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A different approach to logic: absolute logic

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Abstract

The paper is about 'absolute logic': an approach to logic that differs from the standard first-order logic and other known approaches. It should be a new approach the author has created proposing to obtain a general and unifying approach to logic and a faithful model of human mathematical deductive process. In first-order logic there exist two different concepts of term and formula, in place of these two concepts in our approach we have just one notion of expression. The set-builder notation is enclosed as an expression-building pattern. In our system we can easily express second-order, third order and any-order conditions. The meaning of a sentence will depend solely on the meaning of the symbols it contains, it will not depend on external 'structures'. Our deductive system is based on a very simple definition of proof and provides a good model of human mathematical deductive process. The soundness and consistency of the system are proved. We also discuss how our system relates to the most know types of paradoxes, from the discussion no specific vulnerability to paradoxes comes out. The paper provides both the theoretical material and a fully documented example of deduction.

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1. Introduction

This paper outlines a system or approach to mathematical logic which is different from the standard one. By 'the standard approach to logic' I mean the one presented in chapter 2 of Enderton's book [2] and there named 'First-Order Logic'. The same approach is also outlined in chapter 2 of Mendelson's book [4], where it is named 'Quantification Theory'.

We now list the features of our system, pointing out the differences and improvements with respect to standard logic.

In first-order logic there exist two different concepts of term and formula, in place of these two concepts in our approach we have just one notion of expression. Each expression is referred to a certain 'context'. A context can be seen as a (possibly empty) sequence of ordered pairs (x, φ), where x is a variable and φ is itself an expression. Given a context $k = (x_1, \varphi_1) \ldots (x_m, \varphi_m)$ we call a 'state on k' a function which assigns 'allowable values' (we'll explain this later) to the variables $x_1, \ldots, x_m$. If t is an expression with respect to context k and σ is a state on k, we'll be able to define the meaning of t with respect to k and σ, which we'll denote by $\#(k, t, \sigma)$.
Our approach requires to build all at the same time, contexts, expressions, states and meanings. We'll call sentences those expressions which are related to an empty context and whose meaning is true or false. The meaning of a sentence depends solely on the meaning of the symbols it contains, it doesn't depend on external 'structures'.

In first-order logic we have terms and formulas and we cannot apply a predicate to one or more formulas, this seems a clear limitation. With our system we can apply predicates to formulas.

When we specify a set in mathematics we often use the 'set-builder notation'. Examples of sets defined with this notation are $\{x \in \mathbb{N} |\ \exists y \in \mathbb{N} : x = 2y\}$, $\{x \in \mathbb{R} | x = x^2\}$, and so on. In our system the set-builder notation is enclosed as an expression-building pattern, and this is an advantage over standard logic.

Of course in our approach we allow connectives and quantifiers, but unlike first-order logic these are at the same level of other operators, such as equality, membership and more. While the set-builder notation is necessarily present with its role, connectives and quantifiers as 'operators' are not strictly mandatory and are part of a broader category. For instance the universal quantifier simply applies an operation of logical conjunction to a set of conditions, and so it can be classified as an operator.

In first-order logic variables range over individuals, but in mathematics there are statements in which both quantifiers over individuals and quantifiers over sets of individuals occur. One simple example is the following condition:

for each subset X of $\mathbb{N}$ and for each $x \in \mathbb{N}$ we have $x \in X$ or $x \notin X$.

Another example is the condition in which we state that every bounded, non empty set of real numbers has a supremum. Formalisms better suited to express such conditions are second-order logic and type theory, but these systems have a certain level of complexity and are based on different types of variable. In our system we can express the conditions we mentioned above, and we absolutely don't need different types of variables, the set to which the quantifier refers is explicitly written in the expression, this ultimately makes things easier and allows a more general approach. If we read the statement of a theorem in a mathematics book, usually in this statement some variables are introduced, and when introducing them often the set in which they are varying is explicitly specified, so from this point of view our approach is consistent with the actual processes of mathematics.

We have called our system 'absolute logic' and we have just seen the main reason for this: our logic is not a first-order or second-order or n-order logic, it doesn't involve types, so it can be called an 'absolute logic'.

Let's examine how our system behaves when giving a meaning and possibly a truth value to expressions. Standard logic doesn't plainly associate meanings and truth values to formulas. It introduces some related notion as the concepts of 'structure' (defined in section 2.2 of Enderton's book), truth in a structure, validity, satisfiability. Within first-order logic a structure is used, first of all, to define the collection of things to which a quantifier refers to. Moreover, some symbols such as connectives and quantifiers have a fixed meaning, while for other symbols the meaning is given by the structure. Notions such as validity and satisfiability reveal a question-based approach: 'what happens when we change the meaning of some symbols?' Although this may be an interesting perspective, this is not our approach, understanding what happens when we change the meaning of the symbols does not have a primary interest for us, although it's quite obvious that we'll also try to enunciate some results that are valid regardless of the meaning of the symbols. In this regard, if we had this perspective, in the first place it would have to be discussed if there are anyway symbols (e.g. connectives, quantifiers and others too) whose meaning cannot change.

Therefore, if a symbol is in our system, it has his own meaning, and we don't feature a notion of structure like the one of first-order logic. Also, the set of expressions in our language depends on the meaning of symbols. We'll simply speak of the meaning of an expression and when possible of the truth value of that meaning. As we've already said, the meaning of a sentence will depend solely on the meaning of the symbols it contains, it will not depend on external 'structures'.

Our deductive system seeks to provide a good model of human mathematical deductive process. The concept of proof we'll feature is probably the most simple and intuitive that comes to mind, we try to anticipate some of it.

If we define S as the set of sentences then an axiom is a subset of S, an n-ary rule is a subset of S^{n+1}. If φ is a sentence then a proof of φ is a sequence $(\psi_1, \ldots, \psi_m)$ of sentences such that

- there exists an axiom A such that $\psi_1 \in A$;
- if $m > 1$ then for each $j = 2 \ldots m$ one of the following holds
    - there exists an axiom A such that $\psi_j \in A$,
    - there exists an n-ary rule R and $i_1, \ldots, i_n < j$ such that $(\psi_{i_1}, \ldots, \psi_{i_n}, \psi_j) \in R$;
- $\psi_m = \varphi$.

Our deductive system, in order to do its job, needs to track the various hypotheses we have introduced along our proof. In a fixed moment of our reasoning we have a sequence of active hypotheses, and we need to be able to apply one of our rules. To this end our axioms and rules need to be properly constructed.

As regards the soundness of the system, it is proved at the beginning of chapter 4. Consistency, proved in chapter 7, is a direct consequence of soundness. We also discuss (in chapter 7) how the system relates with some well known paradoxes. I will not say this system can prevent any possible form of paradox, since it cannot prevent anyone to conceive something which is unsettling or contradictory. Anyway the system, as far as I can evaluate from that discussion, is not significantly affected by paradoxes, in other words I suppose it is not more vulnerable to paradoxes than other accepted systems.

We have examined the main features of the system. If the reader will ask what is the basic idea behind a system of this type, in agreement with what I said earlier I could say that the principle is to provide something like a general, absolute and unifying approach to logic and a faithful model of human mathematical deductive process.

This statement about our system of course is not a mathematical statement, so I cannot give a mathematical proof of it. On the other hand, logic exists with the specific primary purpose of being a model to human deduction. In general, suppose we want to provide a mathematical model of some process or reality. The fairness of the model can be judged much more through experience than through mathematics. In fact, mathematics always has to do with models and not directly with reality.

This paper's purpose is to present an approach to logic, but clearly we cannot provide here all possible explanations and comparisons in any way related to the approach itself. The author believes that this paper provides a fairly comprehensive presentation of the approach in question, this introduction includes significant elements of explanation, justification and comparison with the standard approach to logic. Other material in this regard is presented in the subsequent sections (for example in chapter 7).

First-order logic has been around for many decades, but to date no absolute evidence has been found that first-order logic is the best possible logic system. In this regard I may quote a stronger statement at the beginning of Josè Ferreirós' paper 'The road to modern logic – an interpretation' ([3]).

> It will be my contention that, contrary to a frequent assumption (at least among philosophers), First-Order Logic is *not* a 'natural unity', i.e. a system the scope and limits of which could be justified solely by rational argument.

Honestly, in my opinion, the approach to logic I propose seems to be a 'natural unity' much more than first-order logic is, and I did what I thought was reasonable to explain this.

Further investigations on this approach will be conducted, in the future, if and when possible, by the author and/or other people. If any claim of this introduction would seem inappropriate, the author is ready to reconsider and possibly fix it. In any case he believes the most important part of this paper is not in the introduction, but in the subsequent chapters.

The paper is quite long, but the time required to get an idea of the content is not very high. In fact, the author has chosen to include all the proofs, but quite often these are simple proofs. In addition, the most complex part is perhaps definition 3.12 which has a certain complexity, but at a first reading it is not necessary to take care of all the details.

2. Revision history

Version of July 5, 2020:

Former versions of this paper exist and are available as preprint on the web, with a slightly different title: ‘A different approach to logic’. The last of those versions is referred in our bibliography ([5]). The current version was written with the idea of obtaining a shorter and clearer definition of expression (see definition 3.12). Due to these changes in that definition, it was necessary to adapt the subsequent parts of the manuscript which were also reorganized, and other improvements were made.

Compared to the previous version, I decided not to include the part about ‘substitution’. I evaluated it was not a core goal of the manuscript to include a general treatment of substitution. Due to this evaluation and to the complexity of the topic, I decided to give up including this part. Some further discussion about this can be found in chapter 7.

Version of July 24, 2020 (current version):

Changed a part in the introduction that was a bit unclear, replaced with a better explanation.

3. Language: symbols, expressions and sentences, and their meaning

We begin to describe our language and then the expressions that characterize it. In the process of defining expressions we also define their meaning and the context to which the expression refers. The expressions of our language are constructed from some set of symbols according to certain rules. Expressions are sequences of symbols and they have a meaning, 'sentences' are specific expression whose meaning has the property of being true or false. We begin by describing the sets of symbols we need.

First we need a set of symbols $\mathcal{V}$. $\mathcal{V}$ members are also called 'variables' and just play the role of variables in the construction of our expressions (this implies that $\mathcal{V}$ members have no meaning associated).

In addition we need another set of symbols $\mathcal{C}$. $\mathcal{C}$ members are also called 'constants' and have a meaning. For each $c \in \mathcal{C}$ we denote by $\#(c)$ the meaning of c.

Let f be a member of $\mathcal{C}$. Being f endowed with meaning, f is always an expression of our language. However, the meaning of f could also be a function. In this case f can also play the role of an 'operator' in the construction of expressions that are more complex than the simple constant f.

Not all the operators that we need, however, are identifiable as functions. Think to the logical connectives (logical negation, logical implication, quantifiers, etc..), but also to the membership predicate '$\in$' and to the equality predicate '$=$'. The meaning of these operators cannot be mapped to a precise mathematical object, therefore these operators won't have a precise meaning in our language, but we'll need to give meaning to the application of the operator to objects, where the operator is applicable.

In mathematics and in the real world objects can have properties, such as having a certain color, or being true, or being false. A property is therefore something that can be assigned to an object, no object, more than one object. For example, with reference to color, one or more objects are red or have the property 'to be of red color'. But more generally one or more objects have a color. Suppose to indicate, for objects x that have a color, the color of x with $C(x)$. So we can say that C is a property applicable to a class of objects. On the same object class we can indicate with $R(x)$ the condition 'x has the red color'. R is in turn a property applicable to a class of objects, with the characteristic that for all x $R(x)$ is true or false. A property with this additional feature can be called a 'predicate'.

The class of objects to which a property may be assigned may be called the domain of the property. The members of that domain may be individual objects or sequences of objects, for example, if x is an object and X is a set, the condition '$x \in X$' involves two

objects, and then the domain of the membership property consists of the ordered pairs (x, X), where x is an object and X is a set.
Generally we are dealing with properties such that the objects of their domain are all individual objects, or all ordered pairs. Theoretically there may also be properties such that the objects of their domain are sequences of more than two items or even the number of items in sequence may be different in different elements of the domain.

As mentioned above the concept of 'property' is similar to the concept of function, but in mathematics there are properties that are not functions. For example, the condition '$x \in X$' just introduced can be applied to an arbitrary object and an arbitrary set, so the 'membership property' has not a well determined domain and cannot be considered a function in a strict sense.

So to build our language we need another set of symbols $\mathcal{F}$, where each f in $\mathcal{F}$ represents a property P_f. Symbols in $\mathcal{F}$ are also called operators or 'property symbols'. We will not assign a meaning to operators, because a property cannot be mapped to a consistent mathematical object (function or other). However, for each f

- for each positive integer n and $x_1, \ldots, x_n$ arbitrary objects we must know the condition $A_f(x_1, \ldots, x_n)$ that indicates if P_f is applicable to $x_1, \ldots, x_n$;
- for each positive integer n and $x_1, \ldots, x_n$ arbitrary objects such that $A_f(x_1, \ldots, x_n)$ holds we must know the value of $P_f(x_1, \ldots, x_n)$.

Since the concept might be unclear we immediately explain it by specifying what are the most important operators that we may include in our language, providing for each of them the conditions $A_f(x_1, \ldots, x_n)$ and $P_f(x_1, \ldots, x_n)$ (in general $P_f(x_1, \ldots, x_n)$ is a generic value, but in these cases it is a condition, i.e. its value can be true or false).

- Logical conjunction: it's the symbol $\wedge$ and we have
  for $n \neq 2$ $A_\wedge(x_1, \ldots, x_n)$ is false ,
  $A_\wedge(x_1, x_2) =$ (x_1 is true or x_1 is false) and (x_2 is true or x_2 is false),
  $P_\wedge(x_1, x_2) =$ both x_1 and x_2 are true ;
- Logical disjunction: it's the symbol $\vee$ and we have
  for $n \neq 2$ $A_\vee(x_1, \ldots, x_n)$ is false ,
  $A_\vee(x_1, x_2) =$ (x_1 is true or x_1 is false) and (x_2 is true or x_2 is false),
  $P_\vee(x_1, x_2) =$ at least one between x_1 and x_2 is true ;
- Logical implication: it's the symbol $\rightarrow$ and we have
  for $n \neq 2$ $A_\rightarrow(x_1, \ldots, x_n)$ is false ,
  $A_\rightarrow(x_1, x_2) =$ (x_1 is true or x_1 is false) and (x_2 is true or x_2 is false),
  $P_\rightarrow(x_1, x_2) = x_1$ is false or x_2 is true ;
- Double logical implication: it's the symbol $\leftrightarrow$ and we have
  for $n \neq 2$ $A_\leftrightarrow(x_1, \ldots, x_n)$ is false ,

$A_{\leftrightarrow}(x_1, x_2) =$ (x_1 is true or x_1 is false) and (x_2 is true or x_2 is false),
$P_{\leftrightarrow}(x_1, x_2) = P_{\rightarrow}(x_1, x_2)$ and $P_{\rightarrow}(x_2, x_1)$;

- Logical negation: it's the symbol $\neg$ and we have
  for $n > 1$ $A_{\neg}(x_1, \ldots, x_n)$ is false ,
  $A_{\neg}(x_1)$ is true,
  $P_{\neg}(x_1) = x_1$ is false ;

- Universal quantifier: it's the symbol $\forall$ and we have
  for $n > 1$ $A_{\forall}(x_1, \ldots, x_n)$ is false ,
  $A_{\forall}(x_1) = x_1$ is a set and for each x in x_1 (x is true or x is false),
  $P_{\forall}(x_1) =$ for each x in x_1 (x is true) .

- Existential quantifier: it's the symbol $\exists$ and we have
  for $n > 1$ $A_{\exists}(x_1, \ldots, x_n)$ is false ,
  $A_{\exists}(x_1) = x_1$ is a set and for each x in x_1 (x is true or x is false),
  $P_{\exists}(x_1) =$ there exists x in x_1 such that (x is true) .

- Membership predicate: it's the symbol $\in$ and we have
  for $n \neq 2$ $A_{\in}(x_1, \ldots, x_n)$ is false ,
  $A_{\in}(x_1, x_2) = x_2$ is a set,
  $P_{\in}(x_1, x_2) = x_1$ is a member of x_2 ;

- Equality predicate: it's the symbol $=$ and we have
  for $n \neq 2$ $A_{=}(x_1, \ldots, x_n)$ is false ,
  $A_{=}(x_1, x_2)$ is true,
  $P_{=}(x_1, x_2) = x_1$ is equal to x_2 .

We can think and use also other operators, for instance operations between sets such as union or intersection can be represented through an operator, etc. .

In the standard approach to logic, quantifiers are not treated like the other logical connectives, but in this system we mean to separate the operation of applying a quantifier from the operation whereby we build the set to which the quantifier is applied, and therefore the quantifier is treated as the other logical operators (altogether, the universal quantifier is simply an extension of logical conjunction, the existential quantifier is simply an extension of logical disjunction).

With regard to the operation of building a set, we need a specific symbol to indicate that we are doing this, this symbol is the symbol '$\{\}$' which we will consider as a unique symbol.

Besides the set builder symbol, we need parentheses and commas to avoid ambiguity in the reading of our expressions; to this end we use the following symbols: left parenthesis '(', right parenthesis ')', comma ',' and colon ':'. We can indicate this further set of symbols with $\mathcal{Z}$.

To avoid ambiguity in reading our expressions we require that the sets $\mathcal{V}$, $\mathcal{C}$, $\mathcal{F}$ and $\mathcal{Z}$ are disjoint. It's also requested that a symbol does not correspond to any chain of more symbols of the language. More generally, given $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_m$ symbols of our language, and using the symbol '$\|$' to indicate the concatenation of characters and strings, we assume that equality of the two chains $\alpha_1 \| \ldots \| \alpha_n$ and $\beta_1 \| \ldots \| \beta_m$ is achieved when and only when $m = n$ and for each $i = 1 \ldots n$ $\alpha_i = \beta_i$.

While the set $\mathcal{Z}$ will be always the same, the sets $\mathcal{V}$, $\mathcal{C}$, $\mathcal{F}$ may change according to what is the language that we describe. To describe our language it is required to know the sets $\mathcal{V}$, $\mathcal{C}$, $\mathcal{F}$ and the function $\#$ which associates a meaning to every element of $\mathcal{C}$. In other words, our language is identified by the 4-tuple $(\mathcal{V}, \mathcal{F}, \mathcal{C}, \#)$. Since the 'meaning' of an operator is not a mathematical object, operators must be seen as symbols which are tightly coupled with their meaning.

Before we can describe the process of constructing expressions we still need to introduce some notation. In fact in that process we'll use the notion of 'context' and the notion of 'state'. Context and states have a similar form, and here we want to define their common form.

We define $\mathcal{D} = \{\emptyset\} \cup \{\{1, \ldots, m\} |\ m \text{ is a positive integer}\}$.

Suppose x is a function whose domain $dom(x)$ belongs to $\mathcal{D}$. Suppose $C \in \mathcal{D}$ is such that $C \subseteq dom(x)$. Then we define $x_{/C}$ as a function whose domain is C and such that for each $j \in C$ $x_{/C}(j) = x(j)$.

Suppose x and φ are two functions with the same domain D, and $D \in \mathcal{D}$. Then we say that (x, φ) is a '*state-like pair*'.

Given a state-like pair $k = (x, \varphi)$ the domain of x will be also called the *domain of k*. Therefore $dom(k) = dom(x) = dom(\varphi)$.

Furthermore $dom(k) \in \mathcal{D}$ and given $C \in \mathcal{D}$ such that $C \subseteq dom(k)$ we can define $k_{/C} = (x_{/C}, \varphi_{/C})$. Clearly $k_{/C}$ is a state-like pair.

We define $\mathcal{R}(k) = \{k_{/C} |\ C \in \mathcal{D}, C \subseteq dom(k)\}$.

Given another state-like pair h we write $h \sqsubseteq k$ if and only if $h \in \mathcal{R}(k)$.

Suppose $h \in \mathcal{R}(k)$, then there exists $C \in \mathcal{D}$ such that $C \subseteq dom(k)$, $h = k_{/C} = (x_{/C}, \varphi_{/C})$. Therefore $dom(h) = C$ and $k_{/dom(h)} = k_{/C} = h$.

Suppose $h \in \mathcal{R}(k)$ and $g \in \mathcal{R}(h)$. This means there exist $C \in \mathcal{D}$ such that $C \subseteq dom(k)$, $h = k_{/C}$, and there exist $D \in \mathcal{D}$ such that $D \subseteq dom(h)$, $g = h_{/D}$. So $D \subseteq dom(h) = C \subseteq dom(k)$, $g = (k_{/C})_{/D} = (x_{/C}, \varphi_{/C})_{/D} = (x_{/D}, \varphi_{/D}) = k_{/D}$. Therefore $g \in \mathcal{R}(k)$.

Suppose $k = (x, \varphi)$ is a state-like pair whose domain is D. Suppose (y, ψ) is an ordered pair. Then we can define the 'addition' of (y, ψ) to k.
Suppose $D = \{1, \ldots, m\}$, then we define $D' = \{1, \ldots, m+1\}$. We define x' as a function whose domain is D' such that for each $\alpha = 1 \ldots m$ $x'(\alpha) = x(\alpha)$, and $x'(m+1) = y$. We define φ' as a function whose domain is D' such that for each $\alpha = 1 \ldots m$ $\varphi'(\alpha) = \varphi(\alpha)$, $\varphi'(m+1) = \psi$. Then we define $k + (y, \psi) = (x', \varphi')$. Obviously $(k + (y, \psi))_{/\{1,\ldots,m\}} = k$, so $k \in \mathcal{R}(k + (y, \psi))$.
If $D = \emptyset$ then clearly $D' = \{1\}$. We define x' as a function whose domain is D' such that $x'(1) = y$. We define φ' as a function whose domain is D' such that $\varphi'(1) = \psi$. Then we define $k + (y, \psi) = (x', \varphi')$. Obviously $(k + (y, \psi))_{/\emptyset} = \emptyset = k$, so $k \in \mathcal{R}(k + (y, \psi))$.
In both cases $k + (y, \psi)$ is a state-like pair, and $k \in \mathcal{R}(k + (y, \psi))$, which implies $dom(k) \subseteq dom(k + (y, \psi))$.
We have also seen that $(k + (y, \psi))_{/dom(k)} = (k + (y, \psi))_{/D} = k$.

We also define $\epsilon = (\emptyset, \emptyset)$, so ϵ is a state-like pair.

Given a state-like pair $k = (x, \varphi)$ we define $var(k)$ as the image of the function x. In other words if $k = \epsilon$ then $x = \emptyset$, so $var(k) = \emptyset$, otherwise x has a domain $\{1, \ldots, m\}$ and $var(k) = \{x_i | i = 1 \ldots m\}$.

Clearly, if we assume that $k + (y, \psi) = (x', \varphi')$, we can easily see that

$$var(k + (y, \psi)) = \{x'_i | i \in dom(x'_i)\} = \{x_i | i \in dom(x_i)\} \cup \{y\} = var(k) \cup \{y\}.$$

In the next lemma we prove that, when a state-like pair is obtained as $k + (y, \psi)$, then k, y, and ψ are univocally determined.

Lemma 3.1. *Suppose $k_1 = (x_1, \varphi_1)$ is a state-like pair whose domain is D_1, and (y_1, ψ_1) is an ordered pair. Suppose $k_2 = (x_2, \varphi_2)$ is a state-like pair whose domain is D_2, and (y_2, ψ_2) is an ordered pair. Finally suppose $k_1 + (y_1, \psi_1) = k_2 + (y_2, \psi_2)$. Under these assumptions we can prove that $k_1 = k_2, y_1 = y_2, \psi_1 = \psi_2$.*

Proof.
We define $h = k_1 + (y_1, \psi_1) = k_2 + (y_2, \psi_2)$. Since $h = k_1 + (y_1, \psi_1)$ we can have two possibilities:

- $D_1 = \emptyset, D'_1 = \{1\}$ and there exist two functions x'_1 and φ'_1 whose domain is D'_1 such that $h = (x'_1, \varphi'_1)$;
- there exists a positive integer m_1 such that $D_1 = \{1, \ldots, m_1\}, D'_1 = \{1, \ldots, m_1+1\}$ and there exist two functions x'_1 and φ'_1 whose domain is D'_1 such that $h = (x'_1, \varphi'_1)$.

Similarly, since $h = k_2 + (y_2, \psi_2)$ we can have two possibilities:

- $D_2 = \emptyset, D'_2 = \{1\}$ and there exist two functions x'_2 and φ'_2 whose domain is D'_2 such that $h = (x'_2, \varphi'_2)$;
- there exists a positive integer m_2 such that $D_2 = \{1, \ldots, m_2\}, D'_2 = \{1, \ldots, m_2+1\}$ and there exist two functions x'_2 and φ'_2 whose domain is D'_2 such that $h = (x'_2, \varphi'_2)$.

It follows that $(x'_1, \varphi'_1) = h = (x'_2, \varphi'_2)$, so $x'_1 = x'_2$ and $\varphi'_1 = \varphi'_2$, and $D'_1 = D'_2$.

Suppose $D_1 = \emptyset$. This implies that $D'_2 = D'_1 = \{1\}$, thus $D_2 = \emptyset$.
In this case $k_1 = \epsilon = k_2$, $y_1 = x'_1(1) = x'_2(1) = y_2$, $\psi_1 = \varphi'_1(1) = \varphi'_2(1) = \psi_2$.

Suppose there exists a positive integer m_1 such that $D_1 = \{1, \ldots, m_1\}$. This implies that $D'_2 = D'_1 = \{1, \ldots, m_1 + 1\}$, thus $D_2 = \{1, \ldots, m_1\}$.
In this case for each $\alpha = 1 \ldots m_1$ $x_1(\alpha) = x'_1(\alpha) = x'_2(\alpha) = x_2(\alpha)$, $\varphi_1(\alpha) = \varphi'_1(\alpha) = \varphi'_2(\alpha) = \varphi_2(\alpha)$. So $k_1 = (x_1, \varphi_1) = (x_2, \varphi_2) = k_2$; and moreover $y_1 = x'_1(m_1 + 1) = x'_2(m_1 + 1) = y_2$, $\psi_1 = \varphi'_1(m_1 + 1) = \varphi'_2(m_1 + 1) = \psi_2$. ■

Other useful results are the following.

LEMMA 3.2. *Suppose* $h = (x, \varphi)$, $k = (z, \psi)$ *are state-like pairs such that* $h \in \mathcal{R}(k)$ *and for each* $i, j \in dom(k)$ $i \neq j \to z_i \neq z_j$. *Then, for each* $i \in dom(k)$, $j \in dom(h)$ $z_i = x_j \to \psi_i = \varphi_j$.

Proof. Let $i \in dom(k)$, $j \in dom(h)$ and $z_i = x_j$. Clearly $j \in dom(k)$, $x_j = z_j$, thus $z_i = z_j$, $i = j$, $\varphi_j = \psi_j = \psi_i$. ■

LEMMA 3.3. *Suppose* $k = (x, \varphi)$ *and* $h = (y, \psi)$ *are state-like pairs such that for each* $i \in dom(k)$, $j \in dom(h)$ $x_i = y_j \to \varphi_i = \psi_j$. *Suppose* (u, θ) *is an ordered pair and* $u \notin var(k)$, $u \notin var(h)$. *Let* $k' = k + (u, \theta)$ *and* $h' = h + (u, \theta)$. *Let also* $k' = (x', \varphi')$ *and* $h' = (y', \psi')$, *then for each* $i \in dom(k')$, $j \in dom(h')$ $x'_i = y'_j \to \varphi'_i = \psi'_j$.

Proof. Let $i \in dom(k')$, $j \in dom(h')$ such that $x'_i = y'_j$.

Suppose $i \in dom(k)$. If $j \notin dom(h)$ then $x'_i = x_i \in var(k)$, $y'_j = u \notin var(k)$ so $x'_i \neq y'_j$. So $j \in dom(h)$ and $\varphi'_i = \varphi_i = \psi_j = \psi'_j$.

Suppose $i \notin dom(k)$. If $j \in dom(h)$ then $x'_i = u \notin var(h)$ and $y'_j = y_j \in var(h)$, so $x'_i \neq y'_j$. Then obviously also $j \notin dom(h)$ and $\varphi'_i = \theta = \psi'_j$. ■

LEMMA 3.4. *Suppose* $k = (x, \varphi)$ *and* $h = (y, \vartheta)$ *are state-like pairs such that for each* $i \in dom(k)$, $j \in dom(h)$ $x_i = y_j \to \varphi_i = \vartheta_j$. *Suppose* $\kappa = (z, \phi) \sqsubseteq k$ *and* $g = (w, \theta) \sqsubseteq h$. *Then for each* $i \in dom(\kappa)$, $j \in dom(g)$ $z_i = w_j \to \phi_i = \theta_j$.

Proof.

There exists $C \in \mathcal{D}$ such that $C \subseteq dom(k)$, $\kappa = k_{/C} = (x_{/C}, \varphi_{/C})$. Therefore $dom(\kappa) = C \subseteq dom(k)$.

Similarly there exists $D \in \mathcal{D}$ such that $D \subseteq dom(h)$, $g = h_{/D} = (y_{/D}, \vartheta_{/D})$. Therefore $dom(g) = D \subseteq dom(h)$.

Let $i \in dom(\kappa)$, $j \in dom(g)$, $z_i = w_j$, then $i \in dom(k)$, $j \in dom(h)$,

$$x_i = (x_{/C})_i = z_i = w_j = (y_{/D})_j = y_j \ .$$

Then

$$\phi_i = (\varphi_{/C})_i = \varphi_i = \vartheta_j = (\vartheta_{/D})_j = \theta_j \ .$$

■

Lemma 3.5. *Suppose $h = (x, \varphi)$ is a state-like pair, (y, ϕ) is an ordered pair and define $k = h + (y, \phi)$. Suppose $g \in \mathcal{R}(k)$ is such that $g \neq k$. Then $g \in \mathcal{R}(h)$.*

Proof.

Let $D = dom(h)$.

Suppose m is a positive integer and $D = \{1, \ldots, m\}$. Then $k = (x', \varphi')$ has a domain $\{1, \ldots, m+1\}$. Moreover there exists $C \in \mathcal{D}$ such that $C \subseteq \{1, \ldots, m+1\}$ and $g = k_{/C}$. Since $g \neq k$ we must have $C \subseteq \{1, \ldots, m\}$. We have

$$g = k_{/C} = (x'_{/C}, \varphi'_{/C}) = ((x'_{/D})_{/C}, (\varphi'_{/D})_{/C}) = (x_{/C}, \varphi_{/C}) = h_{/C} \ .$$

Now suppose $D = \emptyset$. Then $k = (x', \varphi')$ has a domain $\{1\}$. Moreover there exists $C \in \mathcal{D}$ such that $C \subseteq \{1\}$ and $g = k_{/C}$. Since $g \neq k$ we must have $C = \emptyset$ and $g = (\emptyset, \emptyset) = h$.

In both cases $g \in \mathcal{R}(h)$. ■

Lemma 3.6. *Let x be a function such that $dom(x) \in \mathcal{D}$, let $C, D \in \mathcal{D}$ such that $C \subseteq D \subseteq dom(x)$. Then we can define $x_{/C}$ and $(x_{/D})_{/C}$, and we have $(x_{/D})_{/C} = x_{/C}$.*

Proof. Define $y = x_{/D}$, we have $dom(y) = D$ and for each $j \in D$ $y(j) = x(j)$. Moreover $dom(y_{/C}) = C = dom(x_{/C})$ and for each $j \in dom(C)$ $y_{/C}(j) = y(j) = x(j) = x_{/C}(j)$. ■

Lemma 3.7. *Let $k = (x, \varphi)$ be a state-like pair, let $C, D \in \mathcal{D}$ such that $C \subseteq D \subseteq dom(k)$. Then we can define $k_{/C}$ and $(k_{/D})_{/C}$, and we have $(k_{/D})_{/C} = k_{/C}$.*

Proof.

$$(k_{/D})_{/C} = (x_{/D}, \varphi_{/D})_{/C} = ((x_{/D})_{/C}, (\varphi_{/D})_{/C}) = (x_{/C}, \varphi_{/C}) = k_{/C}.$$

■

Lemma 3.8. *Let g, h, k be state-like pairs, let $g \sqsubseteq h$, $h \sqsubseteq k$. Then $g \sqsubseteq k$.*

Proof.

There exists $C \in \mathcal{D}$ such that $C \subseteq dom(h)$, $g = h_{/C}$.
There exists $D \in \mathcal{D}$ such that $D \subseteq dom(k)$, $h = k_{/D}$.

This implies that $C \subseteq dom(h) = D$, so $g = h_{/C} = (k_{/D})_{/C} = k_{/C}$.

Since $C \subseteq dom(k)$, $g \sqsubseteq k$. ■

Lemma 3.9. *Let g, h and $k = (x, \varphi)$ be state-like pairs such that $g, h \in \mathcal{R}(k)$, $dom(g) \subseteq dom(h)$. Then $g \in \mathcal{R}(h)$.*

Proof. There exists $C \in \mathcal{D}$ such that $C \subseteq dom(k)$, $g = k_{/C}$. And there exists $D \in \mathcal{D}$ such that $D \subseteq dom(k)$, $h = k_{/D}$. It results $C = dom(g) \subseteq dom(h) = D$. Then, clearly

$$g = (x, \varphi)_{/C} = (x_{/C}, \varphi_{/C}) = ((x_{/D})_{/C}, (\varphi_{/D})_{/C}) = (x_{/D}, \varphi_{/D})_{/C} = h_{/C} \ .$$

■

LEMMA 3.10. *Suppose $h = (x, \varphi)$ is a state-like pair, (y, ϕ) is an ordered pair and define $k = h + (y, \phi)$. Then $k_{/dom(h)} = h$.*

Proof.

Let $D = dom(h)$ and $k = (x', \varphi')$. Then $k_{/dom(h)} = (x'_{/D}, \varphi'_{/D}) = (x, \varphi) = h$.

■

We also need some notation referred to generic strings, this notation will be useful when applied to our expressions, which are non-empty strings. If t is a string we can indicate with $\ell(t)$ t's length, i.e. the number of characters in t. If $\ell(t) > 0$ then for each $\alpha \in \{1, \ldots, \ell(t)\}$ at position α within t there is a character, this symbol will be indicated with $t[\alpha]$. We call 'depth of α within t' (briefly $d(t, \alpha)$) the number which is obtained by subtracting the number of right round brackets ')' that occur in t before position α from the number of left round brackets '(' that occur in t before position α .
The following lemma will be useful later within proofs of unique readability. Its proof is so simple that we feel free to omit it.

LEMMA 3.11. *Let ϑ, φ, η be strings with $\ell(\vartheta) > 0$, $\ell(\varphi) > 0$, and let $t = \vartheta \| \varphi \| \eta$; let also $\alpha \in \{1, \ldots, \ell(\varphi)\}$. The following result clearly holds:*

$$d(t, \ell(\vartheta) + \alpha) = d(t, \ell(\vartheta) + 1) + d(\varphi, \alpha).$$

■

We can now describe the process of constructing expressions for our language $\mathcal{L}$. This is an inductive process in which not only we build expressions, but also we associate them with meaning, and in parallel also define the fundamental concept of 'context'. This process will be identified as 'Definition 3.12' although actually it is a process in which we give the definitions and prove properties which are needed in order to set up those definitions.

3.1. Definition process. This section contains only definition 3.12. This definition is an inductive definition process within which we have assumptions, lemmas etc.. Symbols like ■ within this definition are not intended to terminate the definition, they just terminate an assumption or lemma etc. which is internal to the definition.

DEFINITION 3.12. Since this is a complex definition, we will first try to provide an informal idea of the entities we'll define in it. The definition is by induction on positive integers, we now introduce the sets and concepts we'll define for a generic positive integer n (this first listing is not the true definition, it's just to introduce the concepts, to enable the reader to understand their role).

