Scalar Gravitational Theory with Variable Rest Mass

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ABSTRACT

The purpose of this paper is to present a non-tensor theory of gravitation that provides the equivalent equations of motion, but does not result in the issue of black holes, non-localizable energy, or spacetime singularities. The prime assumption is the notion that the rest mass of a particle entering a gravitational potential is reduced in proportion to the energy gained by the velocity increases. One could designate this development as a "catalytic" theory in that gravitation is a vector catalyst, that converts rest energy into kinetic energy. The total mass energy will be considered localized with the individual mass particles, and defined relative to a given observer. No energy will be ascribed to the field, thus there is no stress energy tensor. We will develop the energy mass relation, and show that it can result in the proper orbital precession as demonstrated by GR. The rest mass is not significantly different from that of a particle defined in a stationary asymptotically flat GR space-time, when the defining point particles via the Komar mass. Since rest mass of a particle goes to zero on approaching a Schwarzschild boundary, the formation of black holes becomes problematic.

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INTRODUCTION

Our first assumption will be that the total energy of a particle is localized in the volume of the particle, and that there is no energy content in the fields related to the particle. Such localization does not generally contradict Newtonian physics, [8] but is not an acceptable feature GR, or any gauge theory having continuously infinite number of independent infinitesimal generators.[9]. This is not a critique of GR, in that such definitions are not necessary for the calculation of physical effects, which are calculated unambiguously.

The general conception in regard to the rest mass is that if work is done on a mass to raise it in a gravitational field, the rest mass in increased by the amount of work done. The energy is not stored or extracted from the field.

The rest mass as we are defining it, has similarity to the Komar mass in GR, in that it is dependent on the Gravitational potential. The Komar mass is determined by integrating Einstein's Equation

$$R_t^t = 8\pi\left(T_t^t - 1/2\,T\right), \tag{1.1}$$

over any volume in the 3-space generated with t = constant [10]. In some Kerr configurations this integral can become negative, which some researchers consider to be unphysical [11].

In this conjecture, we will take the mass of a particle to be defined relative to an observer, whose mass is also dependent on the gravitational potential. The total energy is considered to reside locally at the point of a particle. At the Schwarzschild boundary, the rest mass will be found to become zero, implying total conversion of rest mass to photons. Proportionality of mass with gravitational potential as noted, implies a finite universe with a finite gravitational potential, which could be integrated to yield the ambient free space mass of the system.

Observations made in the weak solutions of General Relativity do not appear to be contradicted by the development here, and in this range the physical difference may not be measurable.

We will also address the issue of photon energy and gravitational bending, as the result of the variable rest mass. Some current views of photon energy in GR are contrary to Einstein's Original view that the photon energy is constant and the shift is due slower clock associated with the emission. [12]. Some of the current views of GR are that the photon looses energy to the field on rising. The most common view is that the frequency of a photon is based on the emitting rate of rate of a clock exposed to a gravitational and not a change in the frequency due to rising in the field.

This theory preserves a constant time scale, but ascribes the lower frequency to the lower rest mass of the emitter, which yields photons of lesser energy, and lower frequency. this is perhaps a significant difference between this theory and GR. After taking account of the rest mass change at different elevations, the results of the Pound-Rebka-Snider experiment does indicate that the energy of the photon must be conserved. We would conclude from this that, since the energy of a photon rising in a gravitational potential is unaffected, and since by the Shapiro effect, the velocity increases, there is no reason to believe photons could not always escape a gravitational potential.

With the foregoing, and our conjectured caveat that the rest mass decreases in a gravitational potential, the formation of black holes does not seem possible.

The consequences of this theory is that, for large masses, the result is not the formation of black holes, but rather a mass to Gamma ray converter. It could be suggested that the anomalous defined gamma ray sources emissions of the galactic center, imaged by the ESA/INTEGRAL spacecraft could be from bodies close to the maximum mass [3]. The source of the 511 kev annihilation line could also be the result of electrons falling in on objects near the maximum mass rather than electron-positron annihilation. The determination of the value of the maximum neutron star mass will be important in determining the validity of this theory.

It should be clarified that Scalar Theory in this case refers to the fact that the instantaneous gravitational potential is a linear sum, defined as:

$$\phi = \sum_{n}^{N} \frac{2\mu_n}{r_n}, \tag{1.2}$$

and not a reference to the potential functions $\Box^2\phi = 4\pi G c^2 \rho$, and as such the scope of this paper does not include wave motion, or gravitational radiation. Those issues will be left for later.

