) into the equation (A.6) we find

$$e^{-\nu}(\nu' RR' - 2RR'' - (D-3)R'^2 = -k(D-3) \tag{A.9}$$

or

$$\gamma' RR' + 2\gamma RR'' + (D-3)\gamma R'^2 = k(D-3), \tag{A.10}$$

where

$$\gamma = e^{-\nu}. \tag{A.11}$$

The solution of (A.10) for an ordinary D-dim spacetime (one temporal dimension) corresponding to a $D-2$-dim sphere for the homogeneous space can be written as

$$\gamma = (1 - \frac{16\pi G_D M}{(D-2)\Omega_{D-2}R^{D-3}})\,(\frac{dR}{dr})^{-2} \Rightarrow$$

$$g_{rr} = e^{\nu} = (1 - \frac{16\pi G_D M}{(D-2)\Omega_{D-2} R^{D-3}})^{-1} \; (\frac{dR}{dr})^2. \qquad (A.12)$$

where Ω_{D-2} is the appropriate solid angle in $D-2$-dim and G_D is the D-dim gravitational constant whose units are $(length)^{D-2}$. Thus $G_D M$ has units of $(length)^{D-3}$ as it should. When $D = 4$ as a result that the 2-dim solid angle is $\Omega_2 = 4\pi$ one recovers from eq-(A.12) the 4-dim Schwarzchild solution. The solution in eq-(A.12) is consistent with Gauss law and Poisson's equation in $D-1$ spatial dimensions obtained in the Newtonian limit.

The Ω_D is the volume of the unit D-sphere

$$\Omega_D \;=\; \frac{2\pi^{(D+1)/2}}{\Gamma(\frac{D+1}{2})} \qquad (A.13)$$

Thus, according to (A.8) we get

$$\mu = \ln(k - \frac{\beta_D G_D M}{R^{D-3}}) \;+\; constant. \qquad (A.14)$$

we can set the constant to zero, and this means the line element (A.1) can be written as

$$ds^2 = -(k - \frac{\beta_D G_D M}{R^{D-3}})(dt_1)^2 \;+\; \frac{(dR/dr)^2}{(k - \frac{\beta_D G_D M}{R^{D-3}})}(dr)^2 \;+\; R^2(r)\tilde{g}_{ij} d\xi^i d\xi^j. \quad (A.15)$$

One can verify, taking for instance (A.5), that the equations (A.4)-(A.6) do *not* determine the form $R(r)$. It is also interesting to observe that the only effect of the homogeneous metric $\tilde{g}_{ij}$ is reflected in the $k = \pm 1$ parameter, associated with a positive (negative) constant scalar curvature of the homogeneous $D-2$-dim space.

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