$K(n)$ is the set of 'contexts' at step n. A context k is a state-like pair of the form (x, φ) where x and φ have the same domain $D = \{1, \ldots, m\} \in \mathcal{D}$, and for each $i = 1 \ldots m$ x_i is a variable and φ_i is an expression.

For each $k \in K(n)$ $\Xi(k)$ is the set of 'states' bound to context k. If $n > 1$ and $k \in K(n-1)$ then $\Xi(k)$ has already been defined at step $n-1$ or formerly, otherwise it will be defined at step n.

If $k = (x, \varphi)$ is a context, a state on k is a state-like pair $\sigma = (x, s)$ where (roughly speaking) for each i in the domain of x, φ and s s_i is a member of the meaning of the corresponding expression φ_i .

For each $k \in K(n)$ $E(n, k)$ is the set of expressions bound to step n and context k.

$E(n)$ is the union of $E(n, k)$ for $k \in K(n)$ (this will not be explicitly recalled on each iteration in the definition).

For each $k \in K(n)$, $t \in E(n, k)$, $\sigma \in \Xi(k)$ we'll define $\#(k, t, \sigma)$ which stands for 'the meaning of t bound to k and σ'.

The following set $E_s(n, k)$ should be defined in the same way at each step, we put here its definition, to avoid to repeat that definition each time.
For each $k \in K(n)$ we define $E_s(n, k) = \{t | t \in E(n, k), \forall \sigma \in \Xi(k)\, \#(k, t, \sigma)$ is a set$\}$.

We are now are ready to begin the actual definition process, so we perform the simple initial step of our inductive process.

We define $K(1) = \{\epsilon\}$, $\mathit{\Xi}(\epsilon) = \{\epsilon\}$, $E(1, \epsilon) = \mathcal{C}$.
For each $t \in E(1, \epsilon)$ we define $\#(\epsilon, t, \epsilon) = \#(t)$.

The inductive step is a bit more complex. Suppose all our definitions have been given at step n and let's proceed with step $n + 1$. In this inductive step we'll need some assumptions which will be identified with a title like 'Assumption 2.1.x'. Each assumption is a statement that must be valid at step 1, we suppose is valid at step n and needs to be proved true at step $n + 1$ at the end of our definition process.

The first two assumptions we need are the following.

Assumption 3.1.1. For each $k \in K(n)$ k is a state-like pair and for each $\sigma \in \mathit{\Xi}(k)$ σ is a state-like pair and $dom(\sigma) = dom(k)$.

■

Assumption 3.1.2. For each $k \in K(n)$ $k = \epsilon$ and $\Xi(k) = \{\epsilon\}$ or ($n > 1$ and there exist $m < n$, $h \in K(m)$, $\phi \in E_s(m,h)$, $y \in (\mathcal{V} - var(h))$ such that $k = h + (y,\phi)$, $\Xi(k) = \{\sigma + (y,s) | \, \sigma \in \Xi(h), s \in \#(h,\phi,\sigma)\}$).

∎

We can go on with the inductive step and define

$$K(n)^+ = \{h + (y,\phi) | \, h \in K(n), \phi \in E_s(n,h), y \in (\mathcal{V} - var(h))\} - K(n)\,,$$

$$K(n+1) = K(n) \cup K(n)^+\,.$$

Let $k \in K(n)^+$. Then there exist $h \in K(n), \phi \in E_s(n,h), y \in (\mathcal{V} - var(h))$ such that $k = h + (y,\phi)$. By lemma 3.1 we know that h, ϕ, y are *univocally determined.*

We can assume that $\Xi(k)$ is defined for $k \in K(n)$, and we need to define this for $k \in K(n+1) - K(n)$, i.e. for $k \in K(n)^+$. If $k \in K(n)^+$ there exist $h \in K(n), \phi \in E_s(n,h), y \in (\mathcal{V} - var(h))$ such that $k = h + (y,\phi)$; and h, ϕ, y are univocally determined. So we can define

$$\Xi(k) = \{\sigma + (y,s) | \, \sigma \in \Xi(h), s \in \#(h,\phi,\sigma)\}\,.$$

Another consequence of lemma 3.1 is the following: for each $k \in K(n)^+$ and $\sigma + (y,s)$ in $\Xi(k)$, σ, y and s are *univocally determined.*

To ensure the unique readability of our expressions we need the following assumption (which is clearly satisfied for $n = 1$).

Assumption 3.1.3. For each $t \in E(n)$

- $t[\ell(t)] \neq$ '(' ;
- if $t[\ell(t)] =$ ')' then $d(t,\ell(t)) = 1$, else $d(t,\ell(t)) = 0$;
- for each $\alpha \in \{1, \ldots, \ell(t)\}$ if ($t[\alpha] =$ ':') $\vee$ ($t[\alpha] =$ ',') $\vee$ ($t[\alpha] =$ ')') then $d(t,\alpha) \geqslant 1$.

∎

It is time to define $E(n+1,k)$, for each k in $K(n+1)$. Then for each t in $E(n+1,k)$ and σ in $\Xi(k)$ we need to define $\#(k,t,\sigma)$. We begin to do this by defining some new sets of expressions bound to context k, and for the expressions in each new set we define the proposed value of $\#(k,t,\sigma)$.

For each $k = h + (y,\phi) \in K(n)^+$ we define

$$E_a(n+1,k) = \{y\}.$$

For each $t \in E_a(n+1,k)$, $\sigma = \rho + (y,s) \in \Xi(k)$ we define:

$$\#(k,t,\sigma)_{(n+1,k,a)} = s.$$

We notice that $\epsilon \in K(n)$ and define $E_b(n+1,\epsilon) = \emptyset$.

For each $k = h + (y,\phi) \in K(n) - \{\epsilon\}$ we define

$$E_b(n+1,k) = \{t | t \in E(n,h), t \notin E(n,k)\}.$$

For each $t \in E_b(n+1,k)$, $\sigma = \rho + (y,s) \in \Xi(k)$ we define the proposed value of $\#(k,t,\sigma)$:

$$\#(k,t,\sigma)_{(n+1,k,b)} = \#(h,t,\rho).$$

As a premise to the following definition of $E_c(n+1,k)$, we specify that, given a positive integer m and a set D, we call D^m the set $D \times \cdots \times D$ where D appears m times (when $m = 1$ of course $D^1 = D$), and a function whose domain is a subset of D^m is called a *function with m arguments.*

For each $k \in K(n)$ we define $E_c(n+1,k)$ as the set of the strings $(\varphi)(\varphi_1,\ldots,\varphi_m)$ such that:

- m is a positive integer;
- $\varphi,\varphi_1,\ldots,\varphi_m \in E(n,k)$;
- for each $\sigma \in \Xi(k)$ $\#(k,\varphi,\sigma)$ is a function with m arguments and $(\#(k,\varphi_1,\sigma),\ldots,\#(k,\varphi_m,\sigma))$ is a member of its domain;
- $(\varphi)(\varphi_1,\ldots,\varphi_m) \notin E(n,k)$;
- $(\varphi)(\varphi_1,\ldots,\varphi_m) \notin E_b(n+1,k)$.

This means that for each $t \in E_c(n+1,k)$ there exist a positive integer m and $\varphi,\varphi_1,\ldots,\varphi_m \in E(n)$ such that $t = (\varphi)(\varphi_1,\ldots,\varphi_m)$. In the following lemma we'll show that $m,\varphi,\varphi_1,\ldots,\varphi_m$ are uniquely determined. Within this complex definition this proof of unique readability may be considered as a technical detail, and can be skipped at first reading. The proof will often exploit lemma 3.11 and assumption 3.1.3, they will not be quoted each time they are used.

Lemma 3.1.4. Let $t \in E_c(n+1,k)$ and suppose

- there exist a positive integer m and $\varphi,\varphi_1,\ldots,\varphi_m \in E(n)$: $t = (\varphi)(\varphi_1,\ldots,\varphi_m)$.
- there exist a positive integer p and $\psi,\psi_1,\ldots,\psi_p \in E(n)$: $t = (\psi)(\psi_1,\ldots,\psi_p)$.

Then $p = m$, $\psi = \varphi$ and for each $i \in \{1,\ldots,m\}$ $\psi_i = \varphi_i$.

NB: in the context of this definition we just need to prove this lemma for $t \in E_c(n+1,k)$, but actually we just need to assume t is a generic string, and below in this paper we'll also use this result with reference to a generic string t.

Proof.

We can represent t as $(\varphi)(\varphi_1, \ldots, \varphi_m)$ and as $(\psi)(\psi_1, \ldots, \psi_p)$.

In both representations we see 'explicit occurrences' of the symbols '(' , ')' and ','.

In the first representation there are explicit occurrences of ',' only when $m > 1$. We indicate with q the position of the first explicit occurrence of ')', and the second explicit occurrence of ')' is clearly in position $\ell(t)$. If $m > 1$ we indicate with $q_1, \ldots, q_{m-1}$ the positions of the explicit occurrences of ','.

In the second representation there are explicit occurrences of ',' only when $p > 1$. We indicate with r the position of the first explicit occurrence of ')', and the second explicit occurrence of ')' is clearly in position $\ell(t)$. If $p > 1$ we indicate with $r_1, \ldots, r_{p-1}$ the positions of the explicit occurrences of ','.

We have $d(t, q-1) = d(t, 1+\ell(\varphi)) = d(t, 1+1) + d(\varphi, \ell(\varphi)) = 1 + d(\varphi, \ell(\varphi))$.

If $t[q-1] = \varphi[\ell(\varphi)] = \text{`)'}$ then $d(t, q) = d(t, q-1) - 1 = d(\varphi, \ell(\varphi)) = 1$.
Else $t[q-1] = \varphi[\ell(\varphi)] \notin \{\text{`('}, \text{`)'}\}$, so $d(t, q) = d(t, q-1) = 1 + d(\varphi, \ell(\varphi)) = 1$.

Suppose $q < r$. Obviously $q > 1$, $q - 1 \geqslant 1$, $q - 1 \leqslant r - 2 = \ell(\psi)$; $\psi[q-1] = t[q] = \text{`)'}$.
So $d(t, q) = d(t, 1 + (q-1)) = d(t, 2) + d(\psi, q-1) = 1 + d(\psi, q-1) \geqslant 2$.
This is a contradiction, because we have proved $d(t, q) = 1$. Thus $q \geqslant r$.

In the same way we can prove that $r \geqslant q$, so we have $r = q$.

Clearly $\ell(\psi) = r-2 = q-2 = \ell(\varphi)$, and for each $\alpha = 1 \ldots \ell(\varphi)$ $\varphi[\alpha] = t[\alpha+1] = \psi[\alpha]$.
In other words $\psi = \varphi$.

Of course we have also $d(t, r) = d(t, q) = 1$, $d(t, r+2) = d(t, r) - 1 + 1 = 1$,
$d(t, q+2) = d(t, q) - 1 + 1 = 1$.

We still need to show that $p = m$ and for each $i \in \{1, \ldots, m\}$ $\psi_i = \varphi_i$.

First we examine the case where $m = 1$. We want to show that $p = 1$.

Suppose $p > 1$. In this situation we have

$$d(t, r_1 - 1) = d(t, r + 1 + (r_1 - 1 - (r+1))) = d(t, r + 1 + \ell(\psi_1)) =$$
$$= d(t, r+2) + d(\psi_1, \ell(\psi_1)) = 1 + d(\psi_1, \ell(\psi_1)).$$

If $t[r_1 - 1] = \psi_1[\ell(\psi_1)] = \text{`)'}$ then $d(t, r_1) = d(t, r_1 - 1) - 1 = d(\psi_1, \ell(\psi_1)) = 1$.
Else $t[r_1 - 1] = \psi_1[\ell(\psi_1)] \notin \{\text{`('}, \text{`)'}\}$ so $d(t, r_1) = d(t, r_1 - 1) = 1 + d(\psi_1, \ell(\psi_1)) = 1$.

Moreover we have to consider that

$$\begin{aligned}
\ell(\varphi_1) &= \ell(t) - 1 - (q+1) = \ell(t) - q - 2,\\
r_1 &\leqslant \ell(t) - 1,\\
r_1 - (q+1) &\leqslant \ell(t) - 1 - (q+1) = \ell(t) - q - 2 = \ell(\varphi_1),\\
r_1 &\geqslant q+2,\\
r_1 - (q+1) &\geqslant 1,\\
\varphi_1[r_1 - (q+1)] &= t[r_1] = \text{‘,’},\\
1 = d(t, r_1) &= d(t, q+2) + d(\varphi_1, r_1 - (q+1)) = 1 + d(\varphi_1, r_1 - (q+1)).
\end{aligned}$$

This causes $d(\varphi_1, r_1 - (q+1)) = 0$, but by assumption 3.1.3 we must have $d(\varphi_1, r_1 - (q+1)) \geqslant 1$. So it must be $p = 1$.

Of course

$$\ell(\psi_1) = \ell(t) - 1 - (r+1) = \ell(t) - r - 2 = \ell(t) - q - 2 = \ell(\varphi_1).$$

For each $\alpha = 1 \dots \ell(\varphi_1)$ $\varphi_1[\alpha] = t[q+1+\alpha] = t[r+1+\alpha] = \psi_1[\alpha]$. Therefore $\psi_1 = \varphi_1$.

Now let's discuss the case where $m > 1$.

First we want to prove that for each $i = 1 \dots m-1$ $p > i, d(t, q_i) = 1, r_i = q_i, \psi_i = \varphi_i$.

Let's show that $p > 1, d(t, q_1) = 1, r_1 = q_1, \psi_1 = \varphi_1$.

If $p = 1$ of course $m = 1$, so $p > 1$ holds.

We have that

$$d(t, q_1 - 1) = d(t, q+1+\ell(\varphi_1)) = d(t, q+1+1) + d(\varphi_1, \ell(\varphi_1)) = 1 + d(\varphi_1, \ell(\varphi_1)).$$

If $t[q_1 - 1] = \varphi_1[\ell(\varphi_1)] = \text{‘)’}$ then $d(t, q_1) = d(t, q_1 - 1) - 1 = d(\varphi_1, \ell(\varphi_1)) = 1$. Else $t[q_1 - 1] = \varphi_1[\ell(\varphi_1)] \notin \{\text{‘(’}, \text{‘)’}\}$ so $d(t, q_1) = d(t, q_1 - 1) = 1 + d(\varphi_1, \ell(\varphi_1)) = 1$.

Suppose $q_1 < r_1$, we have

$$\begin{aligned}
\ell(\psi_1) &= r_1 - 1 - (r+1) = r_1 - r - 2,\\
q_1 - r - 1 &< r_1 - r - 1,\\
q_1 - r - 1 &\leqslant \ell(\psi_1),\\
q_1 &> q+1,\\
q_1 &> r+1,\\
q_1 - r - 1 &\geqslant 1,
\end{aligned}$$

and then

$$\begin{aligned}
1 = d(t, q_1) &= d(t, r+1+(q_1 - r - 1)) = d(t, r+2) + d(\psi_1, q_1 - r - 1) =\\
&= 1 + d(\psi_1, q_1 - r - 1).
\end{aligned}$$

So $d(\psi_1, q_1 - r - 1) = 0$. But since $\psi_1[q_1 - r - 1] = t[q_1] = \text{‘,’}$, by assumption 3.1.3 we must have $d(\psi_1, q_1 - r - 1) \geqslant 1$, so we have a contradiction.

Hence $q_1 \geqslant r_1$ and in the same way we can show that $r_1 \geqslant q_1$, therefore $r_1 = q_1$.

At this point we observe that

$$\ell(\varphi_1) = q_1 - 1 - (q+1) = q_1 - q - 2 = r_1 - r - 2 = \ell(\psi_1).$$

Moreover, for each $\alpha = 1 \dots \ell(\varphi_1)$ $\varphi_1[\alpha] = t[q+1+\alpha] = t[r+1+\alpha] = \psi_1[\alpha]$. Therefore $\psi_1 = \varphi_1$.

We have proved that $p > 1, d(t, q_1) = 1, r_1 = q_1, \psi_1 = \varphi_1$, and if $m = 2$ we have also shown that for each $i = 1 \dots m-1$ $p > i, d(t, q_i) = 1, r_i = q_i, \psi_i = \varphi_i$.

Now suppose $m > 2$, let $i = 1 \dots m-2$, suppose we have proved $p > i$, $d(t, q_i) = 1$, $r_i = q_i$, $\psi_i = \varphi_i$, we want to show that $p > i+1, d(t, q_{i+1}) = 1, r_{i+1} = q_{i+1}, \psi_{i+1} = \varphi_{i+1}$.

First of all

$$\begin{aligned} d(t, q_{i+1} - 1) = d(t, q_i + \ell(\varphi_{i+1})) = d(t, q_i + 1) + d(\varphi_{i+1}, \ell(\varphi_{i+1})) = \\ = 1 + d(\varphi_{i+1}, \ell(\varphi_{i+1})). \end{aligned}$$

If $t[q_{i+1} - 1] = \varphi_{i+1}[\ell(\varphi_{i+1})] =$ ')' then
$d(t, q_{i+1}) = d(t, q_{i+1} - 1) - 1 = d(\varphi_{i+1}, \ell(\varphi_{i+1})) = 1$.
Else $t[q_{i+1} - 1] = \varphi_{i+1}[\ell(\varphi_{i+1})] \notin \{$'(', ')'$\}$ so
$d(t, q_{i+1}) = d(t, q_{i+1} - 1) = 1 + d(\varphi_{i+1}, \ell(\varphi_{i+1})) = 1$.

Suppose $p = i+1$. We have $i \leqslant m-2$, $i+2 \leqslant m$, $t[q_{i+1}] =$ ','. And we have also

$$\begin{aligned} \ell(\psi_p) &= \ell(t) - 1 - r_i, \\ q_{i+1} &\leqslant \ell(t) - 1, \\ q_{i+1} - r_i &\leqslant \ell(t) - 1 - r_i = \ell(\psi_p), \\ q_{i+1} - r_i &= q_{i+1} - q_i \geqslant 1, \\ \psi_p[q_{i+1} - r_i] &= t[q_{i+1}] = \text{`,'}, \end{aligned}$$

and

$$\begin{aligned} 1 = d(t, q_{i+1}) = d(t, r_i + (q_{i+1} - r_i)) = d(t, r_i + 1) + d(\psi_p, q_{i+1} - r_i) = \\ = 1 + d(\psi_p, q_{i+1} - r_i). \end{aligned}$$

So $d(\psi_p, q_{i+1} - r_i) = 0$ and this contradicts assumption 3.1.3. Therefore $p > i+1$.

Now suppose $q_{i+1} < r_{i+1}$. In this case

$$\begin{aligned} \ell(\psi_{i+1}) &= r_{i+1} - 1 - r_i, \\ q_{i+1} &\leqslant r_{i+1} - 1, \\ q_{i+1} - r_i &\leqslant r_{i+1} - 1 - r_i = \ell(\psi_{i+1}), \\ q_{i+1} - r_i &= q_{i+1} - q_i \geqslant 1, \\ \psi_{i+1}[q_{i+1} - r_i] &= t[q_{i+1}] = \text{`,'}, \end{aligned}$$

and

$$1 = d(t, q_{i+1}) = d(t, r_i + (q_{i+1} - r_i)) = d(t, r_i + 1) + d(\psi_{i+1}, q_{i+1} - r_i) =$$
$$= 1 + d(\psi_{i+1}, q_{i+1} - r_i).$$

So $d(\psi_{i+1}, q_{i+1} - r_i) = 0$ and this contradicts assumption 3.1.3. Therefore $q_{i+1} \geqslant r_{i+1}$.

In the same way we can prove that $q_{i+1} \leqslant r_{i+1}$, hence $r_{i+1} = q_{i+1}$ is proved.

Moreover

$$\ell(\varphi_{i+1}) = q_{i+1} - 1 - q_i = r_{i+1} - 1 - r_i = \ell(\psi_{i+1}),$$

and for each $\alpha = 1 \ldots \ell(\psi_{i+1})$

$$\psi_{i+1}[\alpha] = t[r_i + \alpha] = t[q_i + \alpha] = \varphi_{i+1}[\alpha].$$

We have proved that for each $i = 1 \ldots m-1$ $p > i, d(t, q_i) = 1, r_i = q_i, \psi_i = \varphi_i$.

So $p \geqslant m$, and in the same way we could prove $m \geqslant p$, therefore $p = m$.

We have seen that $r_{m-1} = q_{m-1}$, it follows

$$\ell(\varphi_m) = \ell(t) - 1 - q_{m-1} = \ell(t) - 1 - r_{m-1} = \ell(\psi_m),$$

and for each $\alpha = 1 \ldots \ell(\psi_m)$

$$\psi_m[\alpha] = t[r_{m-1} + \alpha] = t[q_{m-1} + \alpha] = \varphi_m[\alpha],$$

therefore $\psi_m = \varphi_m$.

So also in the case $m > 1$ it is shown that $p = m$ and for each $i = 1 \ldots m$ $\psi_i = \varphi_i$. ■

For each $t = (\varphi)(\varphi_1, \ldots, \varphi_m) \in E_c(n+1, k)$ we define

$$\#(k, t, \sigma)_{(n+1,k,c)} = \#(k, \varphi, \sigma)(\#(k, \varphi_1, \sigma), \ldots, \#(k, \varphi_m, \sigma)).$$

For each $k \in K(n)$ we define $E_d(n+1, k)$ as the set of the strings $f(\varphi_1, \ldots, \varphi_m)$ such that:

- f belongs to $\mathcal{F}$
- m is a positive integer;
- $\varphi_1, \ldots, \varphi_m \in E(n, k)$;
- for each $\sigma \in \Xi(k)$ $A_f(\#(k, \varphi_1, \sigma), \ldots, \#(k, \varphi_m, \sigma))$ is true;
- $f(\varphi_1, \ldots, \varphi_m) \notin E(n, k)$.
- $f(\varphi_1, \ldots, \varphi_m) \notin E_b(n+1, k)$.

For instance, this means that if the 'logical conjunction' symbol '$\wedge$' belongs to $\mathcal{F}$, φ_1, φ_2 belong to $E(n, k)$, for each $\sigma \in \Xi(k)$ both $\#(k, \varphi_1, \sigma)$ and $\#(k, \varphi_2, \sigma)$ are true or false, $\wedge(\varphi_1, \varphi_2) \notin E(n, k)$, $\wedge(\varphi_1, \varphi_2) \notin E_b(n+1, k)$ then $\wedge(\varphi_1, \varphi_2)$ belongs to $E_d(n+1, k)$.

This implies that for each $t \in E_d(n+1, k)$ there are f in $\mathcal{F}$, a positive integer m and $\varphi_1, \ldots, \varphi_m \in E(n)$ such that $t = f(t_1, \ldots, t_m)$. We will now show that $f, m, \varphi_1, \ldots, \varphi_m$ are uniquely determined. Within this complex definition this proof of unique readability may be considered as a technical detail, and can be skipped at first reading. The proof will often exploit lemma 3.11 and assumption 3.1.3, they will not be quoted each time they are used.

LEMMA 3.1.5. Let $t \in E_d(n+1,k)$ and suppose

- there exist $f \in \mathcal{F}$, a positive integer m and $\varphi_1, \ldots, \varphi_m \in E(n)$: $t = f(\varphi_1, \ldots, \varphi_m)$.
- there exist $g \in \mathcal{F}$, a positive integer p and $\psi_1, \ldots, \psi_p \in E(n)$: $t = g(\psi_1, \ldots, \psi_p)$.

Then $g = f$, $p = m$ and for each $i \in \{1, \ldots, m\}$ $\psi_i = \varphi_i$.

NB: in the context of this definition we just need to prove this lemma for $t \in E_d(n+1,k)$, but actually we just need to assume t is a generic string, and below in this paper we'll also use this result with reference to a generic string t.

Proof.

We can represent t as $f(\varphi_1, \ldots, \varphi_m)$ and as $g(\psi_1, \ldots, \psi_p)$.

In both representations we see 'explicit occurrences' of the symbols '(' , ')' and ','.

In the first representation there are explicit occurrences of ',' only when $m > 1$. If $m > 1$ we indicate with $q_1, \ldots, q_{m-1}$ the positions of the explicit occurrences of ','.

In the second representation there are explicit occurrences of ',' only when $p > 1$. If $p > 1$ we indicate with $r_1, \ldots, r_{p-1}$ the positions of the explicit occurrences of ','.

It is immediate to see that $g = t[1] = f$.

We still need to show that $p = m$ and for each $i \in \{1, \ldots, m\}$ $\psi_i = \varphi_i$.

First we examine the case where $m = 1$. We want to show that $p = 1$.

Suppose $p > 1$. In this situation we have

$$\begin{aligned} d(t, r_1 - 1) &= d(t, 2 + (r_1 - 1 - 2)) = d(t, 2 + \ell(\psi_1)) = \\ &= d(t, 2+1) + d(\psi_1, \ell(\psi_1)) = 1 + d(\psi_1, \ell(\psi_1)). \end{aligned}$$

If $t[r_1 - 1] = \psi_1[\ell(\psi_1)] =$ ')' then $d(t, r_1) = d(t, r_1 - 1) - 1 = d(\psi_1, \ell(\psi_1)) = 1$. Else $t[r_1 - 1] = \psi_1[\ell(\psi_1)] \notin \{\text{'('}, \text{')'}\}$ so $d(t, r_1) = d(t, r_1 - 1) = 1 + d(\psi_1, \ell(\psi_1)) = 1$.

Moreover we have to consider that

$$\begin{aligned} \ell(\varphi_1) &= \ell(t) - 1 - 2 = \ell(t) - 3, \\ r_1 &\leqslant \ell(t) - 1, \\ r_1 - 2 &\leqslant \ell(t) - 1 - 2 = \ell(t) - 3 = \ell(\varphi_1), \\ r_1 &\geqslant 2 + 1, \\ r_1 - 2 &\geqslant 1, \\ \varphi_1[r_1 - 2] &= t[r_1] = \text{','}, \\ 1 &= d(t, r_1) = d(t, 2+1) + d(\varphi_1, r_1 - 2) = 1 + d(\varphi_1, r_1 - 2). \end{aligned}$$

This causes $d(\varphi_1, r_1 - 2) = 0$, but by assumption 3.1.3 we must have $d(\varphi_1, r_1 - 2) \geqslant 1$. So it must be $p = 1$.

Of course

$$\ell(\psi_1) = \ell(t) - 1 - 2 = \ell(\varphi_1).$$

For each $\alpha = 1 \dots \ell(\varphi_1)$ $\varphi_1[\alpha] = t[2 + \alpha] = \psi_1[\alpha]$. Therefore $\psi_1 = \varphi_1$.

Now let's discuss the case where $m > 1$.

First we want to prove that for each $i = 1 \dots m-1$ $p > i, d(t, q_i) = 1, r_i = q_i, \psi_i = \varphi_i$.

Let's show that $p > 1, d(t, q_1) = 1, r_1 = q_1, \psi_1 = \varphi_1$.

If $p = 1$ of course $m = 1$, so $p > 1$ holds.

We have that

$$d(t, q_1 - 1) = d(t, 2 + \ell(\varphi_1)) = d(t, 2 + 1) + d(\varphi_1, \ell(\varphi_1)) = 1 + d(\varphi_1, \ell(\varphi_1)).$$

If $t[q_1 - 1] = \varphi_1[\ell(\varphi_1)] =$ ')' then $d(t, q_1) = d(t, q_1 - 1) - 1 = d(\varphi_1, \ell(\varphi_1)) = 1$. Else $t[q_1 - 1] = \varphi_1[\ell(\varphi_1)] \notin \{\text{'('}, \text{')'}\}$ so $d(t, q_1) = d(t, q_1 - 1) = 1 + d(\varphi_1, \ell(\varphi_1)) = 1$.

Suppose $q_1 < r_1$, we have

$$\begin{aligned} \ell(\psi_1) &= r_1 - 1 - 2 = r_1 - 3, \\ q_1 - 2 &< r_1 - 2, \\ q_1 - 2 &\leqslant \ell(\psi_1), \\ q_1 &> 2, \\ q_1 - 2 &\geqslant 1, \end{aligned}$$

and then

$$\begin{aligned} 1 = d(t, q_1) &= d(t, 2 + (q_1 - 2)) = d(t, 2 + 1) + d(\psi_1, q_1 - 2) = \\ &= 1 + d(\psi_1, q_1 - 2). \end{aligned}$$

So $d(\psi_1, q_1 - 2) = 0$. But since $\psi_1[q_1 - 2] = t[q_1] =$ ',', by assumption 3.1.3 we must have $d(\psi_1, q_1 - 2) \geqslant 1$, so we have a contradiction.

Hence $q_1 \geqslant r_1$ and in the same way we can show that $r_1 \geqslant q_1$, therefore $r_1 = q_1$.

At this point we observe that $\ell(\varphi_1) = q_1 - 1 - 2 = r_1 - 1 - 2 = \ell(\psi_1)$.

Moreover, for each $\alpha = 1 \dots \ell(\varphi_1)$ $\varphi_1[\alpha] = t[2 + \alpha] = \psi_1[\alpha]$.
Therefore $\psi_1 = \varphi_1$.

We have proved that $p > 1, d(t, q_1) = 1, r_1 = q_1, \psi_1 = \varphi_1$, and if $m = 2$ we have also shown that for each $i = 1 \dots m - 1$ $p > i, d(t, q_i) = 1, r_i = q_i, \psi_i = \varphi_i$.

Now suppose $m > 2$, let $i = 1 \dots m - 2$, suppose we have proved $p > i$, $d(t, q_i) = 1$, $r_i = q_i$, $\psi_i = \varphi_i$, we want to show that $p > i + 1, d(t, q_{i+1}) = 1, r_{i+1} = q_{i+1}, \psi_{i+1} = \varphi_{i+1}$.

First of all

$$d(t, q_{i+1} - 1) = d(t, q_i + \ell(\varphi_{i+1})) = d(t, q_i + 1) + d(\varphi_{i+1}, \ell(\varphi_{i+1})) =$$
$$= 1 + d(\varphi_{i+1}, \ell(\varphi_{i+1})).$$

If $t[q_{i+1} - 1] = \varphi_{i+1}[\ell(\varphi_{i+1})] =$ ')' then
$d(t, q_{i+1}) = d(t, q_{i+1} - 1) - 1 = d(\varphi_{i+1}, \ell(\varphi_{i+1})) = 1$.
Else $t[q_{i+1} - 1] = \varphi_{i+1}[\ell(\varphi_{i+1})] \notin \{$'(', ')'$\}$ so
$d(t, q_{i+1}) = d(t, q_{i+1} - 1) = 1 + d(\varphi_{i+1}, \ell(\varphi_{i+1})) = 1$.

Suppose $p = i + 1$. We have $i \leqslant m - 2$, $i + 2 \leqslant m$, $t[q_{i+1}] =$ ','. And we have also

$$\ell(\psi_p) = \ell(t) - 1 - r_i,$$
$$q_{i+1} \leqslant \ell(t) - 1,$$
$$q_{i+1} - r_i \leqslant \ell(t) - 1 - r_i = \ell(\psi_p),$$
$$q_{i+1} - r_i = q_{i+1} - q_i \geqslant 1,$$
$$\psi_p[q_{i+1} - r_i] = t[q_{i+1}] = \text{','},$$

and

$$1 = d(t, q_{i+1}) = d(t, r_i + (q_{i+1} - r_i)) = d(t, r_i + 1) + d(\psi_p, q_{i+1} - r_i) =$$
$$= 1 + d(\psi_p, q_{i+1} - r_i).$$

So $d(\psi_p, q_{i+1} - r_i) = 0$ and this contradicts assumption 3.1.3. Therefore $p > i + 1$.

Now suppose $q_{i+1} < r_{i+1}$. In this case

$$\ell(\psi_{i+1}) = r_{i+1} - 1 - r_i,$$
$$q_{i+1} \leqslant r_{i+1} - 1,$$
$$q_{i+1} - r_i \leqslant r_{i+1} - 1 - r_i = \ell(\psi_{i+1}),$$
$$q_{i+1} - r_i = q_{i+1} - q_i \geqslant 1,$$
$$\psi_{i+1}[q_{i+1} - r_i] = t[q_{i+1}] = \text{','},$$

and

$$1 = d(t, q_{i+1}) = d(t, r_i + (q_{i+1} - r_i)) = d(t, r_i + 1) + d(\psi_{i+1}, q_{i+1} - r_i) =$$
$$= 1 + d(\psi_{i+1}, q_{i+1} - r_i).$$

So $d(\psi_{i+1}, q_{i+1} - r_i) = 0$ and this contradicts assumption 3.1.3. Therefore $q_{i+1} \geqslant r_{i+1}$.

In the same way we can prove that $q_{i+1} \leqslant r_{i+1}$, hence $r_{i+1} = q_{i+1}$ is proved.

Moreover

$$\ell(\varphi_{i+1}) = q_{i+1} - 1 - q_i = r_{i+1} - 1 - r_i = \ell(\psi_{i+1}),$$

and for each $\alpha = 1 \ldots \ell(\psi_{i+1})$

$$\psi_{i+1}[\alpha] = t[r_i + \alpha] = t[q_i + \alpha] = \varphi_{i+1}[\alpha].$$

We have proved that for each $i = 1 \dots m-1$ $p > i, d(t, q_i) = 1, r_i = q_i, \psi_i = \varphi_i$.

So $p \geqslant m$, and in the same way we could prove $m \geqslant p$, therefore $p = m$.

We have seen that $r_{m-1} = q_{m-1}$, it follows

$$\ell(\varphi_m) = \ell(t) - 1 - q_{m-1} = \ell(t) - 1 - r_{m-1} = \ell(\psi_m),$$

and for each $\alpha = 1 \dots \ell(\psi_m)$

$$\psi_m[\alpha] = t[r_{m-1} + \alpha] = t[q_{m-1} + \alpha] = \varphi_m[\alpha],$$

therefore $\psi_m = \varphi_m$.

So also in the case $m > 1$ it is shown that $p = m$ and for each $i = 1 \dots m$ $\psi_i = \varphi_i$. ∎

For each $t = f(\varphi_1, \dots, \varphi_m) \in E_d(n+1, k)$ we define

$$\#(k, t, \sigma)_{(n+1,k,d)} = P_f(\#(k, \varphi_1, \sigma), \dots, \#(k, \varphi_m, \sigma)).$$

Let $k \in K(n)$, m a positive integer, x a function whose domain is $\{1, \dots, m\}$ such that for each $i = 1 \dots m$ $x_i \in \mathcal{V} - var(k)$, and for each $i, j = 1 \dots m$ $i \neq j \rightarrow x_i \neq x_j$, φ a function whose domain is $\{1, \dots, m\}$ such that for each $i = 1 \dots m$ $\varphi_i \in E(n)$, and finally let $\phi \in E(n)$. We write

$$\mathcal{E}(n, k, m, x, \varphi, \phi)$$

to indicate the following condition (where $k'_1 = k + (x_1, \varphi_1)$, and if $m > 1$ for each $i = 1 \dots m-1$ $k'_{i+1} = k'_i + (x_{i+1}, \varphi_{i+1})$):

- $\varphi_1 \in E_s(n, k)$;
- if $m > 1$ then for each $i = 1 \dots m-1$ $k'_i \in K(n) \wedge \varphi_{i+1} \in E_s(n, k'_i)$;
- $k'_m \in K(n) \wedge \phi \in E(n, k'_m)$.