We will be presenting a plausible and very accurate rest mass gravitation relation. If slight differences with experiment, not yet detected, exist, it does not necessarily mean the theory is invalid, but that the initial relation may have a degree of incompleteness.

GENERAL DEVELOPMENT

Our initial assumption is that for a massive particle in a gravitational potential, the total mass of a particle at rest is defined by:

$$M^2 = M_0^2\left(1-2\frac{\mu}{r}\right) \tag{1.3}$$

Where M_0 is the rest mass external to the gravitational potential. If there is a velocity ascribed to the particle then the expression becomes:

$$M^2\left(1-\frac{v^2}{c^2}\right)=M_0^2\left(1-2\frac{\mu}{r}\right) \tag{1.4}$$

Its easy to show that this expression results in the well known Lagrangian within measurable accuracy. i.e.

$$Mc^2 = \left(M_0c^2 - \frac{GMm}{r} + \frac{1}{2}Mv^2\right) \tag{1.5}$$

We will presume that Eq.(1.4) ,is the fundamental relation, but there are subtle relativistic issues related to the interaction term that must be considered.

$$\frac{GMm}{r}, \tag{1.6}$$

First to be noted that the mass terms have to be the relativistic mass. This is obvious from the fact that, if the particles happen to be spinning the kinetic energy must be included, meaning the mass is relativistic mass. In addition each mass experience the other as if it is moving with their relative velocity.

From our knowledge of the Thomas precession, it is known that the distance a particle traveling the circumference of a circle around an attracting potential is shortened by the relativistic contraction. We would assert that if the circumference of a circle is contracted as the result of the relativistic velocity, the radius must also be contracted.

With those considerations the gravitation term in Eq.(1.5) , must be:

$$\frac{GMm}{r} \to \frac{G}{r_0\sqrt{1-v^2/c^2}}\frac{M_0}{\sqrt{1-v^2/c^2}}\frac{m_0}{\sqrt{1-v^2/c^2}} \tag{1.7}$$

or:

$$\frac{GMm}{r} \rightarrow \frac{GMm}{r\sqrt{1-v^2/c^2}^3} \rightarrow M\frac{\mu_{loc}}{r}\left(1+\frac{3}{2}\frac{v^2}{c^2}\right) \qquad (1.8)$$

Orbital Mechanics

We now have a differential expression relating, the velocity, and the distance to the local gravitating mass. We should thus be able to solve for the orbital motion, without need to make assumptions about the force mass relation.

In the following it will be shown that the equations of motion produces orbital relations, equivalent to the weak field GR relations, with the same perihelion advance:

Noting. (1.8), and putting this into E. (1.4), we have:

$$M_0\left[1-\left(\frac{\mu_{loc}}{r}+\frac{\mu_{loc}}{r}\frac{3}{2}\left(\frac{v}{c}\right)^2\right)-\frac{1}{2}\frac{\mu}{r}\frac{\mu}{r}\right]\left[1+\frac{1}{2}\left(\frac{v}{c}\right)^2+\frac{3}{8}\left(\frac{v}{c}\right)^4\right]=M \qquad (1.9)$$

Noting that there is only one significant cross term this becomes:

$$M_0\left[1-\frac{\mu}{r}-\frac{3}{2}\frac{\mu}{r}\frac{v^2}{c^2}-\frac{\mu^2}{2r^2}-\left(\frac{1}{2}\frac{\mu}{r}\frac{v^2}{c^2}\right)+\frac{1}{2}\frac{v^2}{c^2}+\frac{3}{8}\frac{v^4}{c^4}\right]=M \qquad (1.10)$$

We can separate this into:

$$-\frac{M-M_0}{M_0}c^2=c^2\left(-\frac{\mu}{r}-\frac{1}{2}\frac{\mu^2}{r^2}+\frac{1}{2}\frac{v^2}{c^2}\left(1-4\frac{\mu}{r}+\right)+\frac{3}{8}\frac{v^4}{c^4}\right) \qquad (1.11)$$

Setting the left term in this to $\in$, we note that in a conservative system, this term is constant. This is because M_0 is a defined constant and the total energy is constant.