For each $k \in K(n)$ we define $E_e(n+1, k)$ as the set of the strings

$$\{\}(x_1 : \varphi_1, \dots, x_m : \varphi_m, \phi)$$

such that:

- m is a positive integer;
- x is a function whose domain is $\{1, \dots, m\}$ such that for each $i = 1 \dots m$ $x_i \in \mathcal{V} - var(k)$, and for each $i, j = 1 \dots m$ $i \neq j \rightarrow x_i \neq x_j$;
- φ is a function whose domain is $\{1, \dots, m\}$ such that for each $i = 1 \dots m$ $\varphi_i \in E(n)$;
- $\phi \in E(n)$;
- $\mathcal{E}(n, k, m, x, \varphi, \phi)$;
- $\{\}(x_1 : \varphi_1, \dots, x_m : \varphi_m, \phi) \notin E(n, k)$.
- $\{\}(x_1 : \varphi_1, \dots, x_m : \varphi_m, \phi) \notin E_b(n+1, k)$.

This implies that for each $t \in E_e(n+1,k)$ there exist a positive integer m, a function x whose domain is $\{1,\dots,m\}$ such that for each $i = 1\dots m$ $x_i \in \mathcal{V}$, a function φ whose domain is $\{1,\dots,m\}$ such that for each $i = 1\dots m$ $\varphi_i \in E(n)$, and $\phi \in E(n)$ such that $t = \{\}(x_1 : \varphi_1, \dots, x_m : \varphi_m, \phi)$. We will now show that m, x, φ, ϕ are uniquely determined. Within this complex definition this proof of unique readability may be considered as a technical detail, and can be skipped at first reading. The proof will often exploit lemma 3.11 and assumption 3.1.3, they will not be quoted each time they are used.

LEMMA 3.1.6. Let $t \in E_e(n+1,k)$ and suppose

- there exist a positive integer m, a function x whose domain is $\{1,\dots,m\}$ such that for each $i = 1\dots m$ $x_i \in \mathcal{V}$, a function φ whose domain is $\{1,\dots,m\}$ such that for each $i = 1\dots m$ $\varphi_i \in E(n)$, and $\phi \in E(n)$ such that $t = \{\}(x_1 : \varphi_1, \dots, x_m : \varphi_m, \phi)$;
- there exist a positive integer p, a function y whose domain is $\{1,\dots,p\}$ such that for each $i = 1\dots p$ $y_i \in \mathcal{V}$, a function ψ whose domain is $\{1,\dots,p\}$ such that for each $i = 1\dots p$ $\psi_i \in E(n)$, and $\vartheta \in E(n)$ such that $t = \{\}(y_1 : \psi_1, \dots, y_p : \psi_p, \vartheta)$;

Then $p = m, y = x, \psi = \varphi$ and $\vartheta = \phi$.

NB: in the context of this definition we just need to prove this lemma for $t \in E_e(n+1,k)$, but actually we just need to assume t is a generic string, and below in this paper we'll also use this result with reference to a generic string t.

Proof.

We can represent t as $\{\}(x_1 : \varphi_1, \dots, x_m : \varphi_m, \phi)$ and as $\{\}(y_1 : \psi_1, \dots, y_p : \psi_p, \vartheta)$.

In both representations we see 'explicit occurrences' of the symbols ',' and ':'.

In the first representation we indicate with $q_1, \dots, q_m$ the positions of the explicit occurrences of ':' and with $r_1 \dots r_m$ the positions of the explicit occurrences of ','.

In the second representation we indicate with $q'_1, \dots, q'_p$ the positions of the explicit occurrences of ':' and with $r'_1 \dots r'_p$ the positions of the explicit occurrences of ','.

We want to show that for each $i = 1\dots m$

$$p \geqslant i, y_i = x_i, q'_i = q_i, d(t, r_i) = 1, r'_i = r_i, \psi_i = \varphi_i.$$

The first step is to show that $y_1 = x_1, q'_1 = q_1, d(t, r_1) = 1, r'_1 = r_1, \psi_1 = \varphi_1$.

Of course $y_1 = t[3] = x_1$, $q'_1 = 4 = q_1$. Moreover

$$d(t, r_1 - 1) = d(t, q_1 + (r_1 - 1 - q_1)) = d(t, q_1 + \ell(\varphi_1)) = d(t, q_1 + 1) + d(\varphi_1, \ell(\varphi_1)) = \\ = 1 + d(\varphi_1, \ell(\varphi_1)).$$

If $t[r_1 - 1] = \varphi_1[\ell(\varphi_1)] =$ ')' then $d(t, r_1) = d(t, r_1 - 1) - 1 = d(\varphi_1, \ell(\varphi_1)) = 1$. Else $t[r_1 - 1] = \varphi_1[\ell(\varphi_1)] \notin \{\text{'('}, \text{')'}\}$ so $d(t, r_1) = d(t, r_1 - 1) = 1 + d(\varphi_1, \ell(\varphi_1)) = 1$.

Now suppose $r_1 < r'_1$. This would mean that

$$\begin{aligned} \ell(\psi_1) &= r'_1 - 1 - q'_1, \\ r_1 - q'_1 &\leqslant r'_1 - 1 - q'_1 = \ell(\psi_1), \\ r_1 - q'_1 &= r_1 - q_1 \geqslant 1, \\ \psi_1[r_1 - q'_1] &= t[r_1] = \text{`,'}, \end{aligned}$$

and

$$1 = d(t, r_1) = d(t, q'_1 + (r_1 - q'_1)) = d(t, q'_1 + 1) + d(\psi_1, r_1 - q'_1) = 1 + d(\psi_1, r_1 - q'_1).$$

So $d(\psi_1, r_1 - q'_1) = 0$ and this contradicts assumption 3.1.3. Hence $r_1 \geqslant r'_1$, and in the same way we can show that $r'_1 \geqslant r_1$, therefore $r_1 = r'_1$.

At this point we observe that $\ell(\varphi_1) = r_1 - 1 - q_1 = \ell(\psi_1)$.
Moreover, for each $\alpha = 1 \ldots \ell(\psi_1)$ $\psi_1[\alpha] = t[q'_1 + \alpha] = t[q_1 + \alpha] = \varphi_1[\alpha]$, hence $\psi_1 = \varphi_1$.

We have proved that $y_1 = x_1, q'_1 = q_1, d(t, r_1) = 1, r'_1 = r_1, \psi_1 = \varphi_1$. As a consequence, if $m = 1$ we have proved that for each $i = 1 \ldots m$

$$p \geqslant i, y_i = x_i, q'_i = q_i, d(t, r_i) = 1, r'_i = r_i, \psi_i = \varphi_i.$$

Consider the case where $m > 1$. Let $i = 1 \ldots m - 1$, we suppose

$$p \geqslant i, y_i = x_i, q'_i = q_i, d(t, r_i) = 1, r'_i = r_i, \psi_i = \varphi_i,$$

and want to show that

$$p \geqslant i + 1, y_{i+1} = x_{i+1}, q'_{i+1} = q_{i+1}, d(t, r_{i+1}) = 1, r'_{i+1} = r_{i+1}, \psi_{i+1} = \varphi_{i+1}.$$

We can immediately show that $d(t, r_{i+1}) = 1$. In fact $d(t, q_{i+1} + 1) = d(t, r_i) = 1$,

$$\begin{aligned} d(t, r_{i+1} - 1) &= d(t, q_{i+1} + (r_{i+1} - 1 - q_{i+1})) = d(t, q_{i+1} + \ell(\varphi_{i+1})) = \\ &= d(t, q_{i+1} + 1) + d(\varphi_{i+1}, \ell(\varphi_{i+1})) = 1 + d(\varphi_{i+1}, \ell(\varphi_{i+1})). \end{aligned}$$

If $t[r_{i+1} - 1] = \varphi_{i+1}[\ell(\varphi_{i+1})] = \text{`)'}$ then
$d(t, r_{i+1}) = d(t, r_{i+1} - 1) - 1 = d(\varphi_{i+1}, \ell(\varphi_{i+1})) = 1$.
Else $t[r_{i+1} - 1] = \varphi_{i+1}[\ell(\varphi_{i+1})] \notin \{\text{`('}, \text{`)'}\}$ so
$d(t, r_{i+1}) = d(t, r_{i+1} - 1) = 1 + d(\varphi_{i+1}, \ell(\varphi_{i+1})) = 1$.

Suppose $p = i$. In this case

$$\begin{aligned} \ell(\vartheta) &= \ell(t) - 1 - r'_i, \\ r_{i+1} - r'_i &\leqslant \ell(t) - 1 - r'_i = \ell(\vartheta), \\ r_{i+1} - r'_i &= r_{i+1} - r_i \geqslant 1, \\ \vartheta[r_{i+1} - r'_i] &= t[r_{i+1}] = \text{`,'}, \end{aligned}$$

and

$$\begin{aligned} 1 = d(t, r_{i+1}) &= d(t, r'_i + (r_{i+1} - r'_i)) = d(t, r'_i + 1) + d(\vartheta, r_{i+1} - r'_i) = \\ &= 1 + d(\vartheta, r_{i+1} - r'_i). \end{aligned}$$

So $d(\vartheta, r_{i+1} - r'_i) = 0$, and this contradicts assumption 3.1.3. Therefore $p \geqslant i + 1$. It follows immediately that $y_{i+1} = t[r'_i + 1] = t[r_i + 1] = x_{i+1}$ and $q'_{i+1} = r'_i + 2 = q_{i+1}$.

Now we suppose $r_{i+1} < r'_{i+1}$. This would mean that

$$\begin{aligned} \ell(\psi_{i+1}) &= r'_{i+1} - 1 - q'_{i+1}, \\ r_{i+1} - q'_{i+1} &\leqslant r'_{i+1} - 1 - q'_{i+1} = \ell(\psi_{i+1}), \\ r_{i+1} - q'_{i+1} &= r_{i+1} - q_{i+1} \geqslant 1, \\ \psi_{i+1}[r_{i+1} - q'_{i+1}] &= t[r_{i+1}] = \text{`,'}, \end{aligned}$$

and

$$\begin{aligned} 1 = d(t, r_{i+1}) &= d(t, q'_{i+1} + (r_{i+1} - q'_{i+1})) = d(t, q'_{i+1} + 1) + d(\psi_{i+1}, r_{i+1} - q'_{i+1}) = \\ &= 1 + d(\psi_{i+1}, r_{i+1} - q'_{i+1}). \end{aligned}$$

So $d(\psi_{i+1}, r_{i+1} - q'_{i+1}) = 0$ and this contradicts assumption 3.1.3. Hence $r_{i+1} \geqslant r'_{i+1}$. In the same way we can show that $r_{i+1} \leqslant r'_{i+1}$, therefore $r_{i+1} = r'_{i+1}$.

At this point we observe that $\ell(\varphi_{i+1}) = r_{i+1} - 1 - q_{i+1} = \ell(\psi_{i+1})$. Furthermore, for each $\alpha = 1 \dots \ell(\varphi_{i+1})$ $\psi_{i+1}[\alpha] = t[q'_{i+1} + \alpha] = t[q_{i+1} + \alpha] = \varphi_{i+1}[\alpha]$, hence $\psi_{i+1} = \varphi_{i+1}$.

It is shown that for each $i = 1 \dots m$

$$p \geqslant i, y_i = x_i, q'_i = q_i, d(t, r_i) = 1, r'_i = r_i, \psi_i = \varphi_i.$$

So $p \geqslant m$. In the same way we could prove that $m \geqslant p$, so $p = m$. At this stage we have shown that $y = x$ and $\psi = \varphi$, we just need a final step, which is proving that $\vartheta = \phi$. This clearly holds because of

$$\ell(\vartheta) = \ell(t) - 1 - r'_p = \ell(t) - 1 - r_m = \ell(\phi),$$

and for each $\alpha = 1 \dots \ell(\vartheta)$

$$\vartheta[\alpha] = t[r'_p + \alpha] = t[r_m + \alpha] = \phi[\alpha].$$

■

For every $t = \{\}(x_1 : \varphi_1, \dots, x_m : \varphi_m, \phi) \in E_e(n + 1, k)$ we define

$$\#(k, t, \sigma)_{(n+1,k,e)} = \{\#(k'_m, \phi, \sigma'_m)|\ \sigma'_m \in \Xi(k'_m), \sigma \sqsubseteq \sigma'_m\},$$

where $k'_1 = k + (x_1, \varphi_1)$, and if $m > 1$ for each $i = 1 \dots m - 1$ $k'_{i+1} = k'_i + (x_{i+1}, \varphi_{i+1})$.

Notice that the set $\{\#(k'_m, \phi, \sigma'_m)|\ \sigma'_m \in \Xi(k'_m), \sigma \sqsubseteq \sigma'_m\}$ is specified using a standard mathematical notation. We could specify it using a notation closer to the one of our formulas, in this case it could have been written as $\{\}(\sigma'_m \in \Xi(k'_m) : \sigma \sqsubseteq \sigma'_m, \#(k'_m, \phi, \sigma'_m))$.

Actually, it might still be a bit unclear what is the intended meaning of the expression

$$\{\}(x_1 : \varphi_1, \ldots, x_m : \varphi_m, \phi).$$

This is the same meaning that the expression

$$\{\phi|\ x_1 \in \varphi_1, \ldots, x_m \in \varphi_m\}$$

is intended to have when used in most mathematics books.

We have terminated the definition of the 'new sets' (of expressions bound to context k) and the related work, we are now ready to define $E(n+1,k)$ for $k \in K(n+1)$.

If $k \in K(n)^+$ we define

- $E(n+1,k) = E_a(n+1,k)$;
- $E_b(n+1,k) = \emptyset,\ E_c(n+1,k) = \emptyset,\ E_d(n+1,k) = \emptyset,\ E_e(n+1,k) = \emptyset.$

If $k \in K(n)$ we define

- $E(n+1,k) = E(n,k) \cup E_b(n+1,k) \cup E_c(n+1,k) \cup E_d(n+1,k) \cup E_e(n+1,k)$;
- $E_a(n+1,k) = \emptyset.$

For every $k \in K(n+1)$, $t \in E(n+1,k)$ and $\sigma \in \Xi(k)$ we need that $\#(k,t,\sigma)$ is defined. But we also need that the definition is such that for each $w \in \{a,b,c,d,e\}$, if $t \in E_w(n+1,k)$ then $\#(k,t,\sigma)_{(n+1,k,w)} = \#(k,t,\sigma)$.

LEMMA 3.1.7. For each $k \in K(n)$, $w_1, w_2 \in \{b,c,d,e\}$ $E_{w_1}(n+1,k) \cap E_{w_2}(n+1,k) = \emptyset$.

Proof.

We have that for each $w \in \{c,d,e\}$ $E_b(n+1,k) \cap E_w(n+1,k) = \emptyset$.

We also have that

- each expression in $E_c(n+1,k)$ begins with the character '(';
- each expression in $E_d(n+1,k)$ begins with a member of $\mathcal{F}$;
- each expression in $E_e(n+1,k)$ begins with the character '$\{\}$'.

This implies that for each $w_1, w_2 \in \{c,d,e\}$ $E_{w_1}(n+1,k) \cap E_{w_2}(n+1,k) = \emptyset$.

■

Let $k \in K(n)$. If $t \in E(n,k)$ then $\#(k,t,\sigma)$ is already defined, $E_a(n+1,k) = \emptyset$ and for each $w \in \{b,c,d,e\}$ $t \notin E_w(n+1,k)$, so in this case there isn't anything to check.

If $k \in K(n)$ and $t \in E_b(n+1,k) \cup E_c(n+1,k) \cup E_d(n+1,k) \cup E_e(n+1,k)$ then there is just one $w \in \{b,c,d,e\}$ such that $t \in E_w(n+1,k)$, we define $\#(k,t,\sigma) = \#(k,t,\sigma)_{(n+1,k,w)}$ and we have nothing else to check.

If $k \in K(n)^+$ and $t \in E_a(n+1,k)$ then we define $\#(k,t,\sigma) = \#(k,t,\sigma)_{(n+1,k,a)}$ and we have nothing else to check.

In the last part of our definition we need to prove that all the assumptions we have made at step n are true at step $n+1$.

Proof of 3.1.1.

We have to show that for each $k \in K(n+1)$ k is a state-like pair and for each $\sigma \in \Xi(k)$ σ is a state-like pair, and $dom(\sigma) = dom(k)$.

If $k \in K(n)$ this is clearly true because it is precisely our assumption.

If $k \in K(n)^+$ then there exist $h \in K(n), \phi \in E_s(n,h), y \in (\mathcal{V} - var(h))$ such that $k = h + (y, \phi)$. So k is a state-like pair.

For each $\sigma \in \Xi(k)$ $\sigma = \rho + (y,s)$ with $\rho \in \Xi(h), s \in \#(h,\phi,\rho)$, so σ is a state-like pair.

Moreover, we can assume $dom(h) = dom(\rho) = \emptyset$ or $dom(h) = dom(\rho) = \{1,\ldots,m\}$ for a positive integer m. In the first case $dom(\sigma) = \{1\} = dom(k)$, else

$$dom(\sigma) = dom(\rho) \cup \{m+1\} = dom(h) \cup \{m+1\} = dom(k)\ .$$

■

Proof of 3.1.2.

We have to show that for each $k \in K(n+1)$ $k = \epsilon$ and $\Xi(k) = \{\epsilon\}$ or (there exist $m < n+1$, $h \in K(m)$, $\phi \in E_s(m,h)$, $y \in (\mathcal{V} - var(h))$ such that $k = h + (y,\phi)$, $\Xi(k) = \{\sigma + (y,s) |\, \sigma \in \Xi(h), s \in \#(h,\phi,\sigma)\}$).

If $k \in K(n)$ by the inductive hypothesis $k = \epsilon$ and $\Xi(k) = \{\epsilon\}$ or ($n > 1$ and there exist $m < n < n+1$, $h \in K(m)$, $\phi \in E_s(m,h)$, $y \in (\mathcal{V} - var(h))$ such that $k = h + (y,\phi)$, $\Xi(k) = \{\sigma + (y,s) |\, \sigma \in \Xi(h), s \in \#(h,\phi,\sigma)\}$).

Otherwise $k \in K(n)^+$ so there exist $h \in K(n), \phi \in E_s(n,h), y \in (\mathcal{V} - var(h))$ such that $k = h + (y,\phi)$, $\Xi(k) = \{\sigma + (y,s) |\, \sigma \in \Xi(h), s \in \#(h,\phi,\sigma)\}$.

■

Proof of 3.1.3.

We need to prove that for each $k \in K(n+1),\ t \in E(n+1,k)$

- $t[\ell(t)] \neq$ '(' ;
- if $t[\ell(t)] =$ ')' then $d(t,\ell(t)) = 1$, else $d(t,\ell(t)) = 0$;
- for each $\alpha \in \{1,\ldots,\ell(t)\}$ if $(t[\alpha] = \text{':'}) \vee (t[\alpha] = \text{','}) \vee (t[\alpha] = \text{')'})$ then $d(t,\alpha) \geqslant 1$.

We recall that:

- If $k \in K(n)^+$ $E(n+1,k) = E_a(n+1,k)$.
- If $k \in K(n)$ $E(n+1,k) = E(n,k) \cup E_b(n+1,k) \cup E_c(n+1,k) \cup E_d(n+1,k) \cup E_e(n+1,k)$.

Let $k \in K(n)^+$ and $t \in \mathbf{E_a(n+1,k)}$. There exist $h \in K(n)$, $\phi \in E_s(n,h)$, $y \in \mathcal{V} - var(h)$ such that $k = h + (y, \phi)$. We also have $t = y$, so t has just one character, $t[1]$ differs from '(', ':', ',', ')' and $d(t, \ell(t)) = 0$.

Let $k \in K(n)$ and $\mathbf{t} \in \mathbf{E(n,k)}$, this means that $t \in E(n)$. In this case we just need to apply assumption 3.1.3.

Let $k = h + (y, \phi) \in K(n) - \{\epsilon\}$ and $t \in \mathbf{E_b(n+1,k)}$. We have $h \in K(n)$, $t \in E(n,h)$, so we can apply assumption 3.1.3 to finish.

Let $k \in K(n)$ and $t \in \mathbf{E_c(n+1,k)}$. There exist $\varphi, \varphi_1, \ldots, \varphi_m$ in $E(n,k)$ such that $t = (\varphi)(\varphi_1, \ldots, \varphi_m)$.

In this representation of t we see 'explicit occurrences' of the symbols '(' , ')' and ','. There are explicit occurrences of ',' only when $m > 1$. We indicate with q the position of the first explicit occurrence of ')', and the second explicit occurrence of ')' is clearly in position $\ell(t)$. If $m > 1$ we indicate with $q_1, \ldots, q_{m-1}$ the positions of the explicit occurrences of ','.

We have $d(t, q-1) = d(t, 1 + \ell(\varphi)) = d(t, 1+1) + d(\varphi, \ell(\varphi)) = 1 + d(\varphi, \ell(\varphi))$.

If $t[q-1] = \varphi[\ell(\varphi)] =$ ')' then $d(t,q) = d(t,q-1) - 1 = d(\varphi, \ell(\varphi)) = 1$.
Else $t[q-1] = \varphi[\ell(\varphi)] \notin \{\text{'('}, \text{')'}\}$, so $d(t,q) = d(t,q-1) = 1 + d(\varphi, \ell(\varphi)) = 1$.

If $m > 1$ we can prove that for each $i = 1 \ldots m-1$ $d(t, q_i) = 1$.

First of all we agree that $d(t, q+2) = d(t,q) - 1 + 1 = 1$.

We have also that

$$d(t, q_1 - 1) = d(t, q + 1 + \ell(\varphi_1)) = d(t, q+1+1) + d(\varphi_1, \ell(\varphi_1)) = 1 + d(\varphi_1, \ell(\varphi_1)).$$

If $t[q_1 - 1] = \varphi_1[\ell(\varphi_1)] =$ ')' then $d(t, q_1) = d(t, q_1 - 1) - 1 = d(\varphi_1, \ell(\varphi_1)) = 1$.
Else $t[q_1 - 1] = \varphi_1[\ell(\varphi_1)] \notin \{\text{'('}, \text{')'}\}$ so $d(t, q_1) = d(t, q_1 - 1) = 1 + d(\varphi_1, \ell(\varphi_1)) = 1$.

If $m = 2$ we have finished this step. Now suppose $m > 2$. Let $i = 1 \ldots m-2$ and suppose $d(t, q_i) = 1$. We'll show that $d(t, q_{i+1}) = 1$ also holds.

In fact

$$d(t, q_{i+1} - 1) = d(t, q_i + \ell(\varphi_{i+1})) = d(t, q_i + 1) + d(\varphi_{i+1}, \ell(\varphi_{i+1})) =$$
$$= 1 + d(\varphi_{i+1}, \ell(\varphi_{i+1})).$$

If $t[q_{i+1} - 1] = \varphi_{i+1}[\ell(\varphi_{i+1})] =$ ')' then
$d(t, q_{i+1}) = d(t, q_{i+1} - 1) - 1 = d(\varphi_{i+1}, \ell(\varphi_{i+1})) = 1$.

Else $t[q_{i+1}-1]=\varphi_{i+1}[\ell(\varphi_{i+1})]\notin\{`(', `)'\}$ so
$d(t,q_{i+1})=d(t,q_{i+1}-1)=1+d(\varphi_{i+1},\ell(\varphi_{i+1}))=1$.

So it is shown that for each $i=1\dots m-1$ $d(t,q_i)=1$.

We now want to show that $d(t,\ell(t))=1$.

If $m=1$ then

$$d(t,\ell(t)-1)=d(t,q+1+\ell(\varphi_1))=d(t,q+2)+d(\varphi_1,\ell(\varphi_1))=d(t,q)+d(\varphi_1,\ell(\varphi_1))= \\ =1+d(\varphi_m,\ell(\varphi_m)).$$

If $m>1$ then

$$d(t,\ell(t)-1)=d(t,q_{m-1}+\ell(\varphi_m))=d(t,q_{m-1}+1)+d(\varphi_m,\ell(\varphi_m))=1+d(\varphi_m,\ell(\varphi_m)).$$

If $t[\ell(t)-1]=\varphi_m[\ell(\varphi_m)]=`)'$ then $d(t,\ell(t))=d(t,\ell(t)-1)-1=d(\varphi_m,\ell(\varphi_m))=1$. Else $t[\ell(t)-1]=\varphi_m[\ell(\varphi_m)]\notin\{`(', `)'\}$ so $d(t,\ell(t))=d(t,\ell(t)-1)=1+d(\varphi_m,\ell(\varphi_m))=1$.

Let's now examine the facts we have to prove. It is true that $t[\ell(t)]\neq `('$. It's also true that $t[\ell(t)]=`)'$ and $d(t,\ell(t))=1$.

Now let $\alpha\in\{1,\dots,\ell(t)\}$ and ($t[\alpha]=`:'$ or $t[\alpha]=`,'$ or $t[\alpha]=`)'$).

If $\alpha\in\{q,q_1,\dots,q_{m-1},\ell(t)\}$ we have already shown that $d(t,\alpha)=1$. Otherwise there are these alternative possibilities:

a. $(\alpha>1)\wedge(\alpha<q)$,
b. $(m=1)\wedge(\alpha>q+1)\wedge(\alpha<\ell(t))$,
c. $(m>1)\wedge(\alpha>q+1)\wedge(\alpha<q_1)$,
d. $(m>2)\wedge(\exists i=1\dots m-2:(\alpha>q_i)\wedge(\alpha<q_{i+1}))$,
e. $(m>1)\wedge(\alpha>q_{m-1})\wedge(\alpha<\ell(t))$.

In the situation a. $\varphi[\alpha-1]=t[\alpha]$,

$$d(t,\alpha)=d(t,1+(\alpha-1))=d(t,2)+d(\varphi,\alpha-1)=1+d(\varphi,\alpha-1)\geqslant 2.$$

In the situation b. we have

$$q+1<\alpha<\ell(t),$$
$$0<\alpha-(q+1)<\ell(t)-(q+1),$$
$$1\leqslant\alpha-(q+1)\leqslant\ell(t)-q-2=\ell(\varphi_1),$$
$$\varphi_1[\alpha-(q+1)]=t[\alpha],$$

$$d(t,\alpha)=d(t,q+1+(\alpha-(q+1)))=d(t,q+2)+d(\varphi_1,\alpha-(q+1))= \\ =1+d(\varphi_1,\alpha-(q+1))\geqslant 2.$$

In the situation c. we have

$$q+1<\alpha<q_1,$$
$$0<\alpha-(q+1)<q_1-(q+1),$$

$$1 \leqslant \alpha - (q+1) \leqslant q_1 - q - 2 = \ell(\varphi_1),$$
$$\varphi_1[\alpha - (q+1)] = t[\alpha],$$
$$d(t,\alpha) = d(t, q+1+(\alpha-(q+1))) = d(t, q+2) + d(\varphi_1, \alpha-(q+1)) =$$
$$= 1 + d(\varphi_1, \alpha - (q+1)) \geqslant 2.$$

In the situation d. we have

$$q_i < \alpha < q_{i+1},$$
$$0 < \alpha - q_i < q_{i+1} - q_i,$$
$$1 \leqslant \alpha - q_i \leqslant q_{i+1} - q_i - 1 = \ell(\varphi_{i+1}),$$
$$\varphi_{i+1}[\alpha - q_i] = t[\alpha],$$
$$d(t,\alpha) = d(t, q_i + (\alpha - q_i)) = d(t, q_i + 1) + d(\varphi_{i+1}, \alpha - q_i) =$$
$$= 1 + d(\varphi_{i+1}, \alpha - q_i) \geqslant 2.$$

In the situation e. we have

$$q_{m-1} < \alpha < \ell(t),$$
$$0 < \alpha - q_{m-1} < \ell(t) - q_{m-1},$$
$$1 \leqslant \alpha - q_{m-1} \leqslant \ell(t) - q_{m-1} - 1 = \ell(\varphi_m),$$
$$\varphi_m[\alpha - q_{m-1}] = t[\alpha],$$
$$d(t,\alpha) = d(t, q_{m-1} + (\alpha - q_{m-1})) = d(t, q_{m-1}+1) + d(\varphi_m, \alpha - q_{m-1}) =$$
$$= 1 + d(\varphi_m, \alpha - q_{m-1}) \geqslant 2.$$

Let $k \in K(n)$ and $t \in \mathbf{E_d}(\mathbf{n+1}, \mathbf{k})$. There exist $f \in \mathcal{F}$, $\varphi_1, \ldots, \varphi_m$ in $E(n,k)$ such that $t = f(\varphi_1, \ldots, \varphi_m)$.

In this representation we see 'explicit occurrences' of the symbols '(' , ')' and ','. There are explicit occurrences of ',' only when $m > 1$. If $m > 1$ we indicate with $q_1, \ldots, q_{m-1}$ the positions of the explicit occurrences of ','.

It is immediate to see that $d(t,3) = 1$.

If $m > 1$ we can prove that for each $i = 1 \ldots m-1$ $d(t, q_i) = 1$.

We have $d(t, q_1 - 1) = d(t, 2 + \ell(\varphi_1)) = d(t, 2+1) + d(\varphi_1, \ell(\varphi_1)) = 1 + d(\varphi_1, \ell(\varphi_1))$.

If $t[q_1 - 1] = \varphi_1[\ell(\varphi_1)] =$ ')' then $d(t, q_1) = d(t, q_1 - 1) - 1 = d(\varphi_1, \ell(\varphi_1)) = 1$. Else $t[q_1 - 1] = \varphi_1[\ell(\varphi_1)] \notin \{$'(', ')'$\}$, so $d(t, q_1) = d(t, q_1 - 1) = 1 + d(\varphi_1, \ell(\varphi_1)) = 1$.

If $m = 2$ we have finished this step. Now suppose $m > 2$. Let $i = 1 \ldots m-2$ and suppose $d(t, q_i) = 1$. We'll show that $d(t, q_{i+1}) = 1$ also holds.

In fact

$$d(t, q_{i+1} - 1) = d(t, q_i + \ell(\varphi_{i+1})) = d(t, q_i + 1) + d(\varphi_{i+1}, \ell(\varphi_{i+1})) =$$
$$= 1 + d(\varphi_{i+1}, \ell(\varphi_{i+1})).$$

If $t[q_{i+1}-1] = \varphi_{i+1}[\ell(\varphi_{i+1})] =$ ‘)’ then
$d(t, q_{i+1}) = d(t, q_{i+1}-1) - 1 = d(\varphi_{i+1}, \ell(\varphi_{i+1})) = 1$.
Else $t[q_{i+1}-1] = \varphi_{i+1}[\ell(\varphi_{i+1})] \notin \{$‘(’, ‘)’$\}$ so
$d(t, q_{i+1}) = d(t, q_{i+1}-1) = 1 + d(\varphi_{i+1}, \ell(\varphi_{i+1})) = 1$.

So it is shown that for each $i = 1 \dots m-1$ $d(t, q_i) = 1$.

We now want to show that $d(t, \ell(t)) = 1$.

If $m = 1$ then

$$\begin{aligned} d(t, \ell(t)-1) &= d(t, 2+\ell(\varphi_1)) = d(t, 2+1) + d(\varphi_1, \ell(\varphi_1)) = 1 + d(\varphi_1, \ell(\varphi_1)) = \\ &= 1 + d(\varphi_m, \ell(\varphi_m)). \end{aligned}$$

If $m > 1$ then

$$d(t, \ell(t)-1) = d(t, q_{m-1} + \ell(\varphi_m)) = d(t, q_{m-1}+1) + d(\varphi_m, \ell(\varphi_m)) = 1 + d(\varphi_m, \ell(\varphi_m)).$$

If $t[\ell(t)-1] = \varphi_m[\ell(\varphi_m)] =$ ‘)’ then $d(t, \ell(t)) = d(t, \ell(t)-1) - 1 = d(\varphi_m, \ell(\varphi_m)) = 1$.
Else $t[\ell(t)-1] = \varphi_m[\ell(\varphi_m)] \notin \{$‘(’, ‘)’$\}$ so $d(t, \ell(t)) = d(t, \ell(t)-1) = 1 + d(\varphi_m, \ell(\varphi_m)) = 1$.

Let’s now examine the facts we have to prove. It is true that $t[\ell(t)] \neq$ ‘(’. It’s also true that $t[\ell(t)] =$ ‘)’ and $d(t, \ell(t)) = 1$.

Now let $\alpha \in \{1, \dots, \ell(t)\}$ and ($t[\alpha] =$ ‘:’ or $t[\alpha] =$ ‘,’ or $t[\alpha] =$ ‘)’).

If $\alpha \in \{q_1, \dots, q_{m-1}, \ell(t)\}$ we have already shown that $d(t, \alpha) = 1$. Otherwise there are these alternative possibilities:

a. $(m = 1) \wedge (\alpha > 2) \wedge (\alpha < \ell(t))$,
b. $(m > 1) \wedge (\alpha > 2) \wedge (\alpha < q_1)$,
c. $(m > 2) \wedge (\exists i = 1 \dots m-2 : (\alpha > q_i) \wedge (\alpha < q_{i+1}))$,
d. $(m > 1) \wedge (\alpha > q_{m-1}) \wedge (\alpha < \ell(t))$.

In the situation a. we have

$$2 < \alpha < \ell(t),$$

$$0 < \alpha - 2 < \ell(t) - 2,$$

$$1 \leqslant \alpha - 2 \leqslant \ell(t) - 2 - 1 = \ell(\varphi_1),$$

$$\varphi_1[\alpha - 2] = t[\alpha],$$

$$\begin{aligned} d(t, \alpha) &= d(t, 2 + (\alpha - 2)) = d(t, 2+1) + d(\varphi_1, \alpha - 2) = \\ &= 1 + d(\varphi_1, \alpha - 2) \geqslant 2. \end{aligned}$$

In the situation b. we have

$$2 < \alpha < q_1,$$

$$0 < \alpha - 2 < q_1 - 2,$$

$$1 \leqslant \alpha - 2 \leqslant q_1 - 2 - 1 = \ell(\varphi_1),$$

$$\varphi_1[\alpha - 2] = t[\alpha],$$

$$d(t,\alpha) = d(t, 2 + (\alpha - 2)) = d(t, 2+1) + d(\varphi_1, \alpha - 2) =$$
$$= 1 + d(\varphi_1, \alpha - 2) \geqslant 2.$$

In the situation c. we have

$$q_i < \alpha < q_{i+1},$$
$$0 < \alpha - q_i < q_{i+1} - q_i,$$
$$1 \leqslant \alpha - q_i \leqslant q_{i+1} - q_i - 1 = \ell(\varphi_{i+1}),$$
$$\varphi_{i+1}[\alpha - q_i] = t[\alpha],$$
$$d(t,\alpha) = d(t, q_i + (\alpha - q_i)) = d(t, q_i + 1) + d(\varphi_{i+1}, \alpha - q_i) =$$
$$= 1 + d(\varphi_{i+1}, \alpha - q_i) \geqslant 2.$$

In the situation d. we have

$$q_{m-1} < \alpha < \ell(t),$$
$$0 < \alpha - q_{m-1} < \ell(t) - q_{m-1},$$
$$1 \leqslant \alpha - q_{m-1} \leqslant \ell(t) - q_{m-1} - 1 = \ell(\varphi_m),$$
$$\varphi_m[\alpha - q_{m-1}] = t[\alpha],$$
$$d(t,\alpha) = d(t, q_{m-1} + (\alpha - q_{m-1})) = d(t, q_{m-1} + 1) + d(\varphi_m, \alpha - q_{m-1}) =$$
$$= 1 + d(\varphi_m, \alpha - q_{m-1}) \geqslant 2.$$

Let $k \in K(n)$ and $t \in \mathbf{E_e(n+1,k)}$. As a consequence to $t \in E_e(n+1,k)$ there exist

- a positive integer m,
- a function x whose domain is $\{1,\dots,m\}$ such that for each $i = 1 \dots m$ $x_i \in \mathcal{V} - var(k)$, and for each $i,j = 1 \dots m$ $i \neq j \to x_i \neq x_j$,
- a function φ whose domain is $\{1,\dots,m\}$ such that for each $i = 1 \dots m$ $\varphi_i \in E(n)$,
- $\phi \in E(n)$

such that $t = \{\}(x_1 : \varphi_1, \dots, x_m : \varphi_m, \phi)$.

In this representation we see 'explicit occurrences' of the symbols ',' and ':'. We indicate with $q_1, \dots, q_m$ the positions of the explicit occurrences of ':' and with $r_1 \dots r_m$ the positions of the explicit occurrences of ','. The only explicit occurrence of ')' has the position $\ell(t)$.

We want to show that for each $i = 1 \dots m$ $d(t, q_i) = 1, d(t, r_i) = 1$ and that $d(t, \ell(t)) = 1$.

It is obvious that $d(t, q_1) = 1$. Moreover

$$d(t, r_1 - 1) = d(t, q_1 + \ell(\varphi_1)) = d(t, q_1 + 1) + d(\varphi_1, \ell(\varphi_1)) =$$
$$= 1 + d(\varphi_1, \ell(\varphi_1)).$$

If $t[r_1 - 1] = \varphi_1[\ell(\varphi_1)] =$ ')' then $d(t, r_1) = d(t, r_1 - 1) - 1 = d(\varphi_1, \ell(\varphi_1)) = 1$. Else $t[r_1 - 1] = \varphi_1[\ell(\varphi_1)] \notin \{\text{'('}, \text{')'}\}$ so $d(t, r_1) = d(t, r_1 - 1) = 1 + d(\varphi_1, \ell(\varphi_1)) = 1$.

If $m = 1$ we have shown that for each $i = 1 \dots m$ $d(t, q_i) = 1, d(t, r_i) = 1$. Now suppose $m > 1$, let $i = 1 \dots m-1$ and suppose $d(t, q_i) = 1, d(t, r_i) = 1$. We show that $d(t, q_{i+1}) = 1, d(t, r_{i+1}) = 1$.

We have $q_{i+1} = r_i + 2$ and it is immediate that $d(t, q_{i+1}) = 1$. Moreover

$$\begin{aligned} d(t, r_{i+1} - 1) &= d(t, q_{i+1} + \ell(\varphi_{i+1})) = \\ &= d(t, q_{i+1} + 1) + d(\varphi_{i+1}, \ell(\varphi_{i+1})) = 1 + d(\varphi_{i+1}, \ell(\varphi_{i+1})). \end{aligned}$$

If $t[r_{i+1} - 1] = \varphi_{i+1}[\ell(\varphi_{i+1})] =$ ‘)’ then
$d(t, r_{i+1}) = d(t, r_{i+1} - 1) - 1 = d(\varphi_{i+1}, \ell(\varphi_{i+1})) = 1$.
Else $t[r_{i+1} - 1] = \varphi_{i+1}[\ell(\varphi_{i+1})] \notin \{$‘(’, ‘)’$\}$ so
$d(t, r_{i+1}) = d(t, r_{i+1} - 1) = 1 + d(\varphi_{i+1}, \ell(\varphi_{i+1})) = 1$.

Furthermore

$$\begin{aligned} d(t, \ell(t) - 1) &= d(t, r_m + \ell(\phi)) = \\ &= d(t, r_m + 1) + d(\phi, \ell(\phi)) = 1 + d(\phi, \ell(\phi)). \end{aligned}$$

If $t[\ell(t) - 1] = \phi[\ell(\phi)] =$ ‘)’ then $d(t, \ell(t)) = d(t, \ell(t) - 1) - 1 = d(\phi, \ell(\phi)) = 1$.
Else $t[\ell(t) - 1] = \phi[\ell(\phi)] \notin \{$‘(’, ‘)’$\}$ so $d(t, \ell(t)) = d(t, \ell(t) - 1) = 1 + d(\phi, \ell(\phi)) = 1$.

Let’s now examine the facts we have to prove. It is true that $t[\ell(t)] \neq$ ‘(’. It’s also true that $t[\ell(t)] =$ ‘)’ and $d(t, \ell(t)) = 1$.

Now let $\alpha \in \{1, \dots, \ell(t)\}$ and ($t[\alpha] =$ ‘:’ or $t[\alpha] =$ ‘,’ or $t[\alpha] =$ ‘)’).

If $\alpha \in \{q_1, \dots, q_m, r_1, \dots, r_m, \ell(t)\}$ we have already shown that $d(t, \alpha) = 1$. Otherwise there are these alternative possibilities:

a. $\exists i = 1 \dots m$ such that $q_i < \alpha < r_i$,
b. $r_m < \alpha < \ell(t)$.

In the situation a. we have

$$q_i < \alpha < r_i,$$
$$0 < \alpha - q_i < r_i - q_i,$$
$$1 \leqslant \alpha - q_i \leqslant r_i - q_i - 1 = \ell(\varphi_i),$$
$$\varphi_i[\alpha - q_i] = t[\alpha],$$

$$\begin{aligned} d(t, \alpha) &= d(t, q_i + (\alpha - q_i)) = d(t, q_i + 1) + d(\varphi_i, \alpha - q_i) = \\ &= 1 + d(\varphi_i, \alpha - q_i) \geqslant 2. \end{aligned}$$

In the situation b. we have

$$r_m < \alpha < \ell(t),$$
$$0 < \alpha - r_m < \ell(t) - r_m,$$
$$1 \leqslant \alpha - r_m \leqslant \ell(t) - r_m - 1 = \ell(\phi),$$
$$\phi[\alpha - r_m] = t[\alpha],$$

$$\begin{aligned} d(t,\alpha) &= d(t, r_m + (\alpha - r_m)) = d(t, r_m + 1) + d(\phi, \alpha - r_m) = \\ &= 1 + d(\phi, \alpha - r_m) \geqslant 2. \end{aligned}$$

■

4. Proofs and deductive methodology

In this chapter we will define deductive systems and proofs and we will introduce other concepts and results related to our deductive methodology. Given a language $\mathcal{L}$, with $\mathcal{L} = (\mathcal{V}, \mathcal{F}, \mathcal{C}, \#)$, we begin with some preliminary definitions.

Let $K = \bigcup_{n \geqslant 1} K(n)$.

For each $k \in K$ let

$$E(k) = \bigcup_{n \geqslant 1 : k \in K(n)} E(n, k) ,$$

$$E_s(k) = \{t | t \in E(k), \forall \sigma \in \Xi(k) \ \#(k, t, \sigma) \text{ is a set } \} .$$

Let $E = \bigcup_{k \in K} E(k)$; E is the set of all expressions in our language.

One expression $t \in E(k)$ is a 'sentence with respect to k' when for each $\sigma \in \Xi(k)$ $\#(k, t, \sigma)$ is true or $\#(k, t, \sigma)$ is false.

We define $S(k) = \{t | t \in E(k), t \text{ is a sentence with respect to } k\}$.

For each $t \in E(\epsilon)$ we define $\#(t) = \#(\epsilon, t, \epsilon)$.

A sentence with respect to ϵ will simply be called a 'sentence'.

At this point we can define what is a proof in our language. To define this we need to define the notions of axiom and rule.

An *axiom* is a set A such that

- $A \subseteq S(\epsilon)$
- for each $\varphi \in A$ $\#(\varphi)$ holds.

The property 'for each $\varphi \in A$ $\#(\varphi)$ holds' states that axiom A is 'sound'.

Given a positive integer n we indicate with $S(\epsilon)^n$ the set of all n-tuples $(\varphi_1, \ldots, \varphi_n)$ for $\varphi_1, \ldots, \varphi_n \in S(\epsilon)$. An n-ary *rule* is a set $R \subseteq S(\epsilon)^{n+1}$ such that

- for each $(\varphi_1, \ldots, \varphi_n, \varphi) \in R$ if $\#(\varphi_1), \ldots, \#(\varphi_n)$ hold then $\#(\varphi)$ holds.

The property 'for each $(\varphi_1, \ldots, \varphi_n, \varphi) \in R$ if $\#(\varphi_1), \ldots, \#(\varphi_n)$ hold then $\#(\varphi)$ holds' states that rule R is 'sound'.

Both in the definition of axiom and rule we have included a requirement of soundness.

A deductive system is built on top of our language $\mathcal{L}$, and is identified by a pair $(\mathcal{A}, \mathcal{R})$ where $\mathcal{A}$ is a set of axioms in $\mathcal{L}$ and $\mathcal{R}$ is a set of rules in $\mathcal{L}$.

Given a language $\mathcal{L}$, $\mathcal{D} = (\mathcal{A}, \mathcal{R})$ deductive system in $\mathcal{L}$, φ, $\psi_1, \ldots, \psi_m$ sentences in $\mathcal{L}$, we say that $(\psi_1, \ldots, \psi_m)$ is a *proof* of φ in $\mathcal{D}$ if and only if

- there exists $A \in \mathcal{A}$ such that $\psi_1 \in A$;
- if $m > 1$ then for each $j = 2 \ldots m$ one of the following holds
  - there exists $A \in \mathcal{A}$ such that $\psi_j \in A$,
  - there exist an n-ary rule $R \in \mathcal{R}$ and $i_1, \ldots, i_n < j$ such that $(\psi_{i_1}, \ldots, \psi_{i_n}, \psi_j) \in R$;
- $\psi_m = \varphi$.

Given $\mathcal{D} = (\mathcal{A}, \mathcal{R})$ deductive system in $\mathcal{L}$ and φ sentence in $\mathcal{L}$ we say that φ *is derivable in* $\mathcal{D}$ and write $\vdash_{\mathcal{D}} \varphi$ if and only if there exist $\psi_1, \ldots, \psi_m$ sentences in $\mathcal{L}$ such that $(\psi_1, \ldots, \psi_m)$ is a proof of φ in $\mathcal{D}$.

A deductive system $\mathcal{D} = (\mathcal{A}, \mathcal{R})$ is said to be *sound* if and only if for each φ sentence in $\mathcal{L}$ if $\vdash_{\mathcal{D}} \varphi$ then $\#(\varphi)$ holds. In the next lemma we easily prove that each of our systems is sound.

LEMMA 4.1. *Let $\mathcal{D} = (\mathcal{A}, \mathcal{R})$ be a deductive system in $\mathcal{L}$. Then $\mathcal{D}$ is sound.*

Proof.

Let φ be a sentence in $\mathcal{L}$. Suppose $\vdash_{\mathcal{D}} \varphi$. There exist $\psi_1, \ldots, \psi_m$ sentences in $\mathcal{L}$ such that $(\psi_1, \ldots, \psi_m)$ is a proof of φ in $\mathcal{D}$. We can show that for each $j = 1 \ldots m$ $\#(\psi_j)$ holds.

There exists $A \in \mathcal{A}$ such that $\psi_1 \in A$, so $\#(\psi_1)$ holds.

If $m > 1$ suppose $j = 2 \ldots m$.

If there exists $A \in \mathcal{A}$ such that $\psi_j \in A$ then $\#(\psi_j)$ holds.

Otherwise there exist an n-ary rule $R \in \mathcal{R}$ and $i_1, \ldots, i_n < j$ such that

$$(\psi_{i_1}, \ldots, \psi_{i_n}, \psi_j) \in R \ .$$

Since $\#(\psi_{i_1}), \ldots, \#(\psi_{i_n})$ all hold then $\#(\psi_j)$ also holds. ■

At the beginning of chapter 3 we have introduced the logical connectives. In our deductions, expressions will make an extensive use of the logical connectives, so we assume that all of these symbols: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \forall, \exists$ are in our set $\mathcal{F}$. For each of these operators f $A_f(x_1, \ldots, x_n)$ and $P_f(x_1, \ldots, x_n)$ are defined as specified at the beginning of chapter 3.

We now need to introduce some other fundamental notions and results relevant to our deductive methodology.

Lemma 4.2. *For each n positive integer such that $n \geqslant 2$, $k \in K(n)$: $k \neq \epsilon$ there exists $m < n$ such that $k \in K(m)^+$.*

Proof.

We prove this by induction on n. Clearly if $k \in K(2)$ and $k \neq \epsilon$ then $k \in K(1)^+$.

Let $n \geqslant 2$, $k \in K(n+1)$: $k \neq \epsilon$. Clearly if $k \in K(n)^+$ our proof is finished. Otherwise $k \in K(n)$ and in this case we can apply the inductive hypothesis. ■

Lemma 4.3. *For each n positive integer such that $n \geqslant 2$, $k \in K(n)$: $k \neq \epsilon$*

- *there exist $m < n$, $h \in K(m), \phi \in E_s(m,h), y \in (\mathcal{V} - var(h))$ such that $k = h + (y,\phi)$, $\Xi(k) = \{\sigma + (y,s) | \, \sigma \in \Xi(h), s \in \#(h,\phi,\sigma)\}$;*
- *for each $g \in K(n), \theta \in E_s(n,g), z \in (\mathcal{V} - var(g))$ such that $k = g + (z,\theta)$ $\Xi(k) = \{\sigma + (z,s) | \, \sigma \in \Xi(g), s \in \#(g,\theta,\sigma)\}$.*

Proof.

The first part clearly follows from lemma 4.2. The second part holds because we have $g = h, z = y, \theta = \phi$. ■

Lemma 4.4. *For each n positive integer such that $n \geqslant 2$, $k \in K(n) : k \neq \epsilon$, $\sigma \in \Xi(k)$, $h \in \mathcal{R}(k)$ such that $h \neq k$, it results $\sigma_{/dom(h)} \in \Xi(h)$.*

Proof.

We prove this by induction on n. Let $k \in K(2)$: $k \neq \epsilon$, $\sigma \in \Xi(k)$, $h \in \mathcal{R}(k)$ such that $h \neq k$. Clearly $k \in K(1)^+$, so there exist $g \in K(1)$, $\phi \in E_s(1,g)$, $y \in \mathcal{V} - var(g)$ such that $k = g + (y,\phi)$. By lemma 3.5 we obtain that $h \in \mathcal{R}(g)$. Since $g = \epsilon$ then also $h = \epsilon$, so $\sigma_{/dom(h)} = \sigma_{/\emptyset} = \epsilon \in \Xi(\epsilon) = \Xi(h)$.

In order to perform the inductive step, let $k \in K(n+1) : k \neq \epsilon$, $\sigma \in \Xi(k)$, $h \in \mathcal{R}(k)$ such that $h \neq k$. By lemma 4.2 there exists $m \leqslant n$ such that $k \in K(m)^+$. Then there exist $g \in K(m), \phi \in E_s(m,g), y \in (\mathcal{V} - var(g))$ such that $k = g + (y,\phi)$. Moreover

$$\Xi(k) = \{\rho + (y,s) | \, \rho \in \Xi(g), s \in \#(g,\phi,\rho)\} .$$

Therefore there exist $\rho \in \Xi(g), s \in \#(g,\phi,\rho)$ such that $\sigma = \rho + (y,s)$. By assumption 3.1.1 and lemma 3.10 we have that $\sigma_{/dom(g)} = \sigma_{/dom(\rho)} = \rho$.

If $h = g$ then $\sigma_{/dom(h)} = \sigma_{/dom(g)} = \rho \in \Xi(h)$.

Otherwise we have $h \neq g$. Since $k = g + (y,\phi)$, $h \in \mathcal{R}(k)$, $h \neq k$ by lemma 3.5 we have that $h \in \mathcal{R}(g)$. If $g = \epsilon$ we would have $h = \epsilon = g$, so $g \neq \epsilon$. This implies that $m \geqslant 2$. By our inductive hypothesis we obtain $\rho_{/dom(h)} \in \Xi(h)$. Now

$$\sigma_{/dom(h)} = (\sigma_{/dom(g)})_{/dom(h)} = \rho_{/dom(h)} \in \Xi(h).$$

■

Lemma 4.5. *For each n positive integer $k = (x,\varphi) \in K(n)$, for each $i,j \in dom(k)$ $i \neq j \to x_i \neq x_j$.*

Proof.

The initial step is trivially verified.

Let n be a positive integer, let $k \in K(n+1)$, we want to verify that
for each $i, j \in dom(k)$ $i \neq j \to x_i \neq x_j$.

If $k \in K(n)$ this is obvioulsy verified.

Otherwise $k \in K(n)^+$, so there exist $h \in K(n), \phi \in E_s(n,h), y \in (\mathcal{V} - var(h))$ such that $k = h + (y, \phi)$. In this case given $i, j \in dom(k)$ such that $i \neq j$, if both $i, j \in dom(h)$ then clearly $x_i \neq x_j$. Otherwise $i \notin dom(h)$ or $j \notin dom(h)$. Let's consider the case where $i \notin dom(h)$ (the other is trivially analogous), here $j \in dom(h)$, so $x_i = y \neq x_j$. ■

LEMMA 4.6. *For each n positive integer, $k = (x, \varphi) \in K(n)$, $\sigma = (z, \xi) \in \Xi(k)$, we have $z = x$ (and so $var(\sigma) = image(z) = image(x) = var(k)$).*

Proof.

The initial step is trivially verified.

Let n be a positive integer, let $k = (x, \varphi) \in K(n+1)$, let $\sigma = (z, \xi) \in \Xi(k)$. If $k \in K(n)$ then $z = x$.

Otherwise $k \in K(n)^+$, so there exist $h = (w, \eta) \in K(n), \phi \in E_s(n,h), y \in (\mathcal{V} - var(h))$ such that $k = h + (y, \phi)$ and

$$\Xi(k) = \{\rho + (y, s) |\, \rho \in \Xi(h), s \in \#(h, \phi, \rho)\}\,.$$

There exists $\rho = (u, \nu) \in \Xi(h), s \in \#(h, \phi, \rho)$ such that $\sigma = \rho + (y, s)$. Using our inductive hypothesis we can state that $u = w$, so x is obtained by adding to the function w a couple (j, y), and z is obtained by adding to the function u the same couple (j, y). Therefore $z = x$. ■

LEMMA 4.7. *For each n positive integer $k = (x, \varphi) \in K(n)$, $\sigma = (z, \xi) \in \Xi(k)$, for each $i, j \in dom(\sigma)$ $i \neq j \to z_i \neq z_j$.*

Proof.

From lemma 4.5 it follows that for each $i, j \in dom(k)$ $i \neq j \to x_i \neq x_j$. Since $dom(\sigma) = dom(k)$ and $z = x$ it is proved that for each $i, j \in dom(\sigma)$ $i \neq j \to z_i \neq z_j$. ■

LEMMA 4.8. *For each n positive integer, $k = (x, \varphi), h = (y, \psi) \in K(n)$ if $h \sqsubseteq k$ then for each $i \in dom(k)$, $j \in dom(h)$ $x_i = y_j \to \varphi_i = \psi_j$.*

Proof.

From lemma 4.5 it follows that for each $i, j \in dom(k)$ $i \neq j \to x_i \neq x_j$. With this we can apply lemma 3.2 and obtain that
for each $i \in dom(k)$, $j \in dom(h)$ $x_i = y_j \to \varphi_i = \psi_j$. ■

LEMMA 4.9. *For each n positive integer, $h, k \in K(n)$, $\sigma = (x, \eta) \in \Xi(k)$, $\rho = (y, \theta) \in \Xi(h)$, if $\rho \sqsubseteq \sigma$ then for each $i \in dom(\sigma)$, $j \in dom(\rho)$ $x_i = y_j \to \eta_i = \theta_j$.*

Proof.

From lemma 4.7 it follows that for each for each $i, j \in dom(\sigma)$ $i \neq j \to x_i \neq x_j$. With this we can apply lemma 3.2 and obtain that
for each $i \in dom(\sigma)$, $j \in dom(\rho)$ $x_i = y_j \to \eta_i = \theta_j$. ∎

LEMMA 4.10. *For each n positive integer such that $n \geqslant 2$, $k \in K(n)$, $t \in E(n,k)$ such that $t \notin \mathcal{C}$ one of the following two alternatives holds:*

- *$t \in E_a(n,k) \cup E_c(n,k) \cup E_d(n,k) \cup E_e(n,k)$;*
- *$n > 2$ and there exist m positive integer such that $2 \leqslant m < n$, $h \in K(m)$ such that $h \sqsubseteq k$, $t \in E_a(m,h) \cup E_c(m,h) \cup E_d(m,h) \cup E_e(m,h)$ and for each $\sigma \in \Xi(k)$ $\sigma_{/dom(h)} \in \Xi(h)$ and $\#(k,t,\sigma) = \#(h,t,\sigma_{/dom(h)})$.*

Proof.

Of course we begin with the case $n = 2$. Let $k \in K(2)$, $t \in E(2,k)$ such that $t \notin \mathcal{C}$. We have $K(2) = K(1) \cup K(1)^+$.

If $k \in K(1)^+$ we have $E(2,k) = E_a(2,k)$, so $t \in E_a(2,k)$.

If $k \in K(1)$ we have

$$E(2,k) = E(1,k) \cup E_b(2,k) \cup E_c(2,k) \cup E_d(2,k) \cup E_e(2,k)\ .$$

Since $k = \epsilon$ we have

$$E(2,k) = \mathcal{C} \cup E_c(2,k) \cup E_d(2,k) \cup E_e(2,k)\ .$$

Therefore in this case we have

$$t \in E_c(2,k) \cup E_d(2,k) \cup E_e(2,k)\ .$$

Let now $n \geqslant 2$ and we try to prove the result for $n+1$. So let $k \in K(n+1)$, $t \in E(n+1,k)$ such that $t \notin \mathcal{C}$. We have $K(n+1) = K(n) \cup K(n)^+$.

If $k \in K(n)^+$ we have $E(n+1,k) = E_a(n+1,k)$ so $t \in E_a(n+1,k)$.

We now need to examine the case $k \in K(n)$. Here we have

$$E(n+1,k) = E(n,k) \cup E_b(n+1,k) \cup E_c(n+1,k) \cup E_d(n+1,k) \cup E_e(n+1,k)\ .$$

If $t \in E_c(n+1,k) \cup E_d(n+1,k) \cup E_e(n+1,k)$ then our result is verified.

If $t \in E(n,k)$ and we can apply our inductive hypothesis, which leads to two alternatives:

- $t \in E_a(n,k) \cup E_c(n,k) \cup E_d(n,k) \cup E_e(n,k)$;
- $n > 2$ and there exist m positive integer such that $2 \leqslant m < n$, $h \in K(m)$ such that $h \sqsubseteq k$, $t \in E_a(m,h) \cup E_c(m,h) \cup E_d(m,h) \cup E_e(m,h)$ and for each $\sigma \in \Xi(k)$ $\sigma_{/dom(h)} \in \Xi(h)$ and $\#(k,t,\sigma) = \#(h,t,\sigma_{/dom(h)})$.

In the first case we observe that $2 \leqslant n < n+1$, $k \in K(n)$, $k \sqsubseteq k$. Moreover for each $\sigma \in \Xi(k)$ $\sigma_{/dom(k)} = \sigma \in \Xi(k)$ and $\#(k,t,\sigma) = \#(k,t,\sigma_{/dom(k)})$.

So in the first case our result is verified.

Let's examine the second case. Here $2 \leqslant m < n < n+1$, $h \in K(m)$, $h \sqsubseteq k$, for each $\sigma \in \Xi(k)$ $\sigma_{/dom(h)} \in \Xi(h)$ and $\#(k,t,\sigma) = \#(h,t,\sigma_{/dom(h)})$. So everything is as expected and our result is verified in this case too.

We have still one case to examine, which is the case of $t \in E_b(n+1,k)$. Here we have $k \neq \epsilon$ so by assumption 3.1.2 there exist $m < n$, $h \in K(m)$, $\phi \in E_s(m,h)$, $y \in (\mathcal{V} - var(h))$ such that $k = h + (y,\phi)$. Moreover by the definition of $E_b(n+1,k)$ we know that $t \in E(n,h)$. So we can apply our inductive hypothesis, which again leads to two alternatives:

- $t \in E_a(n,h) \cup E_c(n,h) \cup E_d(n,h) \cup E_e(n,h)$;
- $n > 2$ and there exist p positive integer such that $2 \leqslant p < n$, $g \in K(p)$ such that $g \sqsubseteq h$, $t \in E_a(p,g) \cup E_c(p,g) \cup E_d(p,g) \cup E_e(p,g)$ and for each $\rho \in \Xi(h)$ $\rho_{/dom(g)} \in \Xi(g)$ and $\#(h,t,\rho) = \#(g,t,\rho_{/dom(g)})$.

In the first case we observe that $2 \leqslant n < n+1$, $h \in K(n)$, $h \sqsubseteq k$, $t \in E_a(n,h) \cup E_c(n,h) \cup E_d(n,h) \cup E_e(n,h)$, moreover for each $\sigma = \rho + (y,s) \in \Xi(k)$ we have

- $\#(k,t,\sigma) = \#(h,t,\rho)$,
- $\sigma_{/dom(h)} = \sigma_{/dom(\rho)} = \rho$,
- therefore $\#(k,t,\sigma) = \#(h,t,\sigma_{/dom(h)})$.

Let's examine the second case. Here $2 \leqslant p < n < n+1$, $g \in K(p)$, $g \sqsubseteq h \sqsubseteq k$, $t \in E_a(p,g) \cup E_c(p,g) \cup E_d(p,g) \cup E_e(p,g)$. Moreover for each $\sigma = \rho + (y,s) \in \Xi(k)$ we have

- $\#(k,t,\sigma) = \#(h,t,\rho)$,
- $\sigma_{/dom(h)} = \sigma_{/dom(\rho)} = \rho$,
- $\sigma_{/dom(g)} = (\sigma_{/dom(h)})_{/dom(g)} = \rho_{/dom(g)}$,
- $\#(k,t,\sigma) = \#(h,t,\rho) = \#(g,t,\rho_{/dom(g)}) = \#(g,t,\sigma_{/dom(g)})$.

■

LEMMA 4.11. *For each n positive integer, $k \in K(n)$, $t \in E(n,k)$ if $t \in \mathcal{C}$ then for each $\sigma \in \Xi(k)$ $\#(k,t,\sigma) = \#(t)$.*

Proof.

Let's verify the result for $n = 1$. Here $k = \epsilon$, for each $\sigma \in \Xi(\epsilon)$ $\sigma = \epsilon$ so $\#(k,t,\sigma) = \#(\epsilon,t,\epsilon) = \#(t)$.

Now let's examine the inductive step. Given $k \in K(n+1)$, $t \in E(n+1,k)$ such that $t \in \mathcal{C}$ and $\sigma \in \Xi(k)$ we want to show that $\#(k,t,\sigma) = \#(t)$.

If $k \in K(n)^+$ then $t \in E_a(n+1,k)$, but since $t \in \mathcal{C}$ this cannot happen, so $k \in K(n)^+$ cannot happen.

Therefore $k \in K(n)$ and $t \in E(n,k) \cup E_b(n+1,k) \cup E_c(n+1,k) \cup E_d(n+1,k) \cup E_e(n+1,k)$.

Since $t \in \mathcal{C}$ it follows that $t \in E(n,k) \cup E_b(n+1,k)$.

If $t \in E(n,k)$ clearly $\#(k,t,\sigma) = \#(t)$ holds by the inductive hypothesis.

If $t \in E_b(n+1,k)$ then we have $k \neq \epsilon$ so by assumption 3.1.2 there exist $m < n$, $h \in K(m)$, $\phi \in E_s(m,h)$, $y \in (\mathcal{V} - var(h))$ such that $k = h + (y,\phi)$, $\Xi(k) = \{\rho + (y,s) | \rho \in \Xi(h), s \in \#(h,\phi,\rho)\})$. Moreover by the definition of $E_b(n+1,k)$ we know that $t \in E(n,h)$.

Clearly there exist $\rho \in \Xi(h)$, $s \in \#(h,\phi,\rho)$ such that $\sigma = \rho + (y,s)$ and $\#(k,t,\sigma) = \#(h,t,\rho)$. By the inductive hypothesis $\#(h,t,\rho) = \#(t)$, so $\#(k,t,\sigma) = \#(t)$. ∎

LEMMA 4.12. *Let $k = (x,\varphi), h = (y,\psi) \in K(n)$ such that for each $i \in dom(k)$, $j \in dom(h)$ $x_i = y_j \to \varphi_i = \psi_j$. Let $t \in E(n,k) \cap E(n,h)$. Let $\sigma = (x,s) \in \Xi(k)$, $\rho = (y,r) \in \Xi(h)$ such that for each $i \in dom(\sigma)$, $j \in dom(\rho)$ $x_i = y_j \to s_i = r_j$. Then $\#(k,t,\sigma) = \#(h,t,\rho)$.*

Proof.

We prove this by induction on a positive integer n.

Let's verify the initial step. Here we have $k = (x,\varphi), h = (y,\psi) \in K(1)$ such that for each $i \in dom(k)$, $j \in dom(h)$ $x_i = y_j \to \varphi_i = \psi_j$. This implies $h = \epsilon = k$. We have $t \in E(1,\epsilon) = \mathcal{C}$. We have $\sigma = (x,s) \in \Xi(\epsilon)$, $\rho = (y,r) \in \Xi(\epsilon)$ such that for each $i \in dom(\sigma)$, $j \in dom(\rho)$ $x_i = y_j \to s_i = r_j$. Of course this implies $\sigma = \epsilon = \rho$. Then by lemma 4.11 $\#(k,t,\sigma) = \#(t) = \#(h,t,\rho)$.

Let us see the inductive step, that is given a positive integer n we assume the result is true for each $m \leqslant n$ and we try to prove it for $n+1$. In other words what we are trying to prove is that for each $k = (x,\varphi), h = (y,\psi) \in K(n+1)$ such that for each $i \in dom(k)$, $j \in dom(h)$ $x_i = y_j \to \varphi_i = \psi_j$ and for each $t \in E(n+1,k) \cap E(n+1,h)$, $\sigma = (x,z) \in \Xi(k)$, $\rho = (y,r) \in \Xi(h)$ such that for each $i \in dom(\sigma)$, $j \in dom(\rho)$ $x_i = y_j \to z_i = r_j$ we have $\#(k,t,\sigma) = \#(h,t,\rho)$.

If $t \in \mathcal{C}$ then by lemma 4.11 $\#(k,t,\sigma) = \#(t) = \#(h,t,\rho)$.

Otherwise since $k \in K(n+1)$ and $t \in E(n+1,k)$ we can apply lemma 4.10 and obtain these two following alternative possibilities:

- $t \in E_a(n+1,k) \cup E_c(n+1,k) \cup E_d(n+1,k) \cup E_e(n+1,k)$;
- $n+1 > 2$ and there exist m positive integer such that $2 \leqslant m < n+1$, $\kappa \in K(m)$ such that $\kappa \sqsubseteq k$, $t \in E_a(m,\kappa) \cup E_c(m,\kappa) \cup E_d(m,\kappa) \cup E_e(m,\kappa)$ and for each $\sigma \in \Xi(k)$ $\sigma_{/dom(\kappa)} \in \Xi(\kappa)$ and $\#(k,t,\sigma) = \#(\kappa,t,\sigma_{/dom(\kappa)})$.

Since $h \in K(n+1)$ and $t \in E(n+1,h)$ we can also use lemma 4.10 to obtain these two other following alternative possibilities:

- $t \in E_a(n+1,h) \cup E_c(n+1,h) \cup E_d(n+1,h) \cup E_e(n+1,h)$;

- there exist p positive integer such that $2 \leqslant p < n+1$, $g \in K(p)$ such that $g \sqsubseteq h$, $t \in E_a(p,g) \cup E_c(p,g) \cup E_d(p,g) \cup E_e(p,g)$ and for each $\rho \in \Xi(h)$ $\rho_{/dom(g)} \in \Xi(g)$ and $\#(h,t,\rho) = \#(g,t,\rho_{/dom(g)})$.

So we have three possible cases to examine. The first is

- $t \in E_a(n+1,k) \cup E_c(n+1,k) \cup E_d(n+1,k) \cup E_e(n+1,k)$ and
- $t \in E_a(n+1,h) \cup E_c(n+1,h) \cup E_d(n+1,h) \cup E_e(n+1,h)$.

The second case is

- $t \in E_a(n+1,k) \cup E_c(n+1,k) \cup E_d(n+1,k) \cup E_e(n+1,k)$ and
- $n+1 > 2$ and there exist p positive integer such that $2 \leqslant p < n+1$, $g \in K(p)$ such that $g \sqsubseteq h$, $t \in E_a(p,g) \cup E_c(p,g) \cup E_d(p,g) \cup E_e(p,g)$ and for each $\rho \in \Xi(h)$ $\rho_{/dom(g)} \in \Xi(g)$ and $\#(h,t,\rho) = \#(g,t,\rho_{/dom(g)})$.

Another case to examine would be the following

- $n+1 > 2$ and there exist m positive integer such that $2 \leqslant m < n+1$, $\kappa \in K(m)$ such that $\kappa \sqsubseteq k$, $t \in E_a(m,\kappa) \cup E_c(m,\kappa) \cup E_d(m,\kappa) \cup E_e(m,\kappa)$ and for each $\sigma \in \Xi(k)$ $\sigma_{/dom(\kappa)} \in \Xi(\kappa)$ and $\#(k,t,\sigma) = \#(\kappa,t,\sigma_{/dom(\kappa)})$ and
- $t \in E_a(n+1,h) \cup E_c(n+1,h) \cup E_d(n+1,h) \cup E_e(n+1,h)$.

Anyway this case is practically equal to the second one, so we don't need to consider it. Finally the third case is the following.

- $n+1 > 2$ and there exist m positive integer such that $2 \leqslant m < n+1$, $\kappa \in K(m)$ such that $\kappa \sqsubseteq k$, $t \in E_a(m,\kappa) \cup E_c(m,\kappa) \cup E_d(m,\kappa) \cup E_e(m,\kappa)$ and for each $\sigma \in \Xi(k)$ $\sigma_{/dom(\kappa)} \in \Xi(\kappa)$ and $\#(k,t,\sigma) = \#(\kappa,t,\sigma_{/dom(\kappa)})$ and
- $n+1 > 2$ and there exist p positive integer such that $2 \leqslant p < n+1$, $g \in K(p)$ such that $g \sqsubseteq h$, $t \in E_a(p,g) \cup E_c(p,g) \cup E_d(p,g) \cup E_e(p,g)$ and for each $\rho \in \Xi(h)$ $\rho_{/dom(g)} \in \Xi(g)$ and $\#(h,t,\rho) = \#(g,t,\rho_{/dom(g)})$.

We now examine the three different cases we have distinguished. We start with the first one, where we have four different subcases:
$t \in E_a(n+1,k),\ t \in E_c(n+1,k),\ t \in E_d(n+1,k),\ t \in E_e(n+1,k)$.

We start with the subcase $t \in E_a(n+1,k)$. We must have $t \in E_a(n+1,h)$.

If $k \in K(n)$ then $E_a(n+1,k) = \emptyset$ so $k \in K(n)^+$ and there exist $\kappa \in K(n)$, $\theta \in E_s(n,\kappa)$, $u \in (\mathcal{V} - var(\kappa))$ such that $k = \kappa + (u,\theta)$, $E_a(n+1,k) = \{u\}$. Since $\sigma \in \Xi(k)$ there exist $\xi \in \Xi(\kappa)$, $s \in \#(\kappa,\theta,\xi)$ such that $\sigma = \xi + (u,s)$,
$\#(k,t,\sigma) = \#(k,t,\sigma)_{(n+1,k,a)} = s$.

If $h \in K(n)$ then $E_a(n+1,h) = \emptyset$ so $h \in K(n)^+$ and there exist $\vartheta \in K(n)$, $\mu \in E_s(n,\vartheta)$, $v \in (\mathcal{V} - var(\vartheta))$ such that $h = \vartheta + (v,\mu)$, $E_a(n+1,k) = \{v\}$. Since $\rho \in \Xi(h)$ there exist $\zeta \in \Xi(\vartheta)$, $q \in \#(\vartheta,\mu,\zeta)$ such that $\rho = \zeta + (v,q)$,
$\#(h,t,\rho) = \#(h,t,\rho)_{(n+1,h,a)} = q$.