Using the procedures for finding as outlined in Robertson & Noonan,[4] the perihelion precession, in agreement with GR is:

$$\sigma = \left(\frac{1}{2}\mu u_0 - \frac{3}{2}\mu u_0 + 2\mu u_0 + 2\mu u^2 \right) = \left(\mu u_0 + 2\mu u^2 \right) \sim 3\frac{\mu}{p} \tag{1.12}$$

The detailed calculations for this are included in Appendix I.

Photon Energy

From the defining relation of this theory Eq.(1.4), the view of the Pound-Rebka-Snider, Mossbauer effect experiment (1960–1965)[6] changes. Instead of the photon losing energy as the photon rises in the tower, the emission of the photon at the bottom of the tower, is from a less massive generator, and at a lower frequency. The generated frequency plus the added Doppler frequency provided by the velocity of the source in the experiment equals the frequency at the top, thus the photon loses no energy in the flight up the tower.

$$\nu_B \left(1 - \frac{\mu}{r} \right) + \nu_D = \nu_T \, . \tag{1.13}$$

This could be a departure from General Relativity, or not depending on the various interpretations, The original Einstein view, was that the energy was not changed in transit but the generation was at an altered frequency due to the time shift. Most current views are that there is an energy change in transit and in the case of a black hole the entirety of the energy is lost before escapement. Our view is more consistent with the original Einstein view, however, but the change is due to the lower mass of the generator at the lower position rather than a time dilation, time is constant.

Proper Deflection and Velocity of Light

In order to accommodate the change in the rest mass associated with this theory, there are other consequences, that happen as the result, the most notable is the change in the velocity of light. The purpose of the following exercise is to determine the change in value of the relative velocity of light inside a gravitational field, as observed by an external observer.

To that end we will make a few assumptions that are consistent with the theory.

Assumptions

1) The frequency of an atom, and thus extended to an atomic clock, is proportional to the rest mass. This assumption is equivalent to the potential time dilation of General Relativity, however in this case, the change is the result, not of a change in the time scale, but a change in the rest mass.
2) The physical dimensions of a material object at rest are invariant in a gravitational potential.

In order to determine the speed of light shift in a gravitational potential we will use a thought experiment based on the above assumptions. One can take the results either way. The results lead to the assumptions, or the assumptions lead to the results.

1) An laser interferometer is set up in an elevator on the top floor of a building, with a standing wave, having an integral number of wavelength across a resonating cavity.
2) The apparatus is lowered to the bottom floor.
3) We will make the assumption that there is no observable difference in the number of standing waves in the resonating cavity. This is also required by the equivalence principle

Using our assumptions, and Eq. the frequency has decreased as a result of the decreased rest mass of the system at the lower position, and is:

$$\nu = \nu_0\left(1 - \frac{\mu_{loc}}{r_{loc}}\right) \tag{1.14}$$

If the frequency has declined by the potential factor then the wavelength would extend beyond the interferometer space, if there were not an equivalent reduction of the wavelength by the same factor.

$$\lambda = \lambda_0 \left(1 - \frac{\mu_{loc}}{r_{loc}} \right) \tag{1.15}$$

Since the product of $\nu\lambda$ is the velocity of light, we have for a change in the velocity:

$$c \ = \ \lambda_0 \nu_0 \left(1 - \frac{\mu_{loc}}{r_{loc}} \right) \left(1 - \frac{\mu_{loc}}{r_{loc}} \right) \ = \ c_0 \left(1 - \frac{\mu_{loc}}{r_{loc}} \right)^2 \tag{1.16}$$

We know this is the proper value or relative velocity by way of the measured Shapiro effect. This relative velocity also leads to the proper, deflection of light, in agreement with GR by way of Fermat's principle, as shown by Blandford et al [7].

Momentum Considerations

Although the energy of a photon rising or falling in a gravitational potential is unaffected the same cannot be true of the momentum. With momentum given by:

$$P = \frac{E}{c} = \frac{h}{\lambda} = \frac{h}{\lambda_0} \left(1 - \frac{\mu_{loc}}{r_{loc}} \right)^{-1} . \qquad .(1.17)$$

Thus from the perspective of this theory, for both the photon, and a massive particle, on entering a gravitational potential, the total energy remains constant, but the momentum increases.