Since $t \in E_a(n+1,k)$ we have $t = u$, since $t \in E_a(n+1,h)$ we have $t = v$, therefore $u = v$.

There exists $i \in dom(\sigma)$ such that $u = x_i,\ s = z_i$, there exists $j \in dom(\rho)$ such that $v = y_j,\ q = r_j$.

Therefore $x_i = u = v = y_j$ and $\#(k,t,\sigma) = s = z_i = r_j = q = \#(h,t,\rho)$.

$\diamond$

We now consider the subcase $t \in E_c(n+1,k)$, which implies $k \in K(n)$ and $t \in E_c(n+1,h)$, $h \in K(n)$. Since $t \in E_c(n+1,k)$ there must exist

- a positive integer m,
- $\chi, \chi_1, \ldots, \chi_m \in E(n,k)$

such that

- for each $\eta \in \Xi(k)$ $\#(k,\chi,\eta)$ is a function with m arguments and $(\#(k,\chi_1,\eta),\ldots,\#(k,\chi_m,\eta))$ is a member of its domain,
- $(\chi)(\chi_1,\ldots,\chi_m) \notin E(n,k)$,
- $t = (\chi)(\chi_1,\ldots,\chi_m)$,
- $\#(k,t,\sigma) = \#(k,\chi,\sigma)(\#(k,\chi_1,\sigma),\ldots,\#(k,\chi_m,\sigma))$.

Since $t \in E_c(n+1,h)$ there must exist

- a positive integer q,
- $\vartheta, \vartheta_1, \ldots, \vartheta_q \in E(n,h)$

such that

- for each $\eta \in \Xi(h)$ $\#(h,\vartheta,\eta)$ is a function with q arguments and $(\#(h,\vartheta_1,\eta),\ldots,\#(h,\vartheta_q,\eta))$ is a member of its domain,
- $(\vartheta)(\vartheta_1,\ldots,\vartheta_q) \notin E(n,h)$,
- $t = (\vartheta)(\vartheta_1,\ldots,\vartheta_q)$,
- $\#(h,t,\rho) = \#(h,\vartheta,\rho)(\#(h,\vartheta_1,\rho),\ldots,\#(h,\vartheta_q,\rho))$.

We apply the unique readability lemma 3.1.4 and get $\vartheta = \chi$, $q = m$ and for each $i \in \{1,\ldots,m\}$ $\vartheta_i = \chi_i$. So what we have to show is

$$\#(k,\chi,\sigma)(\#(k,\chi_1,\sigma),\ldots,\#(k,\chi_m,\sigma)) = \#(h,\chi,\rho)(\#(h,\chi_1,\rho),\ldots,\#(h,\chi_m,\rho)).$$

To this end, for each $i \in \{1,\ldots,m\}$, we simply apply the inductive hypothesis and obtain that $\#(k,\chi_i,\sigma) = \#(h,\chi_i,\rho)$. Similarly $\#(k,\chi,\sigma) = \#(h,\chi,\rho)$.

$\diamond$

We now consider the subcase $t \in E_d(n+1,k)$, which implies $k \in K(n)$ and $t \in E_d(n+1,h)$, $h \in K(n)$. Since $t \in E_d(n+1,k)$ there must exist

- $f \in \mathcal{F}$,
- a positive integer m,
- $\chi_1, \ldots, \chi_m \in E(n,k)$

such that

- for each $\eta \in \Xi(k)$ $A_f(\#(k,\chi_1,\eta), \ldots, \#(k,\chi_m,\eta))$ is true,
- $f(\chi_1, \ldots, \chi_m) \notin E(n,k)$,
- $t = f(\chi_1, \ldots, \chi_m)$,
- $\#(k,t,\sigma) = P_f(\#(k,\chi_1,\sigma), \ldots, \#(k,\chi_m,\sigma))$.

Since $t \in E_d(n+1,h)$ there must exist

- $\kappa \in \mathcal{F}$,
- a positive integer q,
- $\vartheta_1, \ldots, \vartheta_q \in E(n,h)$

such that

- for each $\eta \in \Xi(h)$ $A_\kappa(\#(h,\vartheta_1,\eta), \ldots, \#(h,\vartheta_q,\eta))$ is true,
- $\kappa(\vartheta_1, \ldots, \vartheta_q) \notin E(n,h)$,
- $t = \kappa(\vartheta_1, \ldots, \vartheta_q)$,
- $\#(h,t,\rho) = P_\kappa(\#(h,\vartheta_1,\rho), \ldots, \#(h,\vartheta_q,\rho))$.

We apply the unique readability lemma 3.1.5 and get $\kappa = f$, $q = m$ and for each $i \in \{1, \ldots, m\}$ $\vartheta_i = \chi_i$. So what we have to show is

$$P_f(\#(k,\chi_1,\sigma), \ldots, \#(k,\chi_m,\sigma)) = P_f(\#(h,\chi_1,\rho), \ldots, \#(h,\chi_m,\rho)) \ .$$

To this end, for each $i \in \{1, \ldots, m\}$, we simply apply the inductive hypothesis and obtain that $\#(k,\chi_i,\sigma) = \#(h,\chi_i,\rho)$.

$\diamond$

We now consider the subcase $t \in E_e(n+1,k)$ (which implies $k \in K(n)$). We must have $t \in E_e(n+1,h)$ and $h \in K(n)$. Since $t \in E_e(n+1,k)$ there must exist

- a positive integer q;
- a function u whose domain is $\{1, \ldots, q\}$ such that for each $i = 1 \ldots q$ $u_i \in \mathcal{V}{-}var(k)$, and for each $i,j = 1 \ldots q$ $i \neq j \to u_i \neq u_j$;
- a function ϑ whose domain is $\{1, \ldots, q\}$ such that for each $i = 1 \ldots q$ $\vartheta_i \in E(n)$;
- $\theta \in E(n)$

such that

- $\mathcal{E}(n,k,q,u,\vartheta,\theta)$;
- $\{\}(u_1 : \vartheta_1, \ldots, u_q : \vartheta_q, \theta) \notin E(n,k)$;
- $t = \{\}(u_1 : \vartheta_1, \ldots, u_q : \vartheta_q, \theta)$.

Since $t \in E_e(n+1,h)$ there must exist

- a positive integer r;
- a function v whose domain is $\{1,\ldots,r\}$ such that for each $i=1\ldots r$ $v_i \in \mathcal{V}-var(h)$, and for each $i,j=1\ldots r$ $i\neq j \to v_i \neq v_j$;
- a function χ whose domain is $\{1,\ldots,r\}$ such that for each $i=1\ldots r$ $\chi_i \in E(n)$;
- $\zeta \in E(n)$

such that

- $\mathcal{E}(n,h,r,v,\chi,\zeta)$;
- $\{\}(v_1:\chi_1,\ldots,v_r:\chi_r,\zeta) \notin E(n,h)$;
- $t=\{\}(v_1:\chi_1,\ldots,v_r:\chi_r,\zeta)$.

We can apply the unique readability lemma 3.1.6, so $r=q, v=u, \chi=\vartheta, \zeta=\theta$.

We have
$$\#(k,t,\sigma)=\{\#(k'_q,\theta,\sigma'_q)|\ \sigma'_q \in \Xi(k'_q), \sigma \sqsubseteq \sigma'_q\}\ ,$$
where $k'_1=k+(u_1,\vartheta_1)$, and if $q>1$ for each $i=1\ldots q-1$ $k'_{i+1}=k'_i+(u_{i+1},\vartheta_{i+1})$.

We have also
$$\#(h,t,\rho)=\{\#(h'_q,\theta,\rho'_q)|\ \rho'_q \in \Xi(h'_q), \rho \sqsubseteq \rho'_q\}\ ,$$
where $h'_1=h+(u_1,\vartheta_1)$, and if $q>1$ for each $i=1\ldots q-1$ $h'_{i+1}=h'_i+(u_{i+1},\vartheta_{i+1})$.

We want to show that $\#(k,t,\sigma)=\#(h,t,\rho)$, thus we have to show
$$\{\#(k'_q,\theta,\sigma'_q)|\ \sigma'_q \in \Xi(k'_q), \sigma \sqsubseteq \sigma'_q\} = \{\#(h'_q,\theta,\rho'_q)|\ \rho'_q \in \Xi(h'_q), \rho \sqsubseteq \rho'_q\}\ .$$

To prove this we just need to prove the following two assertions:

- for each $\sigma'_q \in \Xi(k'_q)$ such that $\sigma \sqsubseteq \sigma'_q$ there exists $\rho'_q \in \Xi(h'_q)$ such that $\rho \sqsubseteq \rho'_q$ and $\#(h'_q,\theta,\rho'_q)=\#(k'_q,\theta,\sigma'_q)$;
- for each $\rho'_q \in \Xi(h'_q)$ such that $\rho \sqsubseteq \rho'_q$ there exists $\sigma'_q \in \Xi(k'_q)$ such that $\sigma \sqsubseteq \sigma'_q$ and $\#(k'_q,\theta,\sigma'_q)=\#(h'_q,\theta,\rho'_q)$.

It is clearly enough to prove the first one, since the second would be proved by simply substituting variables in the proof of the first. Let $\sigma'_q \in \Xi(k'_q)$ such that $\sigma \sqsubseteq \sigma'_q$, we want to find $\rho'_q \in \Xi(h'_q)$ such that $\rho \sqsubseteq \rho'_q$ and $\#(h'_q,\theta,\rho'_q)=\#(k'_q,\theta,\sigma'_q)$.

Let $\sigma'_1=(\sigma'_q)_{/dom(k'_1)}$. We should be able to prove that:

- $\sigma'_1 \in \Xi(k'_1)$
- there exists $s_1 \in \#(k,\vartheta_1,\sigma)$ such that $\sigma'_1=\sigma+(u_1,s_1)$.

If $q=1$ then $\sigma'_1 \in \Xi(k'_1)$ clearly holds, else we have $k'_q \neq \epsilon$, $\sigma'_q \in \Xi(k'_q)$, $k'_1 \in \mathcal{R}(k'_q)$, $k'_1 \neq k'_q$, so by lemma 4.4 $\sigma'_1=(\sigma'_q)_{/dom(k'_1)} \in \Xi(k'_1)$.

We have $k'_1=k+(u_1,\vartheta_1)$ and $k'_1 \in K(n)$, clearly $k'_1 \neq \epsilon$ and $n \geqslant 2$ also hold. Moreover $k \in K(n)$, $\vartheta_1 \in E_s(n,k)$, $u_1 \in \mathcal{V}-var(k)$, so by lemma 4.3
$$\Xi(k'_1)=\{\xi+(u_1,s)|\ \xi \in \Xi(k), s \in \#(k,\vartheta_1,\xi)\}.$$

Then there exist $\xi \in \Xi(k), s \in \#(k, \vartheta_1, \xi)$ such that $\sigma'_1 = \xi + (u_1, s)$. Here we can see that

$$(\sigma'_1)_{/dom(k)} = (\sigma'_1)_{/dom(\xi)} = \xi$$

and at the same time, since $dom(k) \subseteq dom(k'_1) \subseteq dom(k'_q) = dom(\sigma'_q)$,

$$(\sigma'_1)_{/dom(k)} = ((\sigma'_q)_{/dom(k'_1)})_{/dom(k)} = (\sigma'_q)_{/dom(k)} = (\sigma'_q)_{/dom(\sigma)} = \sigma.$$

Therefore $\xi = \sigma$ and there exists $s \in \#(k, \vartheta_1, \sigma)$ such that $\sigma'_1 = \sigma + (u_1, s)$.

If $q > 1$ then for each $i = 1 \dots q-1$ we can define $\sigma'_{i+1} = (\sigma'_q)_{/dom(k'_{i+1})}$. We should also be able to prove that for each $i = 1 \dots q-1$

- $\sigma'_{i+1} \in \Xi(k'_{i+1})$,
- there exists $s_{i+1} \in \#(k'_i, \vartheta_{i+1}, \sigma'_i)$ such that $\sigma'_{i+1} = \sigma'_i + (u_{i+1}, s_{i+1})$.

If $i+1 = q$ then $\sigma'_{i+1} \in \Xi(k'_{i+1})$ clearly holds, else $i+1 < q$ and $k'_q \neq \epsilon$, $\sigma'_q \in \Xi(k'_q)$, $k'_{i+1} \in \mathcal{R}(k'_q)$, $k'_{i+1} \neq k'_q$, so by lemma 4.4 $\sigma'_{i+1} = (\sigma'_q)_{/dom(k'_{i+1})} \in \Xi(k'_{i+1})$.

We have $k'_{i+1} = k'_i + (u_{i+1}, \vartheta_{i+1})$ and $k'_{i+1} \in K(n)$, clearly $k'_{i+1} \neq \epsilon$ and $n \geqslant 2$ also hold. Moreover $k'_i \in K(n)$, $\vartheta_{i+1} \in E_s(n, k'_i)$, $var(k'_i) = var(k) \cup \{u_1, \dots, u_i\}$, $u_{i+1} \in \mathcal{V} - var(k'_i)$, so by lemma 4.3

$$\Xi(k'_{i+1}) = \{\xi + (u_{i+1}, s) |\, \xi \in \Xi(k'_i), s \in \#(k'_i, \vartheta_{i+1}, \xi)\}.$$

Then there exist $\xi \in \Xi(k'_i), s \in \#(k'_i, \vartheta_{i+1}, \xi)$ such that $\sigma'_{i+1} = \xi + (u_{i+1}, s)$. Here we can see that

$$(\sigma'_{i+1})_{/dom(k'_i)} = (\sigma'_{i+1})_{/dom(\xi)} = \xi$$

and at the same time, since $dom(k'_i) \subseteq dom(k'_{i+1}) \subseteq dom(k'_q) = dom(\sigma'_q)$,

$$(\sigma'_{i+1})_{/dom(k'_i)} = ((\sigma'_q)_{/dom(k'_{i+1})})_{/dom(k'_i)} = (\sigma'_q)_{/dom(k'_i)} = \sigma'_i.$$

Therefore $\xi = \sigma'_i$ and there exists $s \in \#(k'_i, \vartheta_{i+1}, \sigma'_i)$ such that $\sigma'_{i+1} = \sigma'_i + (u_{i+1}, s)$.

Then we define $\rho'_1 = \rho + (u_1, s_1)$, and we should be able to prove that $\rho'_1 \in \Xi(h'_1)$.

We have $\mathcal{E}(n, h, q, u, \vartheta, \theta)$. This implies $\vartheta_1 \in E_s(n, h)$. We have $h'_1 = h + (u_1, \vartheta_1)$ and $h'_1 \in K(n)$, $h'_1 \neq \epsilon$, $n \geqslant 2$, moreover $h \in K(n)$, $u_1 \in \mathcal{V} - var(h)$ and therefore

$$\Xi(h'_1) = \{\xi + (u_1, s) |\, \xi \in \Xi(h), s \in \#(h, \vartheta_1, \xi)\}.$$

Since $\rho \in \Xi(h)$, to prove that $\rho'_1 \in \Xi(h'_1)$ we just need to prove that $s_1 \in \#(h, \vartheta_1, \rho)$. We know that $s_1 \in \#(k, \vartheta_1, \sigma)$. We have $\vartheta_1 \in E(n, k)$, $\vartheta_1 \in E(n, h)$.

With that we can apply the inductive hypothesis and obtain that $\#(k, \vartheta_1, \sigma) = \#(h, \vartheta_1, \rho)$, therefore $s_1 \in \#(h, \vartheta_1, \rho)$ and $\rho'_1 \in \Xi(h'_1)$.

We also notice that $k'_1 = k + (u_1, \vartheta_1)$, $h'_1 = h + (u_1, \vartheta_1)$, so if we set $k'_1 = (x'_1, \varphi'_1)$ and $h'_1 = (y'_1, \psi'_1)$ then by lemma 3.3 for each $\alpha \in dom(k'_1)$, $\beta \in dom(h'_1)$ $(x'_1)_\alpha = (y'_1)_\beta \to (\varphi'_1)_\alpha = (\psi'_1)_\beta$.
Moreover we notice that $\sigma'_1 = \sigma + (u_1, s_1)$, $\rho'_1 = \rho + (u_1, s_1)$, and if we set $\sigma'_1 = (x'_1, z'_1)$, $\rho'_1 = (y'_1, \mu'_1)$ then by lemma 3.3 for each $\alpha \in dom(\sigma'_1)$, $\beta \in dom(\rho'_1)$ $(x'_1)_\alpha = (y'_1)_\beta \to (z'_1)_\alpha = (\mu'_1)_\beta$.

If $q > 1$ then for each $i = 1 \ldots q-1$ we can define $\rho'_{i+1} = \rho'_i + (u_{i+1}, s_{i+1})$ and we expect to be able to prove that $\rho'_{i+1} \in \Xi(h'_{i+1})$.

We have $\mathcal{E}(n, h, q, u, \vartheta, \theta)$ and $h'_{i+1} = h'_i + (u_{i+1}, \vartheta_{i+1})$. This implies $h'_{i+1} \in K(n)$, $h'_{i+1} \neq \epsilon$ and $n \geqslant 2$ holds too. Moreover $h'_i \in K(n)$, $\vartheta_{i+1} \in E_s(n, h'_i)$, and, since $var(h'_i) = var(h) \cup \{u_1, \ldots, u_i\}$, $u_{i+1} \in \mathcal{V} - var(h'_i)$. Therefore

$$\Xi(h'_{i+1}) = \{\xi + (u_{i+1}, s) |\, \xi \in \Xi(h'_i), s \in \#(h'_i, \vartheta_{i+1}, \xi)\}.$$

By inductive hypothesis we can assume that $\rho'_i \in \Xi(h'_i)$, therefore to prove $\rho'_{i+1} \in \Xi(h'_{i+1})$ we just need to prove $s_{i+1} \in \#(h'_i, \vartheta_{i+1}, \rho'_i)$. We know that $s_{i+1} \in \#(k'_i, \vartheta_{i+1}, \sigma'_i)$.

By inductive hypothesis we can also assume that

- if we set $k'_i = (x'_i, \varphi'_i)$ and $h'_i = (y'_i, \psi'_i)$ then for each $\alpha \in dom(k'_i)$, $\beta \in dom(h'_i)$ $(x'_i)_\alpha = (y'_i)_\beta \to (\varphi'_i)_\alpha = (\psi'_i)_\beta$.
- if we set $\sigma'_i = (x'_i, z'_i)$, $\rho'_i = (y'_i, \mu'_i)$ then for each $\alpha \in dom(\sigma'_i)$, $\beta \in dom(\rho'_i)$ $(x'_i)_\alpha = (y'_i)_\beta \to (z'_i)_\alpha = (\mu'_i)_\beta$.

We have $k'_i \in K(n)$, $h'_i \in K(n)$, $\vartheta_{i+1} \in E_s(n, k'_i)$, $\vartheta_{i+1} \in E_s(n, h'_i)$, $\sigma'_i \in \Xi(k'_i)$, $\rho'_i \in \Xi(h'_i)$, so we can apply the inductive hypothesis and obtain that $\#(k'_i, \vartheta_{i+1}, \sigma'_i) = \#(h'_i, \vartheta_{i+1}, \rho'_i)$. Therefore $s_{i+1} \in \#(h'_i, \vartheta_{i+1}, \rho'_i)$ and we have proved $\rho'_{i+1} \in \Xi(h'_{i+1})$.

In this proof that $\rho'_{i+1} \in \Xi(h'_{i+1})$ we have used an inductive hypothesis which we haven't proved, so we need to prove it now. What we need to prove is the following:

- if we set $k'_{i+1} = (x'_{i+1}, \varphi'_{i+1})$ and $h'_{i+1} = (y'_{i+1}, \psi'_{i+1})$ then for each $\alpha \in dom(k'_{i+1})$, $\beta \in dom(h'_{i+1})$ $(x'_{i+1})_\alpha = (y'_{i+1})_\beta \to (\varphi'_{i+1})_\alpha = (\psi'_{i+1})_\beta$.
- if we set $\sigma'_{i+1} = (x'_{i+1}, z'_{i+1})$, $\rho'_{i+1} = (y'_{i+1}, \mu'_{i+1})$ then for each $\alpha \in dom(\sigma'_{i+1})$, $\beta \in dom(\rho'_{i+1})$ $(x'_{i+1})_\alpha = (y'_{i+1})_\beta \to (z'_{i+1})_\alpha = (\mu'_{i+1})_\beta$.

To prove the first item we consider that $k'_{i+1} = k'_i + (u_{i+1}, \vartheta_{i+1})$, $h'_{i+1} = h'_i + (u_{i+1}, \vartheta_{i+1})$, $u_{i+1} \in \mathcal{V} - var(k'_i)$, $u_{i+1} \in \mathcal{V} - var(h'_i)$. So we can apply lemma 3.3 and the first condition is proved.

To prove the second item we consider that $\sigma'_{i+1} = \sigma'_i + (u_{i+1}, s_{i+1})$, $\rho'_{i+1} = \rho'_i + (u_{i+1}, s_{i+1})$, $u_{i+1} \in \mathcal{V} - var(\sigma'_i)$, $u_{i+1} \in \mathcal{V} - var(\rho'_i)$. So we can apply lemma 3.3 and the second condition is proved.

At this point we have defined ρ'_q such that $\rho \sqsubseteq \rho'_q$ and proved that $\rho'_q \in \Xi(h'_q)$. We have also that $k'_q \in K(n)$, $\theta \in E(n, k'_q)$, $h'_q \in K(n)$, $\theta \in E(n, h'_q)$, $\sigma'_q \in \Xi(k'_q)$. Moreover

- if we set $k'_q = (x'_q, \varphi'_q)$ and $h'_q = (y'_q, \psi'_q)$ then for each $\alpha \in dom(k'_q)$, $\beta \in dom(h'_q)$ $(x'_q)_\alpha = (y'_q)_\beta \to (\varphi'_q)_\alpha = (\psi'_q)_\beta$.
- if we set $\sigma'_q = (x'_q, z'_q)$, $\rho'_q = (y'_q, \mu'_q)$ then for each $\alpha \in dom(\sigma'_q)$, $\beta \in dom(\rho'_q)$ $(x'_q)_\alpha = (y'_q)_\beta \to (z'_q)_\alpha = (\mu'_q)_\beta$.

With that, $\#(h'_q, \theta, \rho'_q) = \#(k'_q, \theta, \sigma'_q)$ follows by inductive hypothesis.

$\diamond$

Let's consider the second case. Like in the first, we have four different subcases, they are: $t \in E_a(n+1,k),\ t \in E_c(n+1,k),\ t \in E_d(n+1,k),\ t \in E_e(n+1,k)$.

We start with the subcase $t \in E_a(n+1,k)$. We must have $t \in E_a(p,g)$.

If $k \in K(n)$ then $E_a(n+1,k) = \emptyset$ so $k \in K(n)^+$ and there exist $\kappa \in K(n),\ \theta \in E_s(n,\kappa),\ u \in (\mathcal{V} - var(\kappa))$ such that $k = \kappa + (u,\theta)$, $E_a(n+1,k) = \{u\}$. Since $\sigma \in \Xi(k)$ there exist $\xi \in \Xi(\kappa),\ s \in \#(\kappa,\theta,\xi)$ such that $\sigma = \xi + (u,s)$,
$\#(k,t,\sigma) = \#(k,t,\sigma)_{(n+1,k,a)} = s$.

If $g \in K(p-1)$ then $E_a(p,g) = \emptyset$ so $g \in K(p-1)^+$ and there exist $\vartheta \in K(p-1),\ \mu \in E_s(p-1,\vartheta),\ v \in (\mathcal{V} - var(\vartheta))$ such that $g = \vartheta + (v,\mu)$, $E_a(p,g) = \{v\}$. We have $\rho \in \Xi(h)$ and $\rho_{/dom(g)} \in \Xi(g)$. Let $\eta = \rho_{/dom(g)}$, then there exist $\zeta \in \Xi(\vartheta),\ q \in \#(\vartheta,\mu,\zeta)$ such that $\eta = \zeta + (v,q)$, $\#(g,t,\eta) = \#(g,t,\eta)_{(p,g,a)} = q$.

We have to prove that $\#(k,t,\sigma) = \#(h,t,\rho)$, and since $\#(h,t,\rho) = \#(g,t,\eta)$ it is enough to prove that $\#(k,t,\sigma) = \#(g,t,\eta)$.

Since $t \in E_a(n+1,k)$ we have $t = u$, since $t \in E_a(p,g)$ we have $t = v$, therefore $u = v$.

Since $\eta \sqsubseteq \rho$ we can apply lemma 3.4 to show that if $\eta = (w,\nu)$ then for each $i \in dom(\sigma),\ j \in dom(\eta)\ x_i = w_j \rightarrow z_i = \nu_j$.

There exists $i \in dom(\sigma)$ such that $u = x_i,\ s = z_i$, there exists $j \in dom(\eta)$ such that $v = w_j,\ q = \nu_j$.

Therefore $x_i = u = v = w_j$ and $\#(k,t,\sigma) = s = z_i = \nu_j = q = \#(g,t,\eta)$.

$\diamond$

We now consider the subcase $t \in E_c(n+1,k)$, which implies $k \in K(n)$ and $t \in E_c(p,g)$. Since $t \in E_c(n+1,k)$ there must exist

- a positive integer m,
- $\chi, \chi_1, \ldots, \chi_m \in E(n,k)$

such that

- for each $\xi \in \Xi(k)$ $\#(k,\chi,\xi)$ is a function with m arguments and $(\#(k,\chi_1,\xi), \ldots, \#(k,\chi_m,\xi))$ is a member of its domain,
- $(\chi)(\chi_1, \ldots, \chi_m) \notin E(n,k)$,
- $t = (\chi)(\chi_1, \ldots, \chi_m)$,
- $\#(k,t,\sigma) = \#(k,\chi,\sigma)(\#(k,\chi_1,\sigma), \ldots, \#(k,\chi_m,\sigma))$.

Since $t \in E_c(p,g)$ there must exist

- a positive integer q,
- $\vartheta, \vartheta_1, \ldots, \vartheta_q \in E(p-1,g)$

such that

- for each $\eta \in \Xi(g)$ $\#(g, \vartheta, \eta)$ is a function with q arguments and $(\#(g, \vartheta_1, \eta), \ldots, \#(g, \vartheta_q, \eta))$ is a member of its domain,
- $(\vartheta)(\vartheta_1, \ldots, \vartheta_q) \notin E(p-1, g)$,
- $t = (\vartheta)(\vartheta_1, \ldots, \vartheta_q)$,
- for each $\eta \in \Xi(g)$ $\#(g, t, \eta) = \#(g, \vartheta, \eta)(\#(g, \vartheta_1, \eta), \ldots, \#(g, \vartheta_q, \eta))$.

We have $\#(h, t, \rho) = \#(g, t, \rho_{/dom(g)})$, so if we define $\eta = \rho_{/dom(g)}$ then $\#(h, t, \rho) = \#(g, t, \eta) = \#(g, \vartheta, \eta)(\#(g, \vartheta_1, \eta), \ldots, \#(g, \vartheta_q, \eta))$.

We have to show that $\#(k, t, \sigma) = \#(h, t, \rho)$ and to show this we just need to show that

$$\#(k, \chi, \sigma)(\#(k, \chi_1, \sigma), \ldots, \#(k, \chi_m, \sigma)) = \#(g, \vartheta, \eta)(\#(g, \vartheta_1, \eta), \ldots, \#(g, \vartheta_q, \eta)) \ .$$

We first apply the unique readability lemma 3.1.4 and get $\vartheta = \chi$, $q = m$ and for each $i \in \{1, \ldots, m\}$ $\vartheta_i = \chi_i$. So what we have to show is

$$\#(k, \chi, \sigma)(\#(k, \chi_1, \sigma), \ldots, \#(k, \chi_m, \sigma)) = \#(g, \chi, \eta)(\#(g, \chi_1, \eta), \ldots, \#(g, \chi_m, \eta)) \ .$$

By lemma 3.4 if we set $g = (w, \phi)$ then for each $i \in dom(k)$, $j \in dom(g)$ $x_i = w_j \rightarrow \varphi_i = \phi_j$.

Since $\eta \sqsubseteq \rho$ we can apply lemma 3.4 to also show that if $\eta = (w, \mu)$ then for each $i \in dom(\sigma)$, $j \in dom(\eta)$ $x_i = w_j \rightarrow z_i = \mu_j$.

With that, for each $i \in \{1, \ldots, m\}$, we can apply the inductive hypothesis and obtain that $\#(k, \chi_i, \sigma) = \#(g, \chi_i, \eta)$. Similarly $\#(k, \chi, \sigma) = \#(g, \chi, \eta)$.

$\diamond$

We now consider the subcase $t \in E_d(n+1, k)$, which implies $k \in K(n)$ and $t \in E_d(p, g)$. Since $t \in E_d(n+1, k)$ there must exist

- $f \in \mathcal{F}$,
- a positive integer m,
- $\chi_1, \ldots, \chi_m \in E(n, k)$

such that

- for each $\xi \in \Xi(k)$ $A_f(\#(k, \chi_1, \xi), \ldots, \#(k, \chi_m, \xi))$ is true,
- $f(\chi_1, \ldots, \chi_m) \notin E(n, k)$,
- $t = f(\chi_1, \ldots, \chi_m)$,
- $\#(k, t, \sigma) = P_f(\#(k, \chi_1, \sigma), \ldots, \#(k, \chi_m, \sigma))$.

Since $t \in E_d(p, g)$ there must exist

- $\kappa \in \mathcal{F}$,
- a positive integer q,
- $\vartheta_1, \ldots, \vartheta_q \in E(p-1, g)$

such that

- for each $\eta \in \Xi(g)$ $A_\kappa(\#(g, \vartheta_1, \eta), \ldots, \#(g, \vartheta_q, \eta))$ is true,
- $\kappa(\vartheta_1, \ldots, \vartheta_q) \notin E(p-1, g)$,
- $t = \kappa(\vartheta_1, \ldots, \vartheta_q)$,
- for each $\eta \in \Xi(g)$ $\#(g, t, \eta) = P_\kappa(\#(g, \vartheta_1, \eta), \ldots, \#(g, \vartheta_q, \eta))$.

We have $\#(h, t, \rho) = \#(g, t, \rho_{/dom(g)})$, so if we define $\eta = \rho_{/dom(g)}$ then $\#(h, t, \rho) = \#(g, t, \eta) = P_\kappa(\#(g, \vartheta_1, \eta), \ldots, \#(g, \vartheta_q, \eta))$.

We have to show that $\#(k, t, \sigma) = \#(h, t, \rho)$ and to show this we just need to show that

$$P_f(\#(k, \chi_1, \sigma), \ldots, \#(k, \chi_m, \sigma)) = P_\kappa(\#(g, \vartheta_1, \eta), \ldots, \#(g, \vartheta_q, \eta)) .$$

We first apply the unique readability lemma 3.1.5 and get $\kappa = f$, $q = m$ and for each $i \in \{1, \ldots, m\}$ $\vartheta_i = \chi_i$. So what we have to show is

$$P_f(\#(k, \chi_1, \sigma), \ldots, \#(k, \chi_m, \sigma)) = P_f(\#(g, \chi_1, \eta), \ldots, \#(g, \chi_m, \eta)) .$$

By lemma 3.4 if we set $g = (w, \phi)$ then for each $i \in dom(k)$, $j \in dom(g)$ $x_i = w_j \to \varphi_i = \phi_j$.

Since $\eta \sqsubseteq \rho$ we can apply lemma 3.4 to also show that if $\eta = (w, \mu)$ then for each $i \in dom(\sigma)$, $j \in dom(\eta)$ $x_i = w_j \to z_i = \mu_j$.

With that, for each $i \in \{1, \ldots, m\}$, we can apply the inductive hypothesis and obtain that $\#(k, \chi_i, \sigma) = \#(g, \chi_i, \eta)$, and our proof is finished.

$\diamond$

We now consider the subcase $t \in E_e(n+1, k)$ (which implies $k \in K(n)$). We must have $t \in E_e(p, g)$. Since $t \in E_e(n+1, k)$ there must exist

- a positive integer q;
- a function u whose domain is $\{1, \ldots, q\}$ such that for each $i = 1 \ldots q$ $u_i \in \mathcal{V}{-}var(k)$, and for each $i, j = 1 \ldots q$ $i \neq j \to u_i \neq u_j$;
- a function ϑ whose domain is $\{1, \ldots, q\}$ such that for each $i = 1 \ldots q$ $\vartheta_i \in E(n)$;
- $\theta \in E(n)$

such that

- $\mathcal{E}(n, k, q, u, \vartheta, \theta)$;
- $\{\}(u_1 : \vartheta_1, \ldots, u_q : \vartheta_q, \theta) \notin E(n, k)$;
- $t = \{\}(u_1 : \vartheta_1, \ldots, u_q : \vartheta_q, \theta)$.

Since $t \in E_e(p, g)$ there must exist

- a positive integer r;
- a function v whose domain is $\{1, \ldots, r\}$ such that for each $i = 1 \ldots r$ $v_i \in \mathcal{V}{-}var(g)$, and for each $i, j = 1 \ldots r$ $i \neq j \to v_i \neq v_j$;
- a function χ whose domain is $\{1, \ldots, r\}$ such that for each $i = 1 \ldots r$ $\chi_i \in E(p-1)$;

- $\zeta \in E(p-1)$;

such that

- $\mathcal{E}(p-1, g, r, v, \chi, \zeta)$;
- $\{\}(v_1 : \chi_1, \ldots, v_r : \chi_r, \zeta) \notin E(p-1, g)$;
- $t = \{\}(v_1 : \chi_1, \ldots, v_r : \chi_r, \zeta)$.

Clearly given $\eta \in \Xi(g)$ we have

$$\#(g, t, \eta) = \{\#(g'_r, \zeta, \eta'_r) |\ \eta'_r \in \Xi(g'_r), \eta \sqsubseteq \eta'_r\},$$

where $g'_1 = g + (v_1, \chi_1)$, and if $r > 1$ for each $i = 1 \ldots r-1$ $g'_{i+1} = g'_i + (v_{i+1}, \chi_{i+1})$.

Anyway $t = \{\}(u_1 : \vartheta_1, \ldots, u_q : \vartheta_q, \theta)$ and we can apply the unique readability lemma 3.1.6, so $r = q, v = u, \chi = \vartheta, \zeta = \theta$. Then given $\eta \in \Xi(g)$ we have

$$\#(g, t, \eta) = \{\#(g'_q, \theta, \eta'_q) |\ \eta'_q \in \Xi(g'_q), \eta \sqsubseteq \eta'_q\},$$

where $g'_1 = g + (u_1, \vartheta_1)$, and if $q > 1$ for each $i = 1 \ldots q-1$ $g'_{i+1} = g'_i + (u_{i+1}, \vartheta_{i+1})$.

Given $\sigma = (x, z) \in \Xi(k)$, $\rho = (y, r) \in \Xi(h)$ such that for each $i \in dom(\sigma)$, $j \in dom(\rho)$ $x_i = y_j \to z_i = r_j$ we want to show that $\#(k, t, \sigma) = \#(h, t, \rho)$.

Now we have

$$\#(k, t, \sigma) = \{\#(k'_q, \theta, \sigma'_q) |\ \sigma'_q \in \Xi(k'_q), \sigma \sqsubseteq \sigma'_q\} \ ,$$

where $k'_1 = k + (u_1, \vartheta_1)$, and if $q > 1$ for each $i = 1 \ldots q-1$ $k'_{i+1} = k'_i + (u_{i+1}, \vartheta_{i+1})$.

And we have $\#(h, t, \rho) = \#(g, t, \rho_{/dom(g)})$. If we define $\eta = \rho_{/dom(g)}$ then

$$\#(h, t, \rho) = \#(g, t, \eta) = \{\#(g'_q, \theta, \eta'_q) |\ \eta'_q \in \Xi(g'_q), \eta \sqsubseteq \eta'_q\} \ .$$

Thus we have to show that

$$\{\#(k'_q, \theta, \sigma'_q) |\ \sigma'_q \in \Xi(k'_q), \sigma \sqsubseteq \sigma'_q\} = \{\#(g'_q, \theta, \eta'_q) |\ \eta'_q \in \Xi(g'_q), \eta \sqsubseteq \eta'_q\} \ .$$

To prove this we just need to prove the following two assertions:

- for each $\sigma'_q \in \Xi(k'_q)$ such that $\sigma \sqsubseteq \sigma'_q$ there exists $\eta'_q \in \Xi(g'_q)$ such that $\eta \sqsubseteq \eta'_q$ and $\#(g'_q, \theta, \eta'_q) = \#(k'_q, \theta, \sigma'_q)$;
- for each $\eta'_q \in \Xi(g'_q)$ such that $\eta \sqsubseteq \eta'_q$ there exists $\sigma'_q \in \Xi(k'_q)$ such that $\sigma \sqsubseteq \sigma'_q$ and $\#(k'_q, \theta, \sigma'_q) = \#(g'_q, \theta, \eta'_q)$.

We begin with the first one. Let $\sigma'_q \in \Xi(k'_q)$ such that $\sigma \sqsubseteq \sigma'_q$, we want to find $\eta'_q \in \Xi(g'_q)$ such that $\eta \sqsubseteq \eta'_q$ and $\#(g'_q, \theta, \eta'_q) = \#(k'_q, \theta, \sigma'_q)$. Remember that $\eta = \rho_{/dom(g)}$.

Let $\sigma'_1 = (\sigma'_q)_{/dom(k'_1)}$. We should be able to prove that:

- $\sigma'_1 \in \Xi(k'_1)$
- there exists $s_1 \in \#(k, \vartheta_1, \sigma)$ such that $\sigma'_1 = \sigma + (u_1, s_1)$.

If $q = 1$ then $\sigma'_1 \in \Xi(k'_1)$ clearly holds, else we have $k'_q \neq \epsilon$, $\sigma'_q \in \Xi(k'_q)$, $k'_1 \in \mathcal{R}(k'_q)$, $k'_1 \neq k'_q$, so by lemma 4.4 $\sigma'_1 = (\sigma'_q)_{/dom(k'_1)} \in \Xi(k'_1)$.

We have $k'_1 = k + (u_1, \vartheta_1)$ and $k'_1 \in K(n)$, clearly $k'_1 \neq \epsilon$ and $n \geqslant 2$ also hold. Moreover $k \in K(n)$, $\vartheta_1 \in E_s(n,k)$, $u_1 \in \mathcal{V} - var(k)$, so by lemma 4.3

$$\Xi(k'_1) = \{\xi + (u_1, s) |\, \xi \in \Xi(k), s \in \#(k, \vartheta_1, \xi)\}.$$

Then there exist $\xi \in \Xi(k), s \in \#(k, \vartheta_1, \xi)$ such that $\sigma'_1 = \xi + (u_1, s)$. Here we can see that

$$(\sigma'_1)_{/dom(k)} = (\sigma'_1)_{/dom(\xi)} = \xi$$

and at the same time, since $dom(k) \subseteq dom(k'_1) \subseteq dom(k'_q) = dom(\sigma'_q)$,

$$(\sigma'_1)_{/dom(k)} = ((\sigma'_q)_{/dom(k'_1)})_{/dom(k)} = (\sigma'_q)_{/dom(k)} = (\sigma'_q)_{/dom(\sigma)} = \sigma.$$

Therefore $\xi = \sigma$ and there exists $s \in \#(k, \vartheta_1, \sigma)$ such that $\sigma'_1 = \sigma + (u_1, s)$.

If $q > 1$ then for each $i = 1 \dots q-1$ we can define $\sigma'_{i+1} = (\sigma'_q)_{/dom(k'_{i+1})}$. We should also be able to prove that for each $i = 1 \dots q-1$

- $\sigma'_{i+1} \in \Xi(k'_{i+1})$,
- there exists $s_{i+1} \in \#(k'_i, \vartheta_{i+1}, \sigma'_i)$ such that $\sigma'_{i+1} = \sigma'_i + (u_{i+1}, s_{i+1})$.

If $i+1 = q$ then $\sigma'_{i+1} \in \Xi(k'_{i+1})$ clearly holds, else $i+1 < q$ and $k'_q \neq \epsilon$, $\sigma'_q \in \Xi(k'_q)$, $k'_{i+1} \in \mathcal{R}(k'_q)$, $k'_{i+1} \neq k'_q$, so by lemma 4.4 $\sigma'_{i+1} = (\sigma'_q)_{/dom(k'_{i+1})} \in \Xi(k'_{i+1})$.

We have $k'_{i+1} = k'_i + (u_{i+1}, \vartheta_{i+1})$ and $k'_{i+1} \in K(n)$, clearly $k'_{i+1} \neq \epsilon$ and $n \geqslant 2$ also hold. Moreover $k'_i \in K(n)$, $\vartheta_{i+1} \in E_s(n, k'_i)$, $var(k'_i) = var(k) \cup \{u_1, \dots, u_i\}$, $u_{i+1} \in \mathcal{V} - var(k'_i)$, so by lemma 4.3

$$\Xi(k'_{i+1}) = \{\xi + (u_{i+1}, s) |\, \xi \in \Xi(k'_i), s \in \#(k'_i, \vartheta_{i+1}, \xi)\}.$$

Then there exist $\xi \in \Xi(k'_i), s \in \#(k'_i, \vartheta_{i+1}, \xi)$ such that $\sigma'_{i+1} = \xi + (u_{i+1}, s)$. Here we can see that

$$(\sigma'_{i+1})_{/dom(k'_i)} = (\sigma'_{i+1})_{/dom(\xi)} = \xi$$

and at the same time, since $dom(k'_i) \subseteq dom(k'_{i+1}) \subseteq dom(k'_q) = dom(\sigma'_q)$,

$$(\sigma'_{i+1})_{/dom(k'_i)} = ((\sigma'_q)_{/dom(k'_{i+1})})_{/dom(k'_i)} = (\sigma'_q)_{/dom(k'_i)} = \sigma'_i.$$

Therefore $\xi = \sigma'_i$ and there exists $s \in \#(k'_i, \vartheta_{i+1}, \sigma'_i)$ such that $\sigma'_{i+1} = \sigma'_i + (u_{i+1}, s)$.

Then we define $\eta'_1 = \eta + (u_1, s_1)$, and we should be able to prove that $\eta'_1 \in \Xi(g'_1)$.

We have $\mathcal{E}(p-1, g, r, v, \chi, \zeta)$, or in other words we have $\mathcal{E}(p-1, g, q, u, \vartheta, \theta)$. This implies $\vartheta_1 \in E_s(p-1, g) \subseteq E_s(p, g)$. We have $g'_1 = g + (u_1, \vartheta_1)$ and $g'_1 \in K(p-1) \subseteq K(p)$, $g'_1 \neq \epsilon$, $p \geqslant 2$, moreover $g \in K(p)$, $u_1 \in \mathcal{V} - var(g)$ and therefore

$$\Xi(g'_1) = \{\xi + (u_1, s) |\, \xi \in \Xi(g), s \in \#(g, \vartheta_1, \xi)\}.$$

Since $\eta \in \Xi(g)$, to prove that $\eta'_1 \in \Xi(g'_1)$ we just need to prove that $s_1 \in \#(g, \vartheta_1, \eta)$. We know that $s_1 \in \#(k, \vartheta_1, \sigma)$. We have $\vartheta_1 \in E(n,k)$, $\vartheta_1 \in E(p-1, g) \subseteq E(n, g)$.

We also notice that by lemma 3.4 if we set $g = (w, \phi)$ then for each $i \in dom(k)$, $j \in dom(g)$ $x_i = w_j \to \varphi_i = \phi_j$.

Since $\eta \sqsubseteq \rho$ we can apply lemma 3.4 to also show that if $\eta = (w, \mu)$ then for each $i \in dom(\sigma)$, $j \in dom(\eta)$ $x_i = w_j \to z_i = \mu_j$.

With this we can apply the inductive hypothesis and obtain that $\#(k, \vartheta_1, \sigma) = \#(g, \vartheta_1, \eta)$, therefore $s_1 \in \#(g, \vartheta_1, \eta)$ and $\eta'_1 \in \Xi(g'_1)$.

We also notice that $k'_1 = k + (u_1, \vartheta_1)$, $g'_1 = g + (u_1, \vartheta_1)$, so if we set $k'_1 = (x'_1, \varphi'_1)$ and $g'_1 = (w'_1, \phi'_1)$ then by lemma 3.3 for each $\alpha \in dom(k'_1)$, $\beta \in dom(g'_1)$ $(x'_1)_\alpha = (w'_1)_\beta \to (\varphi'_1)_\alpha = (\phi'_1)_\beta$.
Moreover we notice that $\sigma'_1 = \sigma + (u_1, s_1)$, $\eta'_1 = \eta + (u_1, s_1)$, and if we set $\sigma'_1 = (x'_1, z'_1)$, $\eta'_1 = (w'_1, \mu'_1)$ then by lemma 3.3 for each $\alpha \in dom(\sigma'_1)$, $\beta \in dom(\eta'_1)$ $(x'_1)_\alpha = (w'_1)_\beta \to (z'_1)_\alpha = (\mu'_1)_\beta$.

If $q > 1$ then for each $i = 1 \ldots q - 1$ we can define $\eta'_{i+1} = \eta'_i + (u_{i+1}, s_{i+1})$ and we expect to be able to prove that $\eta'_{i+1} \in \Xi(g'_{i+1})$.

We have $\mathcal{E}(p-1, g, q, u, \vartheta, \theta)$ and $g'_{i+1} = g'_i + (u_{i+1}, \vartheta_{i+1})$. This implies $g'_{i+1} \in K(p-1) \subseteq K(p)$, $g'_{i+1} \neq \epsilon$ and $p \geqslant 2$ holds too. Moreover $g'_i \in K(p)$, $\vartheta_{i+1} \in E_s(p-1, g'_i) \subseteq E_s(p, g'_i)$, and, since $var(g'_i) = var(g) \cup \{u_1, \ldots, u_i\}$, $u_{i+1} \in \mathcal{V} - var(g'_i)$. Therefore

$$\Xi(g'_{i+1}) = \{\xi + (u_{i+1}, s) \,|\, \xi \in \Xi(g'_i), s \in \#(g'_i, \vartheta_{i+1}, \xi)\}.$$

By inductive hypothesis we can assume that $\eta'_i \in \Xi(g'_i)$, therefore to prove $\eta'_{i+1} \in \Xi(g'_{i+1})$ we just need to prove $s_{i+1} \in \#(g'_i, \vartheta_{i+1}, \eta'_i)$. We know that $s_{i+1} \in \#(k'_i, \vartheta_{i+1}, \sigma'_i)$.

By inductive hypothesis we can also assume that

- if we set $k'_i = (x'_i, \varphi'_i)$ and $g'_i = (w'_i, \phi'_i)$ then for each $\alpha \in dom(k'_i)$, $\beta \in dom(g'_i)$ $(x'_i)_\alpha = (w'_i)_\beta \to (\varphi'_i)_\alpha = (\phi'_i)_\beta$;
- if we set $\sigma'_i = (x'_i, z'_i)$ and $\eta'_i = (w'_i, \mu'_i)$ then for each $\alpha \in dom(\sigma'_i)$, $\beta \in dom(\eta'_i)$ $(x'_i)_\alpha = (w'_i)_\beta \to (z'_i)_\alpha = (\mu'_i)_\beta$.

We have $k'_i \in K(n)$, $g'_i \in K(p) \subseteq K(n)$, $\vartheta_{i+1} \in E_s(n, k'_i)$, $\vartheta_{i+1} \in E_s(p, g'_i) \subseteq E_s(n, g'_i)$, $\sigma'_i \in \Xi(k'_i)$, $\eta'_i \in \Xi(g'_i)$, so we can apply the inductive hypothesis and obtain that $\#(k'_i, \vartheta_{i+1}, \sigma'_i) = \#(g'_i, \vartheta_{i+1}, \eta'_i)$. Therefore $s_{i+1} \in \#(g'_i, \vartheta_{i+1}, \eta'_i)$ and we have proved $\eta'_{i+1} \in \Xi(g'_{i+1})$.

In this proof that $\eta'_{i+1} \in \Xi(g'_{i+1})$ we have used an inductive hypothesis which we haven't proved, so we need to prove it now. What we need to prove is the following:

- if we set $k'_{i+1} = (x'_{i+1}, \varphi'_{i+1})$ and $g'_{i+1} = (w'_{i+1}, \phi'_{i+1})$ then for each $\alpha \in dom(k'_{i+1})$, $\beta \in dom(g'_{i+1})$ $(x'_{i+1})_\alpha = (w'_{i+1})_\beta \to (\varphi'_{i+1})_\alpha = (\phi'_{i+1})_\beta$;
- if we set $\sigma'_{i+1} = (x'_{i+1}, z'_{i+1})$ and $\eta'_{i+1} = (w'_{i+1}, \mu'_{i+1})$ then for each $\alpha \in dom(\sigma'_{i+1})$, $\beta \in dom(\eta'_{i+1})$ $(x'_{i+1})_\alpha = (w'_{i+1})_\beta \to (z'_{i+1})_\alpha = (\mu'_{i+1})_\beta$.

To prove the first item we consider that $k'_{i+1} = k'_i + (u_{i+1}, \vartheta_{i+1})$,
$g'_{i+1} = g'_i + (u_{i+1}, \vartheta_{i+1})$, $u_{i+1} \in \mathcal{V} - var(k'_i)$, $u_{i+1} \in \mathcal{V} - var(g'_i)$. So we can apply lemma 3.3 and the first condition is proved.

To prove the second item we consider that $\sigma'_{i+1} = \sigma'_i + (u_{i+1}, s_{i+1})$, $\eta'_{i+1} = \eta'_i + (u_{i+1}, s_{i+1})$, $u_{i+1} \in \mathcal{V}-var(\sigma'_i)$, $u_{i+1} \in \mathcal{V}-var(\eta'_i)$. So we can apply lemma 3.3 and the second condition is proved.

At this point we have defined η'_q such that $\eta \sqsubseteq \eta'_q$ and proved that $\eta'_q \in \Xi(g'_q)$. We have also that $k'_q \in K(n)$, $\theta \in E(n, k'_q)$, $g'_q \in K(p-1) \subseteq K(n)$, $\theta \in E(p-1, g'_q) \subseteq E(n, g'_q)$, $\sigma'_q \in \Xi(k'_q)$. Moreover

- if we set $k'_q = (x'_q, \varphi'_q)$ and $g'_q = (w'_q, \phi'_q)$ then for each $\alpha \in dom(k'_q)$, $\beta \in dom(g'_q)$ $(x'_q)_\alpha = (w'_q)_\beta \to (\varphi'_q)_\alpha = (\phi'_q)_\beta$;
- if we set $\sigma'_q = (x'_q, z'_q)$ and $\eta'_q = (w'_q, \mu'_q)$ then for each $\alpha \in dom(\sigma'_q)$, $\beta \in dom(\eta'_q)$ $(x'_q)_\alpha = (w'_q)_\beta \to (z'_q)_\alpha = (\mu'_q)_\beta$.

With this, $\#(g'_q, \theta, \eta'_q) = \#(k'_q, \theta, \sigma'_q)$ follows by inductive hypothesis.

We now examine the other side of the proof. Let $\eta'_q \in \Xi(g'_q)$ such that $\eta \sqsubseteq \eta'_q$, we want to find $\sigma'_q \in \Xi(k'_q)$ such that $\sigma \sqsubseteq \sigma'_q$ and $\#(k'_q, \theta, \sigma'_q) = \#(g'_q, \theta, \eta'_q)$. Remember that $\eta = \rho_{/dom(g)}$.

Let $\eta'_1 = (\eta'_q)_{/dom(g'_1)}$. We should be able to prove that:

- $\eta'_1 \in \Xi(g'_1)$
- there exists $s_1 \in \#(g, \vartheta_1, \eta)$ such that $\eta'_1 = \eta + (u_1, s_1)$.

If $q = 1$ then $\eta'_1 \in \Xi(g'_1)$ clearly holds, else we have $g'_q \neq \epsilon$, $\eta'_q \in \Xi(g'_q)$, $g'_1 \in \mathcal{R}(g'_q)$, $g'_1 \neq g'_q$, so by lemma 4.4 $\eta'_1 = (\eta'_q)_{/dom(g'_1)} \in \Xi(g'_1)$.

We have $g'_1 = g + (u_1, \vartheta_1)$ and $g'_1 \in K(n)$, clearly $g'_1 \neq \epsilon$ and $n \geqslant 2$ also hold. Moreover $g \in K(n)$, $\vartheta_1 \in E_s(n, g)$, $u_1 \in \mathcal{V} - var(g)$, so by lemma 4.3

$$\Xi(g'_1) = \{\xi + (u_1, s) |\, \xi \in \Xi(g), s \in \#(g, \vartheta_1, \xi)\}.$$

Then there exist $\xi \in \Xi(g), s \in \#(g, \vartheta_1, \xi)$ such that $\eta'_1 = \xi + (u_1, s)$. Here we can see that

$$(\eta'_1)_{/dom(g)} = (\eta'_1)_{/dom(\xi)} = \xi$$

and at the same time, since $dom(g) \subseteq dom(g'_1) \subseteq dom(g'_q) = dom(\eta'_q)$,

$$(\eta'_1)_{/dom(g)} = ((\eta'_q)_{/dom(g'_1)})_{/dom(g)} = (\eta'_q)_{/dom(g)} = (\eta'_q)_{/dom(\eta)} = \eta.$$

Therefore $\xi = \eta$ and there exists $s \in \#(g, \vartheta_1, \eta)$ such that $\eta'_1 = \eta + (u_1, s)$.

If $q > 1$ then for each $i = 1 \dots q-1$ we can define $\eta'_{i+1} = (\eta'_q)_{/dom(g'_{i+1})}$. We should also be able to prove that for each $i = 1 \dots q-1$

- $\eta'_{i+1} \in \Xi(g'_{i+1})$,
- there exists $s_{i+1} \in \#(g'_i, \vartheta_{i+1}, \eta'_i)$ such that $\eta'_{i+1} = \eta'_i + (u_{i+1}, s_{i+1})$.

If $i + 1 = q$ then $\eta'_{i+1} \in \Xi(g'_{i+1})$ clearly holds, else $i + 1 < q$ and $g'_q \neq \epsilon$, $\eta'_q \in \Xi(g'_q)$, $g'_{i+1} \in \mathcal{R}(g'_q)$, $g'_{i+1} \neq g'_q$, so by lemma 4.4 $\eta'_{i+1} = (\eta'_q)_{/dom(g'_{i+1})} \in \Xi(g'_{i+1})$.

We have $g'_{i+1} = g'_i + (u_{i+1}, \vartheta_{i+1})$ and $g'_{i+1} \in K(n)$, clearly $g'_{i+1} \neq \epsilon$ and $n \geqslant 2$ also hold. Moreover $g'_i \in K(n)$, $\vartheta_{i+1} \in E_s(n, g'_i)$, $var(g'_i) = var(g) \cup \{u_1, \ldots, u_i\}$, $u_{i+1} \in \mathcal{V} - var(g'_i)$, so by lemma 4.3

$$\Xi(g'_{i+1}) = \{\xi + (u_{i+1}, s) |\, \xi \in \Xi(g'_i), s \in \#(g'_i, \vartheta_{i+1}, \xi)\}.$$

Then there exist $\xi \in \Xi(g'_i), s \in \#(g'_i, \vartheta_{i+1}, \xi)$ such that $\eta'_{i+1} = \xi + (u_{i+1}, s)$. Here we can see that

$$(\eta'_{i+1})_{/dom(g'_i)} = (\eta'_{i+1})_{/dom(\xi)} = \xi$$

and at the same time, since $dom(g'_i) \subseteq dom(g'_{i+1}) \subseteq dom(g'_q) = dom(\eta'_q)$,

$$(\eta'_{i+1})_{/dom(g'_i)} = ((\eta'_q)_{/dom(g'_{i+1})})_{/dom(g'_i)} = (\eta'_q)_{/dom(g'_i)} = \eta'_i.$$

Therefore $\xi = \eta'_i$ and there exists $s \in \#(g'_i, \vartheta_{i+1}, \eta'_i)$ such that $\eta'_{i+1} = \eta'_i + (u_{i+1}, s)$.

Then we define $\sigma'_1 = \sigma + (u_1, s_1)$, and we should be able to prove that $\sigma'_1 \in \Xi(k'_1)$.

We have $\mathcal{E}(n, k, q, u, \vartheta, \theta)$. This implies $\vartheta_1 \in E_s(n, k)$. We have $k'_1 = k + (u_1, \vartheta_1)$ and $k'_1 \in K(n)$, $k'_1 \neq \epsilon$, $n \geqslant 2$, moreover $k \in K(n)$, $u_1 \in \mathcal{V} - var(k)$ and therefore

$$\Xi(k'_1) = \{\xi + (u_1, s) |\, \xi \in \Xi(k), s \in \#(k, \vartheta_1, \xi)\}.$$

Since $\sigma \in \Xi(k)$, to prove that $\sigma'_1 \in \Xi(k'_1)$ we just need to prove that $s_1 \in \#(k, \vartheta_1, \sigma)$. We know that $s_1 \in \#(g, \vartheta_1, \eta)$. We have $\vartheta_1 \in E(n, k)$, $\vartheta_1 \in E(p-1, g) \subseteq E(n, g)$.

We also notice that by lemma 3.4 if we set $g = (w, \phi)$ then for each $i \in dom(k)$, $j \in dom(g)$ $x_i = w_j \rightarrow \varphi_i = \phi_j$.

Since $\eta \sqsubseteq \rho$ we can apply lemma 3.4 to also show that if $\eta = (w, \mu)$ then for each $i \in dom(\sigma)$, $j \in dom(\eta)$ $x_i = w_j \rightarrow z_i = \mu_j$.

With all this we can apply the inductive hypothesis and obtain that $\#(g, \vartheta_1, \eta) = \#(k, \vartheta_1, \sigma)$, therefore $s_1 \in \#(k, \vartheta_1, \sigma)$ and $\sigma'_1 \in \Xi(k'_1)$.

We also notice that $k'_1 = k + (u_1, \vartheta_1)$, $g'_1 = g + (u_1, \vartheta_1)$, so if we set $k'_1 = (x'_1, \varphi'_1)$ and $g'_1 = (w'_1, \phi'_1)$ then by lemma 3.3 for each $\alpha \in dom(k'_1)$, $\beta \in dom(g'_1)$ $(x'_1)_\alpha = (w'_1)_\beta \rightarrow (\varphi'_1)_\alpha = (\phi'_1)_\beta$.
Moreover we notice that $\sigma'_1 = \sigma + (u_1, s_1)$, $\eta'_1 = \eta + (u_1, s_1)$, and if we set $\sigma'_1 = (x'_1, z'_1)$, $\eta'_1 = (w'_1, \mu'_1)$ then by lemma 3.3 for each $\alpha \in dom(\sigma'_1)$, $\beta \in dom(\eta'_1)$ $(x'_1)_\alpha = (w'_1)_\beta \rightarrow (z'_1)_\alpha = (\mu'_1)_\beta$.

If $q > 1$ then for each $i = 1 \ldots q-1$ we can define $\sigma'_{i+1} = \sigma'_i + (u_{i+1}, s_{i+1})$ and we expect to be able to prove that $\sigma'_{i+1} \in \Xi(k'_{i+1})$.

We have $\mathcal{E}(n, k, q, u, \vartheta, \theta)$ and $k'_{i+1} = k'_i + (u_{i+1}, \vartheta_{i+1})$. This implies $k'_{i+1} \in K(n)$, $k'_{i+1} \neq \epsilon$ and $n \geqslant 2$ holds too. Moreover $k'_i \in K(n)$, $\vartheta_{i+1} \in E_s(n, k'_i)$ and, since $var(k'_i) = var(k) \cup \{u_1, \ldots, u_i\}$, $u_{i+1} \in \mathcal{V} - var(k'_i)$. Therefore

$$\Xi(k'_{i+1}) = \{\xi + (u_{i+1}, s) |\, \xi \in \Xi(k'_i), s \in \#(k'_i, \vartheta_{i+1}, \xi)\}.$$

By inductive hypothesis we can assume that $\sigma'_i \in \Xi(k'_i)$, therefore to prove $\sigma'_{i+1} \in \Xi(k'_{i+1})$ we just need to prove $s_{i+1} \in \#(k'_i, \vartheta_{i+1}, \sigma'_i)$. We know that $s_{i+1} \in \#(g'_i, \vartheta_{i+1}, \eta'_i)$.

By inductive hypothesis we can also assume that

- if we set $k'_i = (x'_i, \varphi'_i)$ and $g'_i = (w'_i, \phi'_i)$ then for each $\alpha \in dom(k'_i)$, $\beta \in dom(g'_i)$ $(x'_i)_\alpha = (w'_i)_\beta \to (\varphi'_i)_\alpha = (\phi'_i)_\beta$;
- if we set $\sigma'_i = (x'_i, z'_i)$ and $\eta'_i = (w'_i, \mu'_i)$ then for each $\alpha \in dom(\sigma'_i)$, $\beta \in dom(\eta'_i)$ $(x'_i)_\alpha = (w'_i)_\beta \to (z'_i)_\alpha = (\mu'_i)_\beta$.

We have $k'_i \in K(n)$, $g'_i \in K(p) \subseteq K(n)$, $\vartheta_{i+1} \in E_s(n, k'_i)$, $\vartheta_{i+1} \in E_s(p, g'_i) \subseteq E_s(n, g'_i)$, $\sigma'_i \in \Xi(k'_i)$, $\eta'_i \in \Xi(g'_i)$, so we can apply the inductive hypothesis and obtain that $\#(g'_i, \vartheta_{i+1}, \eta'_i) = \#(k'_i, \vartheta_{i+1}, \sigma'_i)$. Therefore $s_{i+1} \in \#(k'_i, \vartheta_{i+1}, \sigma'_i)$ and we have proved $\sigma'_{i+1} \in \Xi(k'_{i+1})$.

In this proof that $\sigma'_{i+1} \in \Xi(k'_{i+1})$ we have used an inductive hypothesis which we haven't proved, so we need to prove it now. What we need to prove is the following:

- if we set $k'_{i+1} = (x'_{i+1}, \varphi'_{i+1})$ and $g'_{i+1} = (w'_{i+1}, \phi'_{i+1})$ then for each $\alpha \in dom(k'_{i+1})$, $\beta \in dom(g'_{i+1})$ $(x'_{i+1})_\alpha = (w'_{i+1})_\beta \to (\varphi'_{i+1})_\alpha = (\phi'_{i+1})_\beta$;
- if we set $\sigma'_{i+1} = (x'_{i+1}, z'_{i+1})$ and $\eta'_{i+1} = (w'_{i+1}, \mu'_{i+1})$ then for each $\alpha \in dom(\sigma'_{i+1})$, $\beta \in dom(\eta'_{i+1})$ $(x'_{i+1})_\alpha = (w'_{i+1})_\beta \to (z'_{i+1})_\alpha = (\mu'_{i+1})_\beta$.

To prove the first item we consider that $k'_{i+1} = k'_i + (u_{i+1}, \vartheta_{i+1})$, $g'_{i+1} = g'_i + (u_{i+1}, \vartheta_{i+1})$, $u_{i+1} \in \mathcal{V}{-}var(k'_i)$, $u_{i+1} \in \mathcal{V}{-}var(g'_i)$. So we can apply lemma 3.3 and the first condition is proved.

To prove the second item we consider that $\sigma'_{i+1} = \sigma'_i + (u_{i+1}, s_{i+1})$, $\eta'_{i+1} = \eta'_i + (u_{i+1}, s_{i+1})$, $u_{i+1} \in \mathcal{V}{-}var(\sigma'_i)$, $u_{i+1} \in \mathcal{V}{-}var(\eta'_i)$. So we can apply lemma 3.3 and the second condition is proved.

At this point we have defined σ'_q such that $\sigma \sqsubseteq \sigma'_q$ and proved that $\sigma'_q \in \Xi(k'_q)$. We have also that $k'_q \in K(n)$, $\theta \in E(n, k'_q)$, $g'_q \in K(p-1) \subseteq K(n)$, $\theta \in E(p-1, g'_q) \subseteq E(n, g'_q)$, $\eta'_q \in \Xi(g'_q)$. Moreover

- if we set $k'_q = (x'_q, \varphi'_q)$ and $g'_q = (w'_q, \phi'_q)$ then for each $\alpha \in dom(k'_q)$, $\beta \in dom(g'_q)$ $(x'_q)_\alpha = (w'_q)_\beta \to (\varphi'_q)_\alpha = (\phi'_q)_\beta$;
- if we set $\sigma'_q = (x'_q, z'_q)$ and $\eta'_q = (w'_q, \mu'_q)$ then for each $\alpha \in dom(\sigma'_q)$, $\beta \in dom(\eta'_q)$ $(x'_q)_\alpha = (w'_q)_\beta \to (z'_q)_\alpha = (\mu'_q)_\beta$.

With this, $\#(k'_q, \theta, \sigma'_q) = \#(g'_q, \theta, \eta'_q)$ follows by inductive hypothesis.

$\diamond$

Let's consider the third case. Here $t \in E(m, \kappa) \cap E(p, g)$, with $m, p < n+1$. Of course given $\sigma = (x, z) \in \Xi(k)$, $\rho = (y, r) \in \Xi(h)$ such that for each $i \in dom(\sigma)$, $j \in dom(\rho)$ $x_i = y_j \to z_i = r_j$ we want to show that $\#(k, t, \sigma) = \#(h, t, \rho)$. So we just need to show that $\#(\kappa, t, \sigma_{/dom(\kappa)}) = \#(g, t, \rho_{/dom(g)})$.

By lemma 3.4 if we set $\kappa = (u, \phi)$ and $g = (w, \theta)$ then for each $i \in dom(\kappa)$, $j \in dom(g)$ $u_i = w_j \to \phi_i = \theta_j$.
If we define $q = max\{m, p\}$ then $\kappa, g \in K(q)$, $t \in E(q, \kappa) \cap E(q, g)$ and $q < n+1$.

Moreover let $\sigma' = \sigma_{/dom(\kappa)}$, $\sigma' = (x', z')$, $\rho' = \rho_{/dom(g)}$, $\rho' = (y', r')$. Since $\sigma' \sqsubseteq \sigma$ and $\rho' \sqsubseteq \rho$ by lemma 3.4 we obtain that for each $i \in dom(\sigma')$, $j \in dom(\rho')$ $x'_i = y'_j \to z'_i = r'_j$. By the inductive hypothesis we then obtain $\#(\kappa, t, \sigma') = \#(g, t, \rho')$, and so we have proved $\#(k, t, \sigma) = \#(h, t, \rho)$.

■

Lemma 4.13. *Given*

- *a positive integer* n*;*
- $k \in K(n)$*;*
- $f \in \mathcal{F}$*;*
- *a positive integer* m*;*
- $\varphi_1, \ldots, \varphi_m \in E(n, k)$*;*

such that for each $\sigma \in \Xi(k)$ $A_f(\#(k, \varphi_1, \sigma), \ldots, \#(k, \varphi_m, \sigma))$ *is true,*
we have that $t = f(\varphi_1, \ldots, \varphi_m) \in E(n+1, k)$*.*

Given $\sigma \in \Xi(k)$ *we have also*

$$\#(k, t, \sigma) = P_f(\#(k, \varphi_1, \sigma), \ldots, \#(k, \varphi_m, \sigma)) \ .$$

Proof.

If $t \in E(n, k) \cup E_b(n+1, k)$ then $t \in E(n+1, k)$, else $t \in E_d(n+1, k) \subseteq E(n+1, k)$.

Using lemma 4.10 we have that one of the following alternatives holds:

- $t \in E_a(n+1, k) \cup E_c(n+1, k) \cup E_d(n+1, k) \cup E_e(n+1, k)$;
- there exist p positive integer such that $2 \leqslant p < n+1$, $h \in K(p)$ such that $h \sqsubseteq k$, $t \in E_a(p, h) \cup E_c(p, h) \cup E_d(p, h) \cup E_e(p, h)$ and for each $\sigma \in \Xi(k)$ $\sigma_{/dom(h)} \in \Xi(h)$ and $\#(k, t, \sigma) = \#(h, t, \sigma_{/dom(h)})$.

If the first alternative holds, that is $t \in E_a(n+1, k) \cup E_c(n+1, k) \cup E_d(n+1, k) \cup E_e(n+1, k)$, then clearly $t \in E_d(n+1, k)$. This implies that $\#(k, t, \sigma) = \#(k, t, \sigma)_{(n+1,k,d)}$, so in this case our proof is finished.

Otherwise it must be $t \in E_d(p, h)$. This implies that there exist:

- $g \in \mathcal{F}$
- q positive integer;
- $\psi_1, \ldots, \psi_q \in E(p-1, h)$;

such that

- for each $\rho \in \Xi(h)$ $A_g(\#(h, \psi_1, \rho), \ldots, \#(h, \psi_q, \rho))$ is true;
- $g(\psi_1, \ldots, \psi_q) \notin E(p-1, h)$;
- $t = g(\psi_1, \ldots, \psi_q)$;
- for each $\rho \in \Xi(h)$ $\#(h, t, \rho) = P_g(\#(h, \psi_1, \rho), \ldots, \#(h, \psi_q, \rho))$.

We have $t = f(\varphi_1, \dots, \varphi_m)$, so we can apply the unique readability lemma 3.1.5 and obtain that $g = f$, $q = m$, for each $i \in \{1, \dots, m\}$ $\psi_i = \varphi_i$. Then given $\rho \in \Xi(h)$

$$\#(h, t, \rho) = P_f(\#(h, \varphi_1, \rho), \dots, \#(h, \varphi_m, \rho)) \ .$$

Now given $\sigma \in \Xi(k)$ we want to prove that

$$\#(k, t, \sigma) = P_f(\#(k, \varphi_1, \sigma), \dots, \#(k, \varphi_m, \sigma)) \ .$$

If we define $\rho = \sigma_{/dom(h)} \in \Xi(h)$ then

$$\#(k, t, \sigma) = \#(h, t, \rho) = P_f(\#(h, \varphi_1, \rho), \dots, \#(h, \varphi_m, \rho)) \ .$$

So we want to prove that

$$P_f(\#(h, \varphi_1, \rho), \dots, \#(h, \varphi_m, \rho)) = P_f(\#(k, \varphi_1, \sigma), \dots, \#(k, \varphi_m, \sigma)) \ ,$$

and to prove this it is enough to prove that for each $i \in \{1, \dots, m\}$

$$\#(h, \varphi_i, \rho) = \#(k, \varphi_i, \sigma) \ .$$

Proving this is not difficult. In fact, if $k = (u, \eta)$ and $h = (v, \vartheta)$ then by lemma 4.8 for each $i \in dom(k)$, $j \in dom(h)$ $u_i = v_j \to \eta_i = \vartheta_j$. If $\sigma = (u, \mu)$ and $\rho = (v, \nu)$ then by lemma 4.9 for each $i \in dom(\sigma)$, $j \in dom(\rho)$ $u_i = v_j \to \mu_i = \nu_j$. With this we can apply lemma 4.12 and obtain that $\#(h, \varphi_i, \rho) = \#(k, \varphi_i, \sigma)$. ∎

Lemma 4.14. *Given*

- *a positive integer n;*
- $h = (v, \vartheta),\ k = (u, \eta) \in K(n)$ *such that*
  for each $i \in dom(k)$, $j \in dom(h)$ $u_i = v_j \to \eta_i = \vartheta_j$;
- $\rho = (v, \nu) \in \Xi(h),\ \sigma = (u, \mu) \in \Xi(k)$ *such that*
  for each $i \in dom(\sigma)$, $j \in dom(\rho)$ $u_i = v_j \to \mu_i = \nu_j$;
- *a positive integer m;*
- *a function x whose domain is $\{1, \dots, m\}$ such that for each $i = 1 \dots m$ $x_i \in \mathcal{V} - var(k)$, $x_i \in \mathcal{V} - var(h)$, and for each $i, j = 1 \dots m$ $i \neq j \to x_i \neq x_j$;*
- *a function φ whose domain is $\{1, \dots, m\}$ such that for each $i = 1 \dots m$ $\varphi_i \in E(n)$;*
- $\phi \in E(n)$*;*

such that

- $\mathcal{E}(n, k, m, x, \varphi, \phi)$*;*
- $\mathcal{E}(n, h, m, x, \varphi, \phi)$*;*

and given $t = \{\}(x_1 : \varphi_1, \dots, x_m : \varphi_m, \phi)$, we have that

for each $\sigma'_m \in \Xi(k'_m)$ such that $\sigma \sqsubseteq \sigma'_m$ there exists $\rho'_m \in \Xi(h'_m)$ such that $\rho \sqsubseteq \rho'_m$ and $\#(h'_m, \phi, \rho'_m) = \#(k'_m, \phi, \sigma'_m)$,

where of course

- $k'_1 = k + (x_1, \varphi_1)$*, and if $m > 1$ for each $i = 1 \dots m - 1$ $k'_{i+1} = k'_i + (x_{i+1}, \varphi_{i+1})$,*

- *$h'_1 = h + (x_1, \varphi_1)$, and if $m > 1$ for each $i = 1 \dots m-1$ $h'_{i+1} = h'_i + (x_{i+1}, \varphi_{i+1})$.*

Proof.