CONCLUSION

With simple assumptions regarding the relation between rest mass and distance, proper gravitational dynamics and phenomena can predicted. The proposed theory yields the proper orbital equations, with the proper perihelion advance, deflection of light and gravitational red shift. The gravitational potential exchanges no energy with photons, thus photons cannot be bound in a gravitational field, and the existence of a black hole as defined in GR, and viewed by Einstein [13], would not be possible.

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Appendix I

Details of Perihelion Advance

The general rest mass velocity relation proposed is:

$$M^2\left(1-\frac{v^2}{c^2}\right)=M_0^2\left(1-2\frac{\mu}{r}\right) \qquad (2.1)$$

Where the velocity invariant potential is:

$$M_0^2\left(1-2\frac{\mu}{r}\frac{1}{\sqrt{1-v_2^2/c_2^2}^{\,3}}\right)=M^2\left(1-\frac{v^2}{c^2}\right) \qquad (2.2)$$

Taking square root:

$$M_0\left(1-2\frac{\mu}{r}\frac{1}{\sqrt{1-v_2^2/c_2^2}^{\,3}}\right)^{1/2}\left(1-\frac{v^2}{c^2}\right)^{-1/2}=M \qquad (2.3)$$

Binomial expansions:

$$\begin{aligned}(1-x)^{1/2} &= 1-\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{16}\\ (1+x)^{-1/2} &= 1-\frac{x}{2}+\frac{3}{8}x^2\\ (1-x)^{-1/2} &= 1+\frac{x}{2}+\frac{3}{8}x^2+\frac{5}{16}x^3\end{aligned} \qquad (2.4)$$

The simple expansion would be:

$$\left[1-2\frac{\mu_{loc}}{r_{loc}}\right]^{1/2}=\left[1-\frac{\mu_{loc}}{r_{loc}}-\frac{1}{2}\frac{\mu_{loc}}{r_{loc}}\frac{\mu_{loc}}{r_{loc}}\right] \qquad (2.5)$$

Expanding all the terms in Eq. (2.3).

$$M_0\left(\left(1-\frac{\mu_{loc}}{r_{loc}}\left(1-v_2^2/c_2^2\right)^{-3/2}-\frac{1}{2}\frac{\mu}{r}\frac{\mu_{loc}}{r_{loc}}\right)\left(1+\frac{1}{2}\left(\frac{v}{c}\right)^2+\frac{3}{2}\left(\frac{v}{c}\right)^4\right)\right)=M\,, \qquad (2.6)$$

and:

$$M_0\left(\left[1-\left(\frac{\mu_{loc}}{r_{loc}}+\frac{\mu_{loc}}{r_{loc}}\frac{3}{2}\left(\frac{v}{c}\right)^2\right)-\frac{1}{2}\frac{\mu}{r}\frac{\mu_{loc}}{r_{loc}}\right]\left[1+\frac{1}{2}\left(\frac{v}{c}\right)^2+\frac{3}{8}\left(\frac{v}{c}\right)^4\right]\right)=M \qquad (2.7)$$

There is only one cross term of significant value.

v

$$M_0\left[1-\frac{\mu}{r}-\frac{\mu}{r}\frac{3}{2}\left(\frac{v}{c}\right)^2-\frac{1}{2}\frac{\mu}{r}\frac{\mu}{r}-\left[\frac{\mu}{r}\frac{1}{2}\left(\frac{v}{c}\right)^2\right]+\frac{1}{2}\left(\frac{v}{c}\right)^2+\frac{3}{8}\left(\frac{v}{c}\right)^4\right]=M \qquad (2.8)$$

Simplifying and separating the mass terms:

$$\begin{aligned}
&M_0\left[1-\frac{\mu}{r}-\frac{1}{2}\frac{\mu}{r}\frac{\mu}{r}+\frac{1}{2}\left(\frac{v}{c}\right)^2+\frac{3}{8}\left(\frac{v}{c}\right)^4-\frac{4}{2}\frac{\mu}{r}\left(\frac{v}{c}\right)^2\right]=M\\
&M_0\left[1-\frac{\mu}{r}-\frac{1}{2}\frac{\mu}{r}\frac{\mu}{r}+\frac{1}{2}\left(\frac{v}{c}\right)^2\left(1+\frac{3}{4}\left(\frac{v}{c}\right)^2-4\frac{\mu}{r}\right)\right]=M\\
&M_0+M_0\left[-\frac{\mu}{r}-\frac{1}{2}\frac{\mu}{r}\frac{\mu}{r}+\frac{1}{2}\left(\frac{v}{c}\right)^2\left(1+\frac{3}{4}\left(\frac{v}{c}\right)^2-4\frac{\mu}{r}\right)\right]=M\\
&\frac{M-M_0}{M_0}-\left[-\frac{\mu}{r}-\frac{1}{2}\frac{\mu}{r}\frac{\mu}{r}+\frac{1}{2}\left(\frac{v}{c}\right)^2\left(1+\frac{3}{4}\left(\frac{v}{c}\right)^2-4\frac{\mu}{r}\right)\right]=0
\end{aligned} \qquad (2.9)$$

multiplying by c^2, & noting that in a conservative system where the total energy is constant, the mass term is constant.

$$c^2\,\frac{M-M_0}{M_0}=\in \qquad (2.10)$$

Thus:

$$2\in-\left[-2c^2\left(\frac{\mu}{r}+\frac{1}{2}\frac{\mu}{r}\frac{\mu}{r}\right)+v^2\left(1+\frac{3}{4}\left(\frac{v}{c}\right)^2-4\frac{\mu}{r}\right)\right] \qquad (2.11)$$

The corresponding GR term per Robertson & Noonan.

$$2\in+\frac{2\mu c^2}{r}-v^2+\frac{2h^2\mu}{r^3} \qquad (2.12)$$

Some conventional coordinate transformations:

$$\begin{bmatrix}
u-1/r \quad u^2\left(r^2\,\dot{\theta}\right)^2=u^2h^2 \quad \left(\frac{dr}{dt}\right)^2=h^2\left(\frac{du}{d\theta}\right)^2\\
v^2=\left[\left(\frac{dr}{dt}\right)^2+u^2\left(r^2\,\dot{\theta}\right)^2\right]=\left[h^2\left(\frac{du}{d\theta}\right)^2+u^2h^2\right]
\end{bmatrix} \qquad (2.13)$$

making the substitutions, we have:

$$2\in-\left(-2c^2\left(1+\frac{1}{2}\mu u\right)\mu u+v^2\left(1-4\mu u\right)+v^2\,\frac{3}{4}\left(\frac{v}{c}\right)^2\right) \qquad (2.14)$$

Now taking the derivative with respect to the angular coordinate:

$$\frac{d}{d\theta}\left[2\in-\left(-2c^2\left(1+\frac{1}{2}\mu u\right)\mu u+\left[\left(\frac{dr}{dt}\right)^2+u^2\left(r^2\,\dot{\theta}\right)^2\right]\left(1-4\mu u\right)+v^2\,\frac{3}{4}\left(\frac{v}{c}\right)^2\right)\right],(2.15)$$

or:

$$\frac{d}{d\theta}\left[2\in-\left(\begin{array}{l}-2c^2\left(1+\frac{1}{2}\mu u\right)\mu u\\ +\left[\left(\frac{dr}{dt}\right)^2+u^2\,h^2\right]\left(1-4\mu u\right)\\ +\,v^2\,\frac{3}{4}\left(\frac{v}{c}\right)^2\end{array}\right)\right] \qquad (2.16)$$

Making some substitutions.

$$0=\frac{d}{d\theta}\left[2\in-\left(\begin{array}{l}-2c^2\left(1+\frac{1}{2}\mu u\right)\mu u\\ +h^2\left[\left(\frac{du}{d\theta}\right)^2+u^2\right]\left(1-4\mu u\right)\\ +\frac{3}{4}\frac{h^4}{c^2}\left[h^2\left(\frac{du}{d\theta}\right)^2+u^2\right]^2\end{array}\right)\right] \qquad \frac{\left(\frac{dr}{dt}\right)^2=h^2\left(\frac{du}{d\theta}\right)^2}{v^2\;=\left[h^2\left(\frac{du}{d\theta}\right)^2+u^2\,h^2\right]} \; (2.17)$$

Differentiating the three terms, designating each as A,B, & C:

$$\left[\frac{d}{d\theta}\left(-2c^2\left(1+\frac{1}{2}\mu u\right)\mu\right)=\;-2\mu c^2\left(1+\mu u\right)\frac{du}{d\theta}\;=\;-2h^2\,\frac{du}{d\theta}\,\frac{\mu c^2}{h^2}\left(1+\mu u\right)\right] \quad (A)$$

Parts of the B term:

$$\left[\begin{array}{l} \frac{d}{d\theta} h^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] \\ = \quad h^2\left[\; 2\left(\frac{du}{d\theta}\right)\left(\frac{d^2u}{d\theta^2}\right) + 2u\left(\frac{du}{d\theta}\right)\right] \\ = \left[2h^2\left(\frac{du}{d\theta}\right)\right]\left[\left(\frac{d^2u}{d\theta^2}\right) + u\right] \\ \\ \frac{d}{d\theta}(1-4\mu u) = \quad -4\mu\left(\frac{du}{d\theta}\right) \\ \\ \end{array}\right] \qquad (2.18)$$

So the B term is:

$$\left[\begin{array}{l} \frac{d}{d\theta}\left\{h^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right](1-4\mu u)\right\} \\ \\ \quad = \left[\begin{array}{l} -h^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right]4\mu\left(\frac{du}{d\theta}\right) \\ +\left[2h^2\left(\frac{du}{d\theta}\right)\right]\left[\left(\frac{d^2u}{d\theta^2}\right) + u\right](1-4\mu u) \end{array}\right] \\ \quad = \; 2h^2\left(\frac{du}{d\theta}\right)\left[\begin{array}{l} \left[\left(\frac{du}{d\theta}\right)^2 + u^2\right](-2\mu) \\ +\left[\left(\frac{d^2u}{d\theta^2}\right) + u\right](1-4\mu u) \end{array}\right] \end{array}\right] \quad (\mathrm{B}) \qquad (2.19)$$

And the C term:

$$
\left[\begin{aligned}
&\frac{d}{d\theta}\;\frac{3}{4}\frac{h^4}{c^2}\left[h^2\left(\frac{du}{d\theta}\right)^2+u^2\right]^2 \\
&\qquad = \frac{3}{4}\frac{1}{c^2}h^2h^2\,2\left(\left(\frac{du}{d\theta}\right)^2+u^2\right)\frac{d}{d\theta}\left(\left(\frac{du}{d\theta}\right)^2+u^2\right) \\
&\qquad = \frac{3}{4}\frac{1}{c^2}h^2h^2\,2\left(\left(\frac{du}{d\theta}\right)^2+u^2\right)\left(2\left(\frac{du}{d\theta}\right)\left(\frac{d^2u}{d\theta^2}\right)+2u\left(\frac{du}{d\theta}\right)\right) \\
&\qquad = 2h^2\left(\frac{du}{d\theta}\right)\frac{3}{2}\frac{1}{c^2}h^2\left(\left(\frac{du}{d\theta}\right)^2+u^2\right)\left(\left(\frac{d^2u}{d\theta^2}\right)+u\right) \\
&\qquad = \left(2h^2\left(\frac{du}{d\theta}\right)\right)\frac{3}{4}\left[2\frac{1}{c^2}h^2u^2\right]\left(\left(\frac{d^2u}{d\theta^2}\right)+u\right) \qquad \left(\frac{du}{d\theta}\right)^2 \sim 0
\end{aligned}\right] \tag{(C)(2.20)}
$$

Collecting and factoring a common term gives:

$$
0=-2h^2\left(\frac{du}{d\theta}\right)\left[\begin{array}{ll}
- & \frac{\mu c^2}{h^2}\left(1+\mu u\right) \\
+ & \left[\left(\frac{du}{d\theta}\right)^2+u^2\right]\left(-2\mu\right) \\
+ & \left[\left(\frac{d^2u}{d\theta^2}\right)+u\right]\left(1-4\mu u\right) \\
+ & \left[\left(\frac{d^2u}{d\theta^2}\right)+u\right]\frac{3}{2}\left[\frac{h^2u^2}{c^2}\right]
\end{array}\right] \tag{2.21}
$$

collecting common terms reduces the number of terms:

$$
0=\left[\begin{array}{ll}
- & \frac{\mu c^2}{h^2}\left(1+\mu u\right) \\
+ & \left[\left(\frac{du}{d\theta}\right)^2+u^2\right]\left(-2\mu\right) \\
+ & \left[\left(\frac{d^2u}{d\theta^2}\right)+u\right]\left(1+\frac{3}{2}\left[\frac{h^2u^2}{c^2}\right]-4\mu u\right)
\end{array}\right] \tag{2.22}
$$