Let $\sigma'_m \in \Xi(k'_m)$ such that $\sigma \sqsubseteq \sigma'_m$, we want to find $\rho'_m \in \Xi(h'_m)$ such that $\rho \sqsubseteq \rho'_m$ and $\#(h'_m, \phi, \rho'_m) = \#(k'_m, \phi, \sigma'_m)$.

If $m = 1$ then σ'_1 is defined, else let $\sigma'_1 = (\sigma'_m)_{/dom(k'_1)}$. We should be able to prove that:

- $\sigma'_1 \in \Xi(k'_1)$
- there exists $s_1 \in \#(k, \varphi_1, \sigma)$ such that $\sigma'_1 = \sigma + (x_1, s_1)$.

If $m = 1$ then $\sigma'_1 \in \Xi(k'_1)$ clearly holds, else we have $k'_m \neq \epsilon$, $\sigma'_m \in \Xi(k'_m)$, $k'_1 \in \mathcal{R}(k'_m)$, $k'_1 \neq k'_m$, so by lemma 4.4 $\sigma'_1 = (\sigma'_m)_{/dom(k'_1)} \in \Xi(k'_1)$.

We have $k'_1 = k + (x_1, \varphi_1)$ and $k'_1 \in K(n)$, clearly $k'_1 \neq \epsilon$ and $n \geqslant 2$ also hold. Moreover $k \in K(n)$, $\varphi_1 \in E_s(n, k)$, $x_1 \in \mathcal{V} - var(k)$, so by lemma 4.3

$$\Xi(k'_1) = \{\xi + (x_1, s) |\, \xi \in \Xi(k), s \in \#(k, \varphi_1, \xi)\}.$$

Then there exist $\xi \in \Xi(k), s \in \#(k, \varphi_1, \xi)$ such that $\sigma'_1 = \xi + (x_1, s)$. Here we can see that

$$(\sigma'_1)_{/dom(k)} = (\sigma'_1)_{/dom(\xi)} = \xi$$

and at the same time, since $dom(k) \subseteq dom(k'_1) \subseteq dom(k'_m) = dom(\sigma'_m)$,

$$(\sigma'_1)_{/dom(k)} = ((\sigma'_m)_{/dom(k'_1)})_{/dom(k)} = (\sigma'_m)_{/dom(k)} = (\sigma'_m)_{/dom(\sigma)} = \sigma.$$

Therefore $\xi = \sigma$ and there exists $s \in \#(k, \varphi_1, \sigma)$ such that $\sigma'_1 = \sigma + (x_1, s)$.

If $m > 1$ let $i = 1 \dots m-1$, if $i+1 = m$ then $\sigma'_{i+1} = \sigma'_m$ is defined, else we can define $\sigma'_{i+1} = (\sigma'_m)_{/dom(k'_{i+1})}$. We should also be able to prove that

- $\sigma'_{i+1} \in \Xi(k'_{i+1})$,
- there exists $s_{i+1} \in \#(k'_i, \varphi_{i+1}, \sigma'_i)$ such that $\sigma'_{i+1} = \sigma'_i + (x_{i+1}, s_{i+1})$.

If $i+1 = m$ then $\sigma'_{i+1} \in \Xi(k'_{i+1})$ clearly holds, else $i+1 < m$ and $k'_m \neq \epsilon$, $\sigma'_m \in \Xi(k'_m)$, $k'_{i+1} \in \mathcal{R}(k'_m)$, $k'_{i+1} \neq k'_m$, so by lemma 4.4 $\sigma'_{i+1} = (\sigma'_m)_{/dom(k'_{i+1})} \in \Xi(k'_{i+1})$.

We have $k'_{i+1} = k'_i + (x_{i+1}, \varphi_{i+1})$ and $k'_{i+1} \in K(n)$, clearly $k'_{i+1} \neq \epsilon$ and $n \geqslant 2$ also hold. Moreover $k'_i \in K(n)$, $\varphi_{i+1} \in E_s(n, k'_i)$, $var(k'_i) = var(k) \cup \{x_1, \dots, x_i\}$, $x_{i+1} \in \mathcal{V} - var(k'_i)$, so by lemma 4.3

$$\Xi(k'_{i+1}) = \{\xi + (x_{i+1}, s) |\, \xi \in \Xi(k'_i), s \in \#(k'_i, \varphi_{i+1}, \xi)\}.$$

Then there exist $\xi \in \Xi(k'_i), s \in \#(k'_i, \varphi_{i+1}, \xi)$ such that $\sigma'_{i+1} = \xi + (x_{i+1}, s)$. Here we can see that

$$(\sigma'_{i+1})_{/dom(k'_i)} = (\sigma'_{i+1})_{/dom(\xi)} = \xi.$$

At the same time, if $i + 1 = m$ then

$$(\sigma'_{i+1})_{/dom(k'_i)} = (\sigma'_m)_{/dom(k'_i)} = \sigma'_i.$$

Else since $dom(k'_i) \subseteq dom(k'_{i+1}) \subseteq dom(k'_m) = dom(\sigma'_m)$,

$$(\sigma'_{i+1})_{/dom(k'_i)} = ((\sigma'_m)_{/dom(k'_{i+1})})_{/dom(k'_i)} = (\sigma'_m)_{/dom(k'_i)} = \sigma'_i.$$

Therefore $\xi = \sigma'_i$ and there exists $s \in \#(k'_i, \varphi_{i+1}, \sigma'_i)$ such that $\sigma'_{i+1} = \sigma'_i + (x_{i+1}, s)$.

Then we define $\rho'_1 = \rho + (x_1, s_1)$, and we should be able to prove that $\rho'_1 \in \Xi(h'_1)$.

We have $\mathcal{E}(n, h, m, x, \varphi, \phi)$. This implies $\varphi_1 \in E_s(n, h)$. We have $h'_1 = h + (x_1, \varphi_1)$ and $h'_1 \in K(n)$, $h'_1 \neq \epsilon$, $n \geqslant 2$, moreover $h \in K(n)$, $x_1 \in \mathcal{V} - var(h)$ and therefore

$$\Xi(h'_1) = \{\xi + (x_1, s) |\, \xi \in \Xi(h), s \in \#(h, \varphi_1, \xi)\}.$$

Since $\rho \in \Xi(h)$, to prove that $\rho'_1 \in \Xi(h'_1)$ we just need to prove that $s_1 \in \#(h, \varphi_1, \rho)$. We know that $s_1 \in \#(k, \varphi_1, \sigma)$. We have $\varphi_1 \in E(n, k)$, $\varphi_1 \in E(n, h)$. We have also that for each $i \in dom(k)$, $j \in dom(h)$ $u_i = v_j \to \eta_i = \vartheta_j$ and for each $i \in dom(\sigma)$, $j \in dom(\rho)$ $u_i = v_j \to \mu_i = \nu_j$. With this we can apply lemma 4.12 and obtain that $\#(k, \varphi_1, \sigma) = \#(h, \varphi_1, \rho)$, therefore $s_1 \in \#(h, \varphi_1, \rho)$.

We also notice that $k'_1 = k + (x_1, \varphi_1)$, $h'_1 = h + (x_1, \varphi_1)$, so if we set $k'_1 = (u'_1, \eta'_1)$ and $h'_1 = (v'_1, \vartheta'_1)$ then by lemma 3.3 for each $\alpha \in dom(k'_1)$, $\beta \in dom(h'_1)$ $(u'_1)_\alpha = (v'_1)_\beta \to (\eta'_1)_\alpha = (\vartheta'_1)_\beta$.
Moreover we notice that $\sigma'_1 = \sigma + (x_1, s_1)$, $\rho'_1 = \rho + (x_1, s_1)$, and if we set $\sigma'_1 = (u'_1, \mu'_1)$, $\rho'_1 = (v'_1, \nu'_1)$ then by lemma 3.3 for each $\alpha \in dom(\sigma'_1)$, $\beta \in dom(\rho'_1)$ $(u'_1)_\alpha = (v'_1)_\beta \to (\mu'_1)_\alpha = (\nu'_1)_\beta$.

If $m > 1$ then for each $i = 1 \dots m-1$ we can define $\rho'_{i+1} = \rho'_i + (x_{i+1}, s_{i+1})$ and we expect to be able to prove that $\rho'_{i+1} \in \Xi(h'_{i+1})$.

We have $\mathcal{E}(n, h, m, x, \varphi, \phi)$ and $h'_{i+1} = h'_i + (x_{i+1}, \varphi_{i+1})$. This implies $h'_{i+1} \in K(n)$, $h'_{i+1} \neq \epsilon$ and $n \geqslant 2$ holds too. Moreover $h'_i \in K(n)$, $\varphi_{i+1} \in E_s(n, h'_i)$, and, since $var(h'_i) = var(h) \cup \{x_1, \dots, x_i\}$, $x_{i+1} \in \mathcal{V} - var(h'_i)$. Therefore

$$\Xi(h'_{i+1}) = \{\xi + (x_{i+1}, s) |\, \xi \in \Xi(h'_i), s \in \#(h'_i, \varphi_{i+1}, \xi)\}.$$

By inductive hypothesis we can assume that $\rho'_i \in \Xi(h'_i)$, therefore to prove $\rho'_{i+1} \in \Xi(h'_{i+1})$ we just need to prove $s_{i+1} \in \#(h'_i, \varphi_{i+1}, \rho'_i)$. We know that $s_{i+1} \in \#(k'_i, \varphi_{i+1}, \sigma'_i)$.

By inductive hypothesis we can also assume that

- if we set $k'_i = (u'_i, \eta'_i)$ and $h'_i = (v'_i, \vartheta'_i)$ then for each $\alpha \in dom(k'_i)$, $\beta \in dom(h'_i)$ $(u'_i)_\alpha = (v'_i)_\beta \to (\eta'_i)_\alpha = (\vartheta'_i)_\beta$;
- if we set $\sigma'_i = (u'_i, \mu'_i)$ and $\rho'_i = (v'_i, \nu'_i)$ then for each $\alpha \in dom(\sigma'_i)$, $\beta \in dom(\rho'_i)$ $(u'_i)_\alpha = (v'_i)_\beta \to (\mu'_i)_\alpha = (\nu'_i)_\beta$.

We have $k'_i \in K(n)$, $h'_i \in K(n)$, $\varphi_{i+1} \in E_s(n, k'_i)$, $\varphi_{i+1} \in E_s(n, h'_i)$, $\sigma'_i \in \Xi(k'_i)$, $\rho'_i \in \Xi(h'_i)$, so we can apply lemma 4.12 and obtain that $\#(k'_i, \varphi_{i+1}, \sigma'_i) = \#(h'_i, \varphi_{i+1}, \rho'_i)$. Therefore $s_{i+1} \in \#(h'_i, \varphi_{i+1}, \rho'_i)$ and we have proved $\rho'_{i+1} \in \Xi(h'_{i+1})$.

In this proof that $\rho'_{i+1} \in \Xi(h'_{i+1})$ we have used an inductive hypothesis which we haven't proved, so we need to prove it now. What we need to prove is the following:

- if we set $k'_{i+1} = (u'_{i+1}, \eta'_{i+1})$ and $h'_{i+1} = (v'_{i+1}, \vartheta'_{i+1})$ then for each $\alpha \in dom(k'_{i+1})$, $\beta \in dom(h'_{i+1})$ $(u'_{i+1})_\alpha = (v'_{i+1})_\beta \to (\eta'_{i+1})_\alpha = (\vartheta'_{i+1})_\beta$;

- if we set $\sigma'_{i+1} = (u'_{i+1}, \mu'_{i+1})$ and $\rho'_{i+1} = (v'_{i+1}, \nu'_{i+1})$ then for each $\alpha \in dom(\sigma'_{i+1})$, $\beta \in dom(\rho'_{i+1})$ $(u'_{i+1})_\alpha = (v'_{i+1})_\beta \to (\mu'_{i+1})_\alpha = (\nu'_{i+1})_\beta$.

To prove the first item we consider that $k'_{i+1} = k'_i + (x_{i+1}, \varphi_{i+1})$, $h'_{i+1} = h'_i + (x_{i+1}, \varphi_{i+1})$, $x_{i+1} \in \mathcal{V} - var(k'_i)$, $x_{i+1} \in \mathcal{V} - var(h'_i)$. So we can apply lemma 3.3 and the first condition is proved.

To prove the second item we consider that $\sigma'_{i+1} = \sigma'_i + (x_{i+1}, s_{i+1})$, $\rho'_{i+1} = \rho'_i + (x_{i+1}, s_{i+1})$, $x_{i+1} \in \mathcal{V} - var(\sigma'_i)$, $x_{i+1} \in \mathcal{V} - var(\rho'_i)$. So we can apply lemma 3.3 and the second condition is proved.

At this point we have defined ρ'_m such that $\rho \sqsubseteq \rho'_m$ and proved that $\rho'_m \in \Xi(h'_m)$. We have also that $k'_m \in K(n)$, $\phi \in E(n, k'_m)$, $h'_m \in K(n)$, $\phi \in E(n, h'_m)$, $\sigma'_m \in \Xi(k'_m)$. Moreover

- if we set $k'_m = (u'_m, \eta'_m)$ and $h'_m = (v'_m, \vartheta'_m)$ then for each $\alpha \in dom(k'_m)$, $\beta \in dom(h'_m)$ $(u'_m)_\alpha = (v'_m)_\beta \to (\eta'_m)_\alpha = (\vartheta'_m)_\beta$;
- if we set $\sigma'_m = (u'_m, \mu'_m)$ and $\rho'_m = (v'_m, \nu'_m)$ then for each $\alpha \in dom(\sigma'_m)$, $\beta \in dom(\rho'_m)$ $(u'_m)_\alpha = (v'_m)_\beta \to (\mu'_m)_\alpha = (\nu'_m)_\beta$.

With this, $\#(h'_m, \phi, \rho'_m) = \#(k'_m, \phi, \sigma'_m)$ follows by lemma 4.12.

■

Lemma 4.15. *Given*

- *a positive integer n;*
- *$k \in K(n)$;*
- *a positive integer m;*
- *a function x whose domain is $\{1, \ldots, m\}$ such that for each $i = 1 \ldots m$ $x_i \in \mathcal{V} - var(k)$, and for each $i, j = 1 \ldots m$ $i \neq j \to x_i \neq x_j$;*
- *a function φ whose domain is $\{1, \ldots, m\}$ such that for each $i = 1 \ldots m$ $\varphi_i \in E(n)$;*
- *$\phi \in E(n)$;*

such that $\mathcal{E}(n, k, m, x, \varphi, \phi)$,
we have that $t = \{\}(x_1 : \varphi_1, \ldots, x_m : \varphi_m, \phi) \in E(n+1, k)$.

Given $\sigma \in \Xi(k)$ we have also

$$\#(k, t, \sigma) = \{\#(k'_m, \phi, \sigma'_m) |\ \sigma'_m \in \Xi(k'_m), \sigma \sqsubseteq \sigma'_m\},$$

where $k'_1 = k + (x_1, \varphi_1)$, and if $m > 1$ for each $i = 1 \ldots m-1$ $k'_{i+1} = k'_i + (x_{i+1}, \varphi_{i+1})$.

Proof.

If $t \in E(n, k) \cup E_b(n+1, k)$ then $t \in E(n+1, k)$, else $t \in E_e(n+1, k) \subseteq E(n+1, k)$.

Using lemma 4.10 we have that one of the following alternatives holds:

- $t \in E_a(n+1, k) \cup E_c(n+1, k) \cup E_d(n+1, k) \cup E_e(n+1, k)$;

- there exist p positive integer such that $2 \leqslant p < n+1$, $h \in K(p)$ such that $h \sqsubseteq k$, $t \in E_a(p,h) \cup E_c(p,h) \cup E_d(p,h) \cup E_e(p,h)$ and for each $\sigma \in \Xi(k)$ $\sigma_{/dom(h)} \in \Xi(h)$ and $\#(k,t,\sigma) = \#(h,t,\sigma_{/dom(h)})$.

If the first alternative holds, that is $t \in E_a(n+1,k) \cup E_c(n+1,k) \cup E_d(n+1,k) \cup E_e(n+1,k)$, then clearly $t \in E_e(n+1,k)$. This implies that $\#(k,t,\sigma) = \#(k,t,\sigma)_{(n+1,k,e)}$, so in this case our proof is finished.

Otherwise it must be $t \in E_e(p,h)$ and $h \in K(p-1)$. This implies that there exist:

- a positive integer q;
- a function y whose domain is $\{1,\dots,q\}$ such that for each $i = 1 \dots q$ $y_i \in \mathcal{V}{-}var(h)$, and for each $i,j = 1 \dots q$ $i \neq j \to y_i \neq y_j$;
- a function ψ whose domain is $\{1,\dots,q\}$ such that for each $i = 1 \dots q$ $\psi_i \in E(p-1)$;
- $\theta \in E(p-1)$;

such that

- $\mathcal{E}(p-1,h,q,y,\psi,\theta)$;
- $\{\}(y_1 : \psi_1,\dots,y_q : \psi_q,\theta) \notin E(p-1,h)$.
- $t = \{\}(y_1 : \psi_1,\dots,y_q : \psi_q,\theta)$.

Clearly given $\rho \in \Xi(h)$ we have

$$\#(h,t,\rho) = \{\#(h'_q,\theta,\rho'_q) |\ \rho'_q \in \Xi(h'_q), \rho \sqsubseteq \rho'_q\},$$

where $h'_1 = h + (y_1,\psi_1)$, and if $q > 1$ for each $i = 1 \dots q-1$ $h'_{i+1} = h'_i + (y_{i+1},\psi_{i+1})$.

Anyway $t = \{\}(x_1 : \varphi_1,\dots,x_m : \varphi_m,\phi)$ and we can apply the unique readability lemma 3.1.6, so $q = m, y = x, \psi = \varphi, \theta = \phi$. Then given $\rho \in \Xi(h)$ we have

$$\#(h,t,\rho) = \{\#(h'_m,\phi,\rho'_m) |\ \rho'_m \in \Xi(h'_m), \rho \sqsubseteq \rho'_m\},$$

where $h'_1 = h + (x_1,\varphi_1)$, and if $m > 1$ for each $i = 1 \dots m-1$ $h'_{i+1} = h'_i + (x_{i+1},\varphi_{i+1})$.

Now given $\sigma \in \Xi(k)$ we want to prove that

$$\#(k,t,\sigma) = \{\#(k'_m,\phi,\sigma'_m) |\ \sigma'_m \in \Xi(k'_m), \sigma \sqsubseteq \sigma'_m\}.$$

If we define $\rho = \sigma_{/dom(h)} \in \Xi(h)$ then

$$\#(k,t,\sigma) = \#(h,t,\rho) = \{\#(h'_m,\phi,\rho'_m) |\ \rho'_m \in \Xi(h'_m), \rho \sqsubseteq \rho'_m\}.$$

So in the end what we need to prove is that

$$\{\#(k'_m,\phi,\sigma'_m) |\ \sigma'_m \in \Xi(k'_m), \sigma \sqsubseteq \sigma'_m\} = \{\#(h'_m,\phi,\rho'_m) |\ \rho'_m \in \Xi(h'_m), \rho \sqsubseteq \rho'_m\}.$$

To prove this we just need to prove the following two assertions:

- for each $\sigma'_m \in \Xi(k'_m)$ such that $\sigma \sqsubseteq \sigma'_m$ there exists $\rho'_m \in \Xi(h'_m)$ such that $\rho \sqsubseteq \rho'_m$ and $\#(h'_m,\phi,\rho'_m) = \#(k'_m,\phi,\sigma'_m)$;
- for each $\rho'_m \in \Xi(h'_m)$ such that $\rho \sqsubseteq \rho'_m$ there exists $\sigma'_m \in \Xi(k'_m)$ such that $\sigma \sqsubseteq \sigma'_m$ and $\#(k'_m,\phi,\sigma'_m) = \#(h'_m,\phi,\rho'_m)$.

Here we want to apply lemma 4.14. This is possible since

- $h, k \in K(n)$, since $h \sqsubseteq k$ if $h = (v, \vartheta),\ k = (u, \eta)$ then by lemma 4.8 for each $i \in dom(k),\ j \in dom(h)\ u_i = v_j \to \eta_i = \vartheta_j$;
- $\rho \in \Xi(h),\ \sigma \in \Xi(k)$, since $\rho \sqsubseteq \sigma$ if $\rho = (v, \nu),\ \sigma = (u, \mu)$ then by lemma 4.9 for each $i \in dom(\sigma),\ j \in dom(\rho)\ u_i = v_j \to \mu_i = \nu_j$;
- x is a function whose domain is $\{1, \ldots, m\}$ such that for each $i = 1 \ldots m\ x_i \in \mathcal{V} - var(k),\ x_i \in \mathcal{V} - var(h)$ and for each $i, j = 1 \ldots m\ i \neq j \to x_i \neq x_j$;
- φ is a function whose domain is $\{1, \ldots, m\}$ such that for each $i = 1 \ldots m\ \varphi_i \in E(n)$;
- $\phi \in E(n)$;
- $\mathcal{E}(n, k, m, x, \varphi, \phi)$;
- $\mathcal{E}(n, h, m, x, \varphi, \phi)$;
- $t = \{\}(x_1 : \varphi_1, \ldots, x_m : \varphi_m, \phi)$.

Clearly $\mathcal{E}(n, h, m, x, \varphi, \phi)$ holds because of $\mathcal{E}(p-1, h, m, x, \varphi, \phi)$. Indeed $\mathcal{E}(p-1, h, m, x, \varphi, \phi)$ implies

- $\varphi_1 \in E_s(p-1, h) \subseteq E_s(n, h)$;
- if $m > 1$ then for each $i = 1 \ldots m-1\ h'_i \in K(p-1) \subseteq K(n) \wedge \varphi_{i+1} \in E_s(p-1, h'_i) \subseteq E_s(n, h'_i)$;
- $h'_m \in K(p-1) \subseteq K(n) \wedge \phi \in E(p-1, h'_m) \subseteq E(n, h'_m)$.

Both of our statements hold because, while lemma 4.14 proves the first one, it can also prove the second by simply switching variable names. ∎

Lemma 4.16. *Let $h \in K,\ \phi \in E_s(h),\ y \in (\mathcal{V} - var(h)),\ k = h + (y, \phi)$. We have $k \in K$, and if $\vartheta \in S(k)$ then*

- *$\{\}(y : \phi, \vartheta) \in E(h)$;*
- *$\forall(\{\}(y : \phi, \vartheta)) \in S(h),\ \exists(\{\}(y : \phi, \vartheta)) \in S(h)$;*
- *$\forall\ \rho \in \Xi(h)\ \#(h, \forall(\{\}(y : \phi, \vartheta)), \rho) = P_\forall(\{\#(k, \vartheta, \sigma) |\ \sigma \in \Xi(k),\ \rho \sqsubseteq \sigma\})$;*
- *$\forall\ \rho \in \Xi(h)\ \#(h, \exists(\{\}(y : \phi, \vartheta)), \rho) = P_\exists(\{\#(k, \vartheta, \sigma) |\ \sigma \in \Xi(k),\ \rho \sqsubseteq \sigma\})$.*

Proof.

Since $\phi \in E_s(h)$ there is a positive integer n such that $h \in K(n),\ \phi \in E_s(n, h)$. This implies that $k \in K(n)^+ \cup K(n) = K(n+1) \subseteq K$.

Let $\vartheta \in S(k)$. There is a positive integer m such that $\vartheta \in E(m, k)$. We define $p = \max\{n+1, m\}$, then we have

- $h \in K(p)$
- $y \in (\mathcal{V} - var(h))$
- $\phi \in E_s(p, h)$
- $k \in K(p),\ \vartheta \in E(p, k)$.

Here we can apply lemma 4.15, in fact in the statement of the lemma we can replace n with p, k with h, m with 1, x with $(1,y)$, φ with $(1,\phi)$, ϕ with ϑ. Every required condition is satisfied, including the condition $\mathcal{E}(p,h,1,(1,y),(1,\phi),\vartheta)$.

So by lemma 4.15 we have that $\{\}(y:\phi,\vartheta) \in E(p+1,h)$ and for each $\rho \in \Xi(h)$

$$\#(h,\{\}(y:\phi,\vartheta),\rho) = \{\#(k,\vartheta,\sigma)|\ \sigma \in \Xi(k),\ \rho \sqsubseteq \sigma\}\ .$$

We want to show that $\forall(\{\}(y:\phi,\vartheta)) \in E(p+2,h)$. To obtain this we can use lemma 4.13, so we just need to show that for each $\rho \in \Xi(h)$ $A_\forall(\#(h,\{\}(y:\phi,\vartheta),\rho))$ holds.
Now $A_\forall(\#(h,\{\}(y:\phi,\vartheta),\rho))$ is equal to

$\#(h,\{\}(y:\phi,\vartheta),\rho)$ is a set and for each $u \in \#(h,\{\}(y:\phi,\vartheta),\rho)$ u is true or u is false.

Clearly $\#(h,\{\}(y:\phi,\vartheta),\rho)$ is a set, furthermore for each $u \in \#(h,\{\}(y:\phi,\vartheta),\rho)$ there is $\sigma \in \Xi(k)$ such that $\rho \sqsubseteq \sigma$ and $u = \#(k,\vartheta,\sigma)$. Since $\vartheta \in S(k)$ u is true or u is false. So $A_\forall(\#(h,\{\}(y:\phi,\vartheta),\rho))$ holds.

We have proved that $\forall(\{\}(y:\phi,\vartheta)) \in E(p+2,h)$. Similarly we can show that $\exists(\{\}(y:\phi,\vartheta)) \in E(p+2,h)$. In fact to show this we just need to prove that for each $\rho \in \Xi(h)$ $A_\exists(\#(h,\{\}(y:\phi,\vartheta),\rho))$ holds, and this is proved since

$$A_\exists(\#(h,\{\}(y:\phi,\vartheta),\rho)) = A_\forall(\#(h,\{\}(y:\phi,\vartheta),\rho))\ .$$

Using lemma 4.13 we can also obtain that for each $\rho \in \Xi(h)$

$$\begin{aligned}\#(h,\forall(\{\}(y:\phi,\vartheta)),\rho) &= P_\forall(\#(h,\{\}(y:\phi,\vartheta),\rho)) = \\ &= P_\forall(\{\#(k,\vartheta,\sigma)|\ \sigma \in \Xi(k),\ \rho \sqsubseteq \sigma\})\ .\end{aligned}$$

$$\begin{aligned}\#(h,\exists(\{\}(y:\phi,\vartheta)),\rho) &= P_\exists(\#(h,\{\}(y:\phi,\vartheta),\rho)) = \\ &= P_\exists(\{\#(k,\vartheta,\sigma)|\ \sigma \in \Xi(k),\ \rho \sqsubseteq \sigma\})\ .\end{aligned}$$

Finally, as we have seen, for each $\rho \in \Xi(h)$

$$\#(h,\forall(\{\}(y:\phi,\vartheta)),\rho) = P_\forall(\{\#(k,\vartheta,\sigma)|\ \sigma \in \Xi(k),\ \rho \sqsubseteq \sigma\})\ ,$$

and $P_\forall(\{\#(k,\vartheta,\sigma)|\ \sigma \in \Xi(k),\ \rho \sqsubseteq \sigma\})$ is clearly true or false.

Hence $\forall(\{\}(y:\phi,\vartheta)) \in S(h)$. Similarly we obtain that $\exists(\{\}(y:\phi,\vartheta)) \in S(h)$. ∎

Definition 4.17. Let $x \in \mathcal{V},\ \varphi \in E$. We define

$$H[x:\varphi] = \varphi \in E_s(\epsilon)\ .$$

If the condition $H[x:\varphi]$ holds then we define $k[x:\varphi] = \epsilon + (x,\varphi)$. Clearly $k[x:\varphi] \in K$. In fact there exists n positive integer such that $\epsilon \in K(n) \wedge \varphi \in E_s(n,\epsilon)$, $x \in \mathcal{V} - var(\epsilon)$, so $k[x:\varphi] = \epsilon + (x,\varphi) \in K(n) \cup K(n)^+ = K(n+1) \subseteq K$.

Moreover $var(k[x:\varphi]) = \{x\}$.

Let m be a positive integer. Let $x_1, \ldots, x_{m+1} \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_{m+1} \in E$. We can assume to have defined $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ and if this holds to have defined also $k[x_1 : \varphi_1, \ldots, x_m : \varphi_m] \in K$, such that

$$var(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]) = \{x_1, \ldots, x_m\} \ .$$

We define

$$H[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}] = H[x_1 : \varphi_1, \ldots, x_m : \varphi_m] \\ \wedge \varphi_{m+1} \in E_s(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]) \ .$$

If $H[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]$ then we define

$$k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}] = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m] + (x_{m+1}, \varphi_{m+1}) \ .$$

Clearly $k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}] \in K$. In fact there exists a positive integer n such that $k[x_1 : \varphi_1, \ldots, x_m : \varphi_m] \in K(n)$ and $\varphi_{m+1} \in E_s(n, k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$, $x_{m+1} \in \mathcal{V} - var(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$, so $k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}] \in K(n) \cup K(n)^+ = K(n+1)$.

Moreover

$$var(k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]) = \{x_1, \ldots, x_{m+1}\} \ .$$

Lemma 4.18. *Let m positive integer, $x_1, \ldots, x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$, $\varphi_1, \ldots, \varphi_m \in E$. Then $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ is defined and if $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ holds then $k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ is also defined and belongs to K. Moreover*

$$var(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]) = \{x_1, \ldots, x_m\} \ .$$

Proof. This is an obvious consequence of the previous definition and can be trivially verified by induction on m. ∎

Remark 4.19. Let m be a positive integer. Let $x_1, \ldots, x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_m \in E$ and assume $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$. In these assumptions we can easily see that for each $i = 1 \ldots m$ $H[x_1 : \varphi_1, \ldots, x_i : \varphi_i]$ holds and so $k[x_1 : \varphi_1, \ldots, x_i : \varphi_i]$ is defined, $k[x_1 : \varphi_1, \ldots, x_i : \varphi_i] \in K$, $var(k[x_1 : \varphi_1, \ldots, x_i : \varphi_i]) = \{x_1, \ldots, x_i\}$.

In fact this is clearly true for $i = m$. Given $i = 2 \ldots m$, if we suppose this is true for i, then we have $H[x_1 : \varphi_1, \ldots, x_{i-1} : \varphi_{i-1}]$, and so the remaining facts also hold.

In these assumptions we can define $k_0 = \epsilon$ and for each $i = 1 \ldots m$ $k_i = k[x_1 : \varphi_1, \ldots, x_i : \varphi_i]$. We have $k_0 \in K$, $var(k_0) = \emptyset$, for each $i = 1 \ldots m$ $k_i \in K$, $var(k_i) = \{x_1, \ldots, x_i\}$. Hereafter we'll often use this kind of simplified notation.

We can also easily see that for each $i = 1 \ldots m$ $\varphi_i \in E_s(k_{i-1})$ and $k_i = k_{i-1} + (x_i, \varphi_i)$, and $dom(k_i) = \{1, \ldots, i\}$.

Definition 4.20. Let m be a positive integer. Let $x_1, \ldots, x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_m \in E$ and assume $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$. Let φ be a member of $S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$. Define

$$\gamma[x_m : \varphi_m, \varphi] = \forall(\{\}(x_m : \varphi_m, \varphi)) \ .$$

By lemma 4.16 we have $\gamma[x_m : \varphi_m, \varphi] \in S(k_{m-1})$.

If $m > 1$ for each $i = 2 \ldots m$ suppose we have defined $\gamma[x_i : \varphi_i, \ldots, x_m : \varphi_m, \varphi]$ as a member of $S(k_{i-1})$ and define

$$\gamma[x_{i-1} : \varphi_{i-1}, \ldots, x_m : \varphi_m, \varphi] = \forall(\{\}(x_{i-1} : \varphi_{i-1}, \gamma[x_i : \varphi_i, \ldots, x_m : \varphi_m, \varphi])) \ .$$

Since $k_{i-1} = k_{i-2} + (x_{i-1}, \varphi_{i-1})$ we can apply lemma 4.16 and obtain that $\gamma[x_{i-1} : \varphi_{i-1}, \ldots, x_m : \varphi_m, \varphi] \in S(k_{i-2})$.

Lemma 4.21. *Let m be a positive integer. Let $x_1, \ldots, x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_m \in E$ and assume $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$. Let $\varphi \in S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$. For each $i = 1 \ldots m$ we have defined $\gamma[x_i : \varphi_i, \ldots, x_m : \varphi_m, \varphi]$ as a member of $S(k_{i-1})$.*

Let $m > 1$, $j = 2 \ldots m$. We have $\gamma[x_j : \varphi_j, \ldots, x_m : \varphi_m, \varphi] \in S(k_{j-1})$. We can show that for each $i = 1 \ldots j-1$

$$\gamma[x_i : \varphi_i, \ldots, x_m : \varphi_m, \varphi] = \gamma[x_i : \varphi_i, \ldots, x_{j-1} : \varphi_{j-1}, \gamma[x_j : \varphi_j, \ldots, x_m : \varphi_m, \varphi]] \ .$$

Proof.