Dividing by the coefficient of the second order term gives:

$$0=\begin{bmatrix} -\ \dfrac{\mu c^2}{h^2}(1+\mu u)/\left(1+\dfrac{3}{2}\left[\dfrac{h^2u^2}{c^2}\right]-4\mu u\right) \\ -\left[\left(\dfrac{du}{d\theta}\right)^2+u^2\right](2\mu)/\left(1+\dfrac{3}{2}\left[\dfrac{h^2u^2}{c^2}\right]-4\mu u\right) \\ +\left[\left(\dfrac{d^2u}{d\theta^2}\right)+u\right] \end{bmatrix} \tag{2.23}$$

or:

$$0=\begin{bmatrix} -\ \dfrac{\mu c^2}{h^2}\left(1+\mu u-\dfrac{3}{2}\left[\dfrac{h^2u^2}{c^2}\right]+4\mu u\right) \\ -\left(2\mu u^2\right)\left(1-\dfrac{3}{2}\left[\dfrac{h^2u^2}{c^2}\right]+4\mu u\right) \\ +\left[\left(\dfrac{d^2u}{d\theta^2}\right)+u\right] \end{bmatrix} \qquad \left(\frac{du}{d\theta}\right)^2\sim 0 \tag{2.24}$$

The equation for a circle is:

$$\left(\frac{d^2u}{d\theta^2}\right)+u=u_0+f \tag{2.25}$$

Where f is a perturbation of the orbit.

The precession, per the procedure of Robertson & Noonan is $\sigma=\frac{1}{2}\frac{\partial}{\partial u}f$.

Where in this case f is:

$$f=\begin{bmatrix} u_0\left(+\mu u-\dfrac{3}{2}\left[\dfrac{h^2u^2}{c^2}\right]+4\mu u\right) \\ \left(2\mu u^2-3\mu\left[\dfrac{h^2u^4}{c^2}\right]+8\mu^2u^3\right) \end{bmatrix} \tag{2.26}$$

Where: $$\frac{\mu c^2}{h^2} = u_0 = \frac{1}{p\left(1-e^2\right)}.$$

So:

$$\sigma = \frac{1}{2}\frac{\partial}{\partial u}\left[\begin{array}{c} u_0\left(+5\mu u - \frac{3}{2}\left[\frac{h^2u^2}{c^2}\right] + 2\mu u^2\right) \\ +\left(-3\mu\left[\frac{h^2u^4}{c^2}\right] + 8\mu^2 u^3\right) \end{array}\right] \tag{2.27}$$

Where σ is a ratio of the perihelion advance to the orbit circumference

$$\sigma = \frac{1}{2}\left[\left(\begin{array}{c} \left(+5\mu u_0 - 3\left[\frac{h^2}{c^2}\right]u_0 u + 4\mu u\right) \\ +\left(-12\mu\left[\frac{h^2u^3}{c^2}\right] + 24\mu^2u^2\right) \end{array}\right)\right] \qquad \frac{\left[\frac{h^2}{c^2}\right] = \frac{\mu}{u_0}}{\left[-\frac{1}{2}12\mu\left[\frac{\mu}{u_0}\right]u^3 + \frac{1}{2}24\mu^2 u\right] \sim 0} \tag{2.28}$$

Then we have for the precession:

$$\sigma = \left(\frac{1}{2}\mu u_0 - \frac{3}{2}\mu u_0 + 2\mu u_0 + 2\mu u^2\right) = \left(\mu u_0 + 2\mu u^2\right) \sim 3\frac{\mu}{p} \tag{2.29}$$

The units are the ratio of the advance to the orbital circumference.

Comparing with the GR value from Robertson & Noonan:

$$\sigma = \frac{1}{2}\frac{\partial}{\partial u}\left(\left(\frac{dr}{dt}\right)^2 2\frac{u}{c^2} + 2\frac{uu^2}{c^2}h^2\right) = 3\frac{u^2}{c^2}h^2 = 3\mu u\left(\frac{u}{u_0}\right) \tag{2.30}$$

Thus our procedure yields the proper perihelion precession.