We show this by induction on i. First we prove the property for $i = j-1$.

$$\begin{aligned}\gamma[x_{j-1} : \varphi_{j-1}, \ldots, x_m : \varphi_m, \varphi] &= \forall(\{\}(x_{j-1} : \varphi_{j-1}, \gamma[x_j : \varphi_j, \ldots, x_m : \varphi_m, \varphi])) = \\ &= \gamma[x_{j-1} : \varphi_{j-1}, \gamma[x_j : \varphi_j, \ldots, x_m : \varphi_m, \varphi]] \ .\end{aligned}$$

Now we assume $j - 1 \geqslant 2$ and $2 \leqslant i \leqslant j-1$. We assume the property is true for i and want to show it holds also for $i-1$. We have

$$\begin{aligned}&\gamma[x_{i-1} : \varphi_{i-1}, \ldots, x_m : \varphi_m, \varphi] = \forall(\{\}(x_{i-1} : \varphi_{i-1}, \gamma[x_i : \varphi_i, \ldots, x_m : \varphi_m, \varphi])) = \\ &= \forall(\{\}(x_{i-1} : \varphi_{i-1}, \gamma[x_i : \varphi_i, \ldots, x_{j-1} : \varphi_{j-1}, \gamma[x_j : \varphi_j, \ldots, x_m : \varphi_m, \varphi]])) = \\ &= \gamma[x_{i-1} : \varphi_{i-1}, \ldots, x_{j-1} : \varphi_{j-1}, \gamma[x_j : \varphi_j, \ldots, x_m : \varphi_m, \varphi]] \ .\end{aligned}$$

■

Lemma 4.22. *Let X be a set, let f, g be functions whose domain is X. Then let $B = \{f(x) |\ x \in X\}$ and $C = \{g(x) |\ x \in X\}$. Suppose for each $x \in X$*

- *$f(x)$ is true or $f(x)$ is false,*
- *$g(x)$ is true or $g(x)$ is false,*
- *$f(x) \leftrightarrow g(x)$.*

Then the following hold

- *$A_\forall(B)$,*
- *$A_\forall(C)$,*

- $P_\forall(B) \leftrightarrow P_\forall(C)$.

Proof.

Clearly B is a set and for each $b \in B$ there exists $x \in X$ such that $b = f(x)$, so b is true or false. So $A_\forall(B)$ holds and similarly $A_\forall(C)$ holds.

Moreover, if $P_\forall(B)$ holds this means that for each $b \in B$ b is true. Let $c \in C$, there exists $x \in X$ such that $c = g(x)$, we have $f(x) \in B$, so $f(x)$ is true and so $g(x)$ is also true. So $P_\forall(C)$ holds. Conversely with the same reasoning we can prove that if $P_\forall(C)$ holds then $P_\forall(B)$ also holds. ∎

THEOREM 4.23. *Let m be a positive integer. Let $x_1, \ldots, x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_m \in E$ and assume $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$. Let $\varphi \in S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$. Then*

$$\begin{aligned}&\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \varphi]) \leftrightarrow \\ &\qquad \leftrightarrow P_\forall(\{\#(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m], \varphi, \sigma)|\ \sigma \in \Xi(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])\})\end{aligned}$$

Proof.

We'll use the symbols $k_0, \ldots, k_m$ with the meaning specified in remark 4.19, so what we need to show is:

$$\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \varphi]) \leftrightarrow P_\forall(\{\#(k_m, \varphi, \sigma)|\ \sigma \in \Xi(k_m)\})\ .$$

To this end we need to show that for each $i = m \ldots 1$ and for each $\rho \in \Xi(k_{i-1})$

$$\#(k_{i-1}, \gamma[x_i : \varphi_i, \ldots, x_m : \varphi_m, \varphi], \rho) \leftrightarrow P_\forall(\{\#(k_m, \varphi, \sigma)|\ \sigma \in \Xi(k_m),\ \rho \sqsubseteq \sigma\})\ .$$

We prove this by induction on i, starting with the case where $i = m$. Here we need to show that for each $\rho \in \Xi(k_{m-1})$

$$\#(k_{m-1}, \gamma[x_m : \varphi_m, \varphi], \rho) \leftrightarrow P_\forall(\{\#(k_m, \varphi, \sigma)|\ \sigma \in \Xi(k_m),\ \rho \sqsubseteq \sigma\})\ .$$

Actually

$$\begin{aligned}\#(k_{m-1}, \gamma[x_m : \varphi_m, \varphi], \rho) &= \#(k_{m-1}, \forall(\{\}(x_m : \varphi_m, \varphi)), \rho) = \\ &= P_\forall(\{\#(k_m, \varphi, \sigma)|\ \sigma \in \Xi(k_m),\ \rho \sqsubseteq \sigma\})\ .\end{aligned}$$

Now suppose $m > 1$, let $i = 2 \ldots m$ and suppose the property holds for i, we show it also holds for $i - 1$. We need to prove that for each $\rho \in \Xi(k_{i-2})$

$$\#(k_{i-2}, \gamma[x_{i-1} : \varphi_{i-1}, \ldots, x_m : \varphi_m, \varphi], \rho) \leftrightarrow P_\forall(\{\#(k_m, \varphi, \sigma)|\ \sigma \in \Xi(k_m),\ \rho \sqsubseteq \sigma\})\ .$$

Using lemma 4.22 we have

$$\begin{aligned}&\#(k_{i-2}, \gamma[x_{i-1} : \varphi_{i-1}, \ldots, x_m : \varphi_m, \varphi], \rho) = \\ &\qquad = \#(k_{i-2}, \forall(\{\}(x_{i-1} : \varphi_{i-1}, \gamma[x_i : \varphi_i, \ldots, x_m : \varphi_m, \varphi])), \rho) = \\ &\qquad = P_\forall(\{\#(k_{i-1}, \gamma[x_i : \varphi_i, \ldots, x_m : \varphi_m, \varphi], \delta)|\ \delta \in \Xi(k_{i-1}),\ \rho \sqsubseteq \delta\}) \leftrightarrow \\ &\qquad \leftrightarrow P_\forall(\{P_\forall(\{\#(k_m, \varphi, \sigma)|\ \sigma \in \Xi(k_m),\ \delta \sqsubseteq \sigma\})|\ \delta \in \Xi(k_{i-1}),\ \rho \sqsubseteq \delta\})\ .\end{aligned}$$

So it comes to showing that

$$P_\forall(\{P_\forall(\{\#(k_m,\varphi,\sigma)|\ \sigma\in\Xi(k_m),\ \delta\sqsubseteq\sigma\})|\ \delta\in\Xi(k_{i-1}),\ \rho\sqsubseteq\delta\})\leftrightarrow$$
$$\leftrightarrow P_\forall(\{\#(k_m,\varphi,\sigma)|\ \sigma\in\Xi(k_m),\ \rho\sqsubseteq\sigma\})\ .$$

Suppose $P_\forall(\{P_\forall(\{\#(k_m,\varphi,\sigma)|\ \sigma\in\Xi(k_m),\ \delta\sqsubseteq\sigma\})|\ \delta\in\Xi(k_{i-1}),\ \rho\sqsubseteq\delta\})$.

This means that for each $\delta\in\Xi(k_{i-1})$ such that $\rho\sqsubseteq\delta$ and for each $\sigma\in\Xi(k_m):\ \delta\sqsubseteq\sigma$ $\#(k_m,\varphi,\sigma)$ holds.

Let $\sigma\in\Xi(k_m):\rho\sqsubseteq\sigma$, we need to prove $\#(k_m,\varphi,\sigma)$.

We define $\delta=\sigma_{/dom(k_{i-1})}$. By lemma 4.4 $\delta\in\Xi(k_{i-1})$. Moreover $\delta,\rho\in\mathcal{R}(\sigma)$ and $dom(\rho)=dom(k_{i-2})\subseteq dom(k_{i-1})=dom(\delta)$. By lemma 3.9 we obtain $\rho\sqsubseteq\delta$. Therefore $\#(k_m,\varphi,\sigma)$ holds.

Conversely suppose $P_\forall(\{\#(k_m,\varphi,\sigma)|\ \sigma\in\Xi(k_m),\ \rho\sqsubseteq\sigma\})$, so that for each $\sigma\in\Xi(k_m):\rho\sqsubseteq\sigma$ $\#(k_m,\varphi,\sigma)$ is true. Let $\delta\in\Xi(k_{i-1})$ be such that $\rho\sqsubseteq\delta$ and let $\sigma\in\Xi(k_m)$ be such that $\delta\sqsubseteq\sigma$. Since $\sigma\in\Xi(k_m)$ and $\rho\sqsubseteq\sigma$ we have $\#(k_m,\varphi,\sigma)$.

This completes the proof that for each $\rho\in\Xi(k_{i-2})$

$$\#(k_{i-2},\gamma[x_{i-1}:\varphi_{i-1},\ldots,x_m:\varphi_m,\varphi],\rho)\leftrightarrow P_\forall(\{\#(k_m,\varphi,\sigma)|\ \sigma\in\Xi(k_m),\ \rho\sqsubseteq\sigma\})\ .$$

We have also finished the proof that for each $i=m\ldots1$ and for each $\rho\in\Xi(k_{i-1})$

$$\#(k_{i-1},\gamma[x_i:\varphi_i,\ldots,x_m:\varphi_m,\varphi],\rho)\leftrightarrow P_\forall(\{\#(k_m,\varphi,\sigma)|\ \sigma\in\Xi(k_m),\ \rho\sqsubseteq\sigma\})\ .$$

It follows that for each $\rho\in\Xi(k_0)$

$$\#(k_0,\gamma[x_i:\varphi_i,\ldots,x_m:\varphi_m,\varphi],\rho)\leftrightarrow P_\forall(\{\#(k_m,\varphi,\sigma)|\ \sigma\in\Xi(k_m),\ \rho\sqsubseteq\sigma\})\ .$$

and clearly this can be rewritten

$$\#(\epsilon,\gamma[x_i:\varphi_i,\ldots,x_m:\varphi_m,\varphi],\epsilon)\leftrightarrow P_\forall(\{\#(k_m,\varphi,\sigma)|\ \sigma\in\Xi(k_m),\ \epsilon\sqsubseteq\sigma\})\ ,$$
$$\#(\gamma[x_i:\varphi_i,\ldots,x_m:\varphi_m,\varphi])\leftrightarrow P_\forall(\{\#(k_m,\varphi,\sigma)|\ \sigma\in\Xi(k_m)\})\ .$$

■

We terminate the chapter with other useful lemmas.

Lemma 4.24. *Let $c\in\mathcal{C}$. For each positive integer n and $k\in K(n)$*

- *$c\in E(n+1,k)$;*
- *for each $\sigma\in\Xi(k)$ $\#(k,c,\sigma)=\#(c)$.*

Proof.

The proof is by induction on n.

For $n=1$ we have $k=\epsilon$ so $c\in E(1,\epsilon)=E(n,k)\subseteq E(n+1,k)$ and for each $\sigma\in\Xi(k)$ $\sigma=\epsilon$, so $\#(k,c,\sigma)=\#(\epsilon,c,\epsilon)=\#(c)$.

Let n be a positive integer and $k\in K(n+1)=K(n)\cup K(n)^+$.

If $k\in K(n)$ then

- $c \in E(n+1,k) \subseteq E(n+2,k)$;
- for each $\sigma \in \Xi(k)$ $\#(k,c,\sigma) = \#(c)$.

Otherwise $k \in K(n)^+$, so there exist $h \in K(n), \phi \in E_s(n,h), y \in (\mathcal{V} - var(h))$ such that $k = h + (y,\phi)$. By the inductive hypothesis

- $c \in E(n+1,h)$;
- for each $\rho \in \Xi(h)$ $\#(h,c,\rho) = \#(c)$.

We have $c \notin E(n+1,k)$, $k = h+(y,\phi) \in K(n+1)-\{\epsilon\}$, so $c \in E_b(n+2,k) \subseteq E(n+2,k)$ and for each $\sigma = \rho + (y,s) \in \Xi(k)$

$$\#(k,c,\sigma) = \#(k,c,\sigma)_{(n+2,k,b)} = \#(h,c,\rho) = \#(c) \ .$$

■

LEMMA 4.25. *Let m be a positive integer, $x_1,\ldots,x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1,\ldots,\varphi_m \in E$, assume $H[x_1 : \varphi_1,\ldots,x_m : \varphi_m]$, define $k = k[x_1 : \varphi_1,\ldots,x_m : \varphi_m]$ and as usual $k_0 = \epsilon$ and for each $i = 1\ldots m$ $k_i = k[x_1 : \varphi_1,\ldots,x_i : \varphi_i]$.*
Then for each $i = 1\ldots m$, $j = i\ldots m$ $x_i \in E(k_j)$.

Proof.

We begin by proving that $x_i \in E(k_i)$. Since $k_i \in K$ there exists a positive integer n such that $k_i \in K(n)$, and since $k_i \neq \epsilon$ we have $n \geqslant 2$. By lemma 4.2 there exists a positive integer $q < n$ such that $k_i \in K(q)^+$. So there exist $h \in K(q), \phi \in E_s(q,h), y \in (\mathcal{V}-var(h))$ such that $k_i = h + (y,\phi)$. We have also $k_i = k_{i-1} + (x_i,\varphi_i)$ so

$$x_i = y \in E_a(q+1,k_i) \subseteq E(q+1,k_i) \subseteq E(k_i) \ .$$

Now let $j = i\ldots m$, $x_i \in E(k_j)$, we want to show that $x_i \in E(k_{j+1})$. Since $k_{j+1} \in K$ there exists a positive integer n such that $k_{j+1} \in K(n)$. There exists a positive integer q such that $x_i \in E(q,k_j)$. Let $p = max\{q,n\}$, then $k_{j+1} \in K(p)$ and $x_i \in E(p,k_j)$.

We can also observe that $k_{j+1} = k_j + (x_{j+1},\varphi_{j+1}) \in K(p) - \{\epsilon\}$, so

$$E_b(p+1,k_{j+1}) = \{t|\ t \in E(p,k_j),\ t \notin E(p,k_{j+1})\}.$$

Clearly if $x_i \in E(p,k_{j+1})$ then $x_i \in E(k_{j+1})$, otherwise $x_i \in E(p,k_j)$ and $x_i \notin E(p,k_{j+1})$, so $x_i \in E_b(p+1,k_{j+1}) \subseteq E(k_{j+1})$. ■

LEMMA 4.26. *Let $k \in K$, $f \in \mathcal{F}$, m a positive integer, $\varphi_1,\ldots,\varphi_m \in E(k)$. Suppose for each $\sigma \in \Xi(k)$ $A_f(\#(k,\varphi_1,\sigma),\ldots,\#(k,\varphi_m,\sigma))$ is true. Then*

- *$f(\varphi_1,\ldots,\varphi_m) \in E(k)$;*
- *for each $\sigma \in \Xi(k)$ $\#(k,f(\varphi_1,\ldots,\varphi_m),\sigma) = P_f(\#(k,\varphi_1,\sigma),\ldots,\#(k,\varphi_m,\sigma))$;*

Proof.

There exists a positive integer n such that $k \in K(n)$ and $\varphi_1, \ldots, \varphi_m \in E(n,k)$. By lemma 4.13 this implies that $f(\varphi_1, \ldots, \varphi_m) \in E(n+1,k)$ and for each $\sigma \in \Xi(k)$

$$\#(k, f(\varphi_1, \ldots, \varphi_m), \sigma) = P_f(\#(k, \varphi_1, \sigma), \ldots, \#(k, \varphi_m, \sigma)) \ .$$

■

LEMMA 4.27. *Suppose the membership predicate symbol $\in$ we defined at the beginning of chapter 3 belongs to $\mathcal{F}$. Suppose $k \in K$, $t, \varphi \in E(k)$ and for each $\sigma \in \Xi(k)$ $\#(k, \varphi, \sigma)$ is a set. Then*

- $\in (t, \varphi) \ \in S(k)$;
- *for each $\sigma \in \Xi(k)$ $\#(k, \in (t, \varphi), \sigma) = P_\in(\#(k, t, \sigma), \#(k, \varphi, \sigma))$.*

Proof.

For each $\sigma \in \Xi(k)$ $\#(k, \varphi, \sigma)$ is a set, so $A_\in(\#(k, t, \sigma), \#(k, \varphi, \sigma))$ holds. Therefore, by lemma 4.26, $\in (t, \varphi) \ \in E(k)$.

Using lemma 4.26 we also obtain that for each $\sigma \in \Xi(k)$

$\#(k, \in (t, \varphi), \sigma) = P_\in(\#(k, t, \sigma), \#(k, \varphi, \sigma)) = \#(k, t, \sigma)$ belongs to $\#(k, \varphi, \sigma)$.

So $\#(k, \in (t, \varphi), \sigma)$ is true or false and $\in (t, \varphi) \ \in S(k)$. ■

We now need to prove lemma 4.28, which is in some way similar to 4.16 but involves the other logical connectives.

LEMMA 4.28. *Let $h \in K$, $\varphi_1, \varphi_2 \in S(h)$. Then*

- $\wedge(\varphi_1, \varphi_2), \vee(\varphi_1, \varphi_2), \rightarrow (\varphi_1, \varphi_2), \leftrightarrow (\varphi_1, \varphi_2), \neg(\varphi_1) \in S(h)$;
- *for each $\rho \in \Xi(h)$ $\#(h, \wedge(\varphi_1, \varphi_2), \rho) = P_\wedge(\#(h, \varphi_1, \rho), \#(h, \varphi_2, \rho))$;*
- *for each $\rho \in \Xi(h)$ $\#(h, \vee(\varphi_1, \varphi_2), \rho) = P_\vee(\#(h, \varphi_1, \rho), \#(h, \varphi_2, \rho))$;*
- *for each $\rho \in \Xi(h)$ $\#(h, \rightarrow (\varphi_1, \varphi_2), \rho) = P_\rightarrow(\#(h, \varphi_1, \rho), \#(h, \varphi_2, \rho))$;*
- *for each $\rho \in \Xi(h)$ $\#(h, \leftrightarrow (\varphi_1, \varphi_2), \rho) = P_\leftrightarrow(\#(h, \varphi_1, \rho), \#(h, \varphi_2, \rho))$;*
- *for each $\rho \in \Xi(h)$ $\#(h, \neg(\varphi_1), \rho) = P_\neg(\#(h, \varphi_1, \rho))$.*

Proof.

For each $\rho \in \Xi(h)$ $\#(h, \varphi_1, \rho)$ is true or $\#(h, \varphi_1, \rho)$ is false; $\#(h, \varphi_2, \rho)$ is true or $\#(h, \varphi_2, \rho)$ is false.

We recall that for each $\rho \in \Xi(h)$ $A_\wedge(\#(h, \varphi_1, \rho), \#(h, \varphi_2, \rho))$, $A_\vee(\#(h, \varphi_1, \rho), \#(h, \varphi_2, \rho))$, $A_\rightarrow(\#(h, \varphi_1, \rho), \#(h, \varphi_2, \rho))$, $A_\leftrightarrow(\#(h, \varphi_1, \rho), \#(h, \varphi_2, \rho))$ are all defined as
($\#(h, \varphi_1, \rho)$ is true or $\#(h, \varphi_1, \rho)$ is false) and ($\#(h, \varphi_2, \rho)$ is true or $\#(h, \varphi_2, \rho)$ is false).

Therefore $A_\wedge(\#(h, \varphi_1, \rho), \#(h, \varphi_2, \rho))$, $A_\vee(\#(h, \varphi_1, \rho), \#(h, \varphi_2, \rho))$, $A_\rightarrow(\#(h, \varphi_1, \rho), \#(h, \varphi_2, \rho))$, $A_\leftrightarrow(\#(h, \varphi_1, \rho), \#(h, \varphi_2, \rho))$ are all true.

And for each $\rho \in \Xi(h)$ $A_\neg(\#(h,\varphi_1,\rho))$ is true.

Then by lemma 4.26

$$\wedge(\varphi_1,\varphi_2), \vee(\varphi_1,\varphi_2), \rightarrow(\varphi_1,\varphi_2), \leftrightarrow(\varphi_1,\varphi_2), \neg(\varphi_1) \in E(h)\ .$$

Moreover for each $\rho \in \Xi(h)$

$$\begin{aligned}
\#(h,\wedge(\varphi_1,\varphi_2),\rho) &= P_\wedge(\#(h,\varphi_1,\rho),\#(h,\varphi_2,\rho));\\
\#(h,\vee(\varphi_1,\varphi_2),\rho) &= P_\vee(\#(h,\varphi_1,\rho),\#(h,\varphi_2,\rho));\\
\#(h,\rightarrow(\varphi_1,\varphi_2),\rho) &= P_\rightarrow(\#(h,\varphi_1,\rho),\#(h,\varphi_2,\rho));\\
\#(h,\leftrightarrow(\varphi_1,\varphi_2),\rho) &= P_\leftrightarrow(\#(h,\varphi_1,\rho),\#(h,\varphi_2,\rho));\\
\#(h,\neg(\varphi_1),\rho) &= P_\neg(\#(h,\varphi_1,\rho))\ .
\end{aligned}$$

so

$$\begin{aligned}
&\#(h,\wedge(\varphi_1,\varphi_2),\rho) \text{ is true or false;}\\
&\#(h,\vee(\varphi_1,\varphi_2),\rho) \text{ is true or false;}\\
&\#(h,\rightarrow(\varphi_1,\varphi_2),\rho) \text{ is true or false;}\\
&\#(h,\leftrightarrow(\varphi_1,\varphi_2),\rho) \text{ is true or false;}\\
&\#(h,\neg(\varphi_1),\rho) \text{ is true or false .}
\end{aligned}$$

Therefore we get

$$\wedge(\varphi_1,\varphi_2), \vee(\varphi_1,\varphi_2), \rightarrow(\varphi_1,\varphi_2), \leftrightarrow(\varphi_1,\varphi_2), \neg(\varphi_1) \in S(h)\ .$$

■

Definition 4.29. For each $k \in K$ we define

$$n_k = min\{n|\ n \text{ positive integer},\ k \in K(n)\}\ .$$

Let m be a positive integer, $x_1,\dots,x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1,\dots,\varphi_m \in E$, assume $H[x_1:\varphi_1,\dots,x_m:\varphi_m]$, define $k = k[x_1:\varphi_1,\dots,x_m:\varphi_m]$ and as usual $k_0 = \epsilon$ and for each $i = 1\dots m$ $k_i = k[x_1:\varphi_1,\dots,x_i:\varphi_i]$. For each $i = 1\dots m$ we also define $n_i = n_{k_i}$.

Lemma 4.30. *Let $k \in K$ and let m positive integer such that $k \in K(m)^+$. Then*

$$n_k = m+1\ .$$

Proof.

We have $k \notin K(m)$ so $n_k \geqslant m+1$ and $k \in K(m+1)$ so $n_k \leqslant m+1$. ■

Lemma 4.31. *Let $m \geqslant 2$, $x_1,\dots,x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1,\dots,\varphi_m \in E$, assume $H[x_1:\varphi_1,\dots,x_m:\varphi_m]$, define $k = k[x_1:\varphi_1,\dots,x_m:\varphi_m]$ and as usual $k_0 = \epsilon$ and for each $i = 1\dots m$ $k_i = k[x_1:\varphi_1,\dots,x_i:\varphi_i]$. Then for each $i = 1\dots m-1$ $n_i < n_{i+1}$.*

Proof.

There exists a positive integer q such that $k_{i+1} \in K(q)^+$ and $n_{i+1} = q + 1$.

There also exist $h \in K(q), \phi \in E_s(q,h), y \in (\mathcal{V} - var(h))$ such that $k_{i+1} = h + (y, \phi)$. And we have also $k_{i+1} = k_i + (x_{i+1}, \varphi_{i+1})$, so $k_i = h \in K(q)$ and therefore $n_i \leqslant q < q + 1 = n_{i+1}$. ∎

LEMMA 4.32. *Let $m \geqslant 2$, $x_1, \ldots, x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_m \in E$, assume $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$, define $k = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ and as usual $k_0 = \epsilon$ and for each $i = 1 \ldots m$ $k_i = k[x_1 : \varphi_1, \ldots, x_i : \varphi_i]$. Then given $i = 1 \ldots m$ and $\varphi \in E(k_i)$ for each $j = i \ldots m$ $\varphi \in E(k_j)$ and for each $\sigma \in \Xi(k_j)$ $\#(k_j, \varphi, \sigma) = \#(k_i, \varphi, \sigma_{/dom(k_i)})$.*

Proof.

We prove this by induction on $j = i \ldots m$. The initial step is trivially verified.

If $i < m$ we need an inductive step. Let $j = i \ldots m - 1$, assume $\varphi \in E(k_j)$ and for each $\rho \in \Xi(k_j)$ $\#(k_j, \varphi, \rho) = \#(k_i, \varphi, \rho_{/dom(k_i)})$. We want to prove that $\varphi \in E(k_{j+1})$ and for each $\sigma \in \Xi(k_{j+1})$ $\#(k_{j+1}, \varphi, \sigma) = \#(k_i, \varphi, \sigma_{/dom(k_i)})$.

Since $\varphi \in E(k_j)$ there exists a positive integer p such that $k_j \in K(p)$ and $\varphi \in E(p, k_j)$. If $\varphi \in E(p, k_{j+1})$ then clearly $\varphi \in E(k_{j+1})$. Otherwise, since $k_{j+1} = k_j + (x_{j+1}, \varphi_{j+1})$, $\varphi \in E_b(p+1, k_{j+1}) \subseteq E(p+1, k_{j+1}) \subseteq E(k_{j+1})$.

At this point we observe that $k_i \sqsubseteq k_{j+1}$, $\varphi \in E(k_i) \cap E(k_{j+1})$, $\sigma \in \Xi(k_{j+1})$, $\sigma_{/dom(k_i)} \in \Xi(k_i)$, $\sigma_{/dom(k_i)} \sqsubseteq \sigma$. Here we can apply lemma 4.12 and obtain that $\#(k_{j+1}, \varphi, \sigma) = \#(k_i, \varphi, \sigma_{/dom(k_i)})$. ∎

5. Building a deductive system

In this chapter we will build a deductive sytem $\mathcal{D} = (\mathcal{A}, \mathcal{R})$, in order to be able to show an example of proof in the next chapter. The deductive system we are building can refer to any language $\mathcal{L} = (\mathcal{V}, \mathcal{F}, \mathcal{C}, \#)$ such that all of these symbols: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \forall, \exists, \in, =$ are in our set $\mathcal{F}$. For each of these operators f $A_f(x_1, \ldots, x_n)$ and $P_f(x_1, \ldots, x_n)$ are defined as specified at the beginning of chapter 3.

We'll now list the set of axioms and rules of our deductive system. For every axiom/rule we first prove a result which ensures the soundness of the axiom/rule and then define properly the axiom/rule itself.

In our proofs we'll frequently use the following simple result.

Lemma 5.1. *Let S be a set and q, r be functions over S such that for each $\sigma \in S$ $q(\sigma)$ and $r(\sigma)$ are true or false (in these assumptions q, r can be called 'predicates over S'). Then*

$$A_\forall(\{q(\sigma)|\ \sigma \in S\}),\ A_\exists(\{q(\sigma)|\ \sigma \in S\})$$

$$P_\forall(\{q(\sigma)|\ \sigma \in S\}) \leftrightarrow \ \textit{for each } \sigma \in S\ q(\sigma),$$

$$P_\exists(\{q(\sigma)|\ \sigma \in S\}) \leftrightarrow \ \textit{there exists } \sigma \in S:\ q(\sigma),$$

$$A_\forall(\{q(\sigma)|\ \sigma \in S, r(\sigma)\}),\ A_\exists(\{q(\sigma)|\ \sigma \in S, r(\sigma)\})$$

$$P_\forall(\{q(\sigma)|\ \sigma \in S, r(\sigma)\}) \leftrightarrow \ \textit{for each } \sigma \in S \textit{ if } r(\sigma) \textit{ then } q(\sigma),$$

$$P_\exists(\{q(\sigma)|\ \sigma \in S, r(\sigma)\}) \leftrightarrow \ \textit{there exists } \sigma \in S : r(\sigma) \textit{ and } q(\sigma).$$

Proof.

Let $x_1 = \{q(\sigma)|\ \sigma \in S\}$.

Clearly x_1 is a set and for each $x \in x_1$ there exists $\sigma \in S$ such that $x = q(\sigma)$, so x is true or false. So $A_\forall(x_1)$ and $A_\exists(x_1)$ both hold.

We suppose $P_\forall(x_1)$ and try to prove for each $\sigma \in S$ $q(\sigma)$.
Let $\sigma \in S$, clearly $q(\sigma) \in x_1$, so $q(\sigma)$ is true.

Conversely we suppose for each $\sigma \in S$ $q(\sigma)$ and try to prove $P_\forall(x_1)$.
Let $x \in x_1$, there exists $\sigma \in S$ such that $x = q(\sigma)$ is true.

We suppose $P_\exists(x_1)$ and try to prove there exists $\sigma \in S$ $q(\sigma)$.
There exists x in x_1 such that (x is true). There exists $\sigma \in S$ such that $x = q(\sigma)$, therefore $q(\sigma)$ is true.

Conversely we suppose there exists $\sigma \in S$ $q(\sigma)$ and try to prove $P_\exists(x_1)$.
Clearly $q(\sigma) \in x_1$ and $q(\sigma)$ is true, so $P_\exists(x_1)$ is proved.

Now, to prove the remaining results, let $x_1 = \{q(\sigma)|\ \sigma \in S,\ r(\sigma)\}$.

Clearly x_1 is a set and for each $x \in x_1$ there exists $\sigma \in S$ such that ($r(\sigma)$ and) $x = q(\sigma)$, so x is true or false. So $A_\forall(x_1)$ and $A_\exists(x_1)$ both hold.

We suppose $P_\forall(x_1)$ and try to prove for each $\sigma \in S$ if $r(\sigma)$ then $q(\sigma)$.
Let $\sigma \in S$ and assume $r(\sigma)$, clearly $q(\sigma) \in x_1$, so $q(\sigma)$ is true.

Conversely we suppose for each $\sigma \in S$ if $r(\sigma)$ then $q(\sigma)$ and try to prove $P_\forall(x_1)$.
Let $x \in x_1$, there exists $\sigma \in S$ such that $r(\sigma)$ and $x = q(\sigma)$ is true.

We suppose $P_\exists(x_1)$ and try to prove there exists $\sigma \in S : r(\sigma)$ and $q(\sigma)$.
There exists x in x_1 such that x is true. So there exists $\sigma \in S$ such that $r(\sigma)$ and $x = q(\sigma)$, therefore $q(\sigma)$ is true.

Conversely we suppose there exists $\sigma \in S : r(\sigma)$ and $q(\sigma)$ and try to prove $P_\exists(x_1)$.
Clearly $q(\sigma) \in x_1$ and $q(\sigma)$ is true, so $P_\exists(x_1)$ is proved. ∎

LEMMA 5.2. *Let m be a positive integer. Let $x_1, \ldots, x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_m \in E$ and assume $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$. Define $k = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ and let $\varphi, \psi \in S(k)$.*

Under these assumptions we have

- $\wedge(\varphi, \psi), \rightarrow(\wedge(\varphi, \psi), \varphi), \rightarrow(\wedge(\varphi, \psi), \psi) \in S(k)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow(\wedge(\varphi, \psi), \varphi)] \in S(\epsilon)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow(\wedge(\varphi, \psi), \psi)] \in S(\epsilon)$.

Moreover $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow(\wedge(\varphi, \psi), \varphi)])$ and $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow(\wedge(\varphi, \psi), \psi)])$ are both true.

Proof.

Using theorem 4.23 and lemma 4.28 we can rewrite $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow(\wedge(\varphi, \psi), \varphi)])$ as follows:

$$P_\forall(\{\#(k, \rightarrow(\wedge(\varphi, \psi), \varphi), \sigma) | \sigma \in \Xi(k)\})$$

$$P_\forall(\{P_\rightarrow(\#(k, \wedge(\varphi, \psi), \sigma), \#(k, \varphi, \sigma)) \mid \sigma \in \Xi(k)\})$$

$$P_\forall(\{P_\rightarrow(P_\wedge(\#(k, \varphi, \sigma), \#(k, \psi, \sigma)), \#(k, \varphi, \sigma)) \mid \sigma \in \Xi(k)\}).$$

This can be expressed as

$$\text{for each } \sigma \in \Xi(k) \text{ if } \#(k, \varphi, \sigma) \text{ and } \#(k, \psi, \sigma) \text{ then } \#(k, \varphi, \sigma),$$

which is clearly true.

In the same way we can prove the truth of

$$\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow(\wedge(\varphi, \psi), \psi)]).$$

∎

Lemma 5.2 permits us to create an axiom $A_{5.2}$ which is the union of two sets of sentences.

Let G_1 be the set of all the sentences $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\wedge(\varphi, \psi), \varphi)]$ such that

- m is a positive integer, $x_1, \ldots, x_m \in \mathcal{V}$, $x_i \neq x_j$ for $i \neq j$, $\varphi_1, \ldots, \varphi_m \in E$, $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$,
- $\varphi, \psi \in S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$.

Let G_2 be the set of all the sentences $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\wedge(\varphi, \psi), \psi)]$ such that

- m is a positive integer, $x_1, \ldots, x_m \in \mathcal{V}$, $x_i \neq x_j$ for $i \neq j$, $\varphi_1, \ldots, \varphi_m \in E$, $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$,
- $\varphi, \psi \in S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$.

Then $A_{5.2}$ is the union of G_1 and G_2, and we assume $A_{5.2} \in \mathcal{A}$.

Lemma 5.3. *Let m be a positive integer. Let $x_1, \ldots, x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_m \in E$ and assume $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$. Define $k = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ and let $\varphi, \psi, \chi \in S(k)$.*

Under these assumptions we have

- $\rightarrow (\varphi, \psi), \rightarrow (\psi, \chi), \rightarrow (\varphi, \chi) \in S(k)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\varphi, \psi)] \in S(\epsilon)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\psi, \chi)] \in S(\epsilon)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\varphi, \chi)] \in S(\epsilon)$.

Moreover if

- $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\varphi, \psi)])$,
- $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\psi, \chi)])$

then $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\varphi, \chi)])$.

Proof.

We can rewrite $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\varphi, \psi)])$ as follows:

$$P_\forall(\{\#(k, \rightarrow (\varphi, \psi), \sigma) |\ \sigma \in \Xi(k)\})$$
$$P_\forall(\{P_\rightarrow (\#(k, \varphi, \sigma), \#(k, \psi, \sigma)) \mid \sigma \in \Xi(k)\}).$$

And we can rewrite $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\psi, \chi)])$ as follows:

$$P_\forall(\{\#(k, \rightarrow (\psi, \chi), \sigma) |\ \sigma \in \Xi(k)\})$$
$$P_\forall(\{P_\rightarrow (\#(k, \psi, \sigma), \#(k, \chi, \sigma)) \mid \sigma \in \Xi(k)\}).$$

In other words for each $\sigma \in \Xi(k)$ if $\#(k, \varphi, \sigma)$ then $\#(k, \psi, \sigma)$, and if $\#(k, \psi, \sigma)$ then $\#(k, \chi, \sigma)$. So, for each $\sigma \in \Xi(k)$, if $\#(k, \varphi, \sigma)$ then $\#(k, \chi, \sigma)$. This can be written as follows:

$$P_\forall(\{P_\rightarrow (\#(k, \varphi, \sigma), \#(k, \chi, \sigma)) \mid \sigma \in \Xi(k)\})$$
$$P_\forall(\{\#(k, \rightarrow (\varphi, \chi), \sigma) |\ \sigma \in \Xi(k)\}),$$

$$\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \to (\varphi, \chi)]).$$

■

Lemma 5.3 allows us to create a rule $R_{5.3}$ which is the set of all 3-tuples

$$\begin{pmatrix} \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \to (\varphi, \psi)], \\ \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \to (\psi, \chi)], \\ \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \to (\varphi, \chi)] \end{pmatrix}$$

such that

- m is a positive integer, $x_1, \ldots, x_m \in \mathcal{V}$, $x_i \neq x_j$ for $i \neq j$, $\varphi_1, \ldots, \varphi_m \in E$, $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$,
- $\varphi, \psi, \chi \in S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$.

We assume $R_{5.3} \in \mathcal{R}$.

Lemma 5.4. *Let m be a positive integer. Let $x_1, \ldots, x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_m \in E$ and assume $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$. Define $k = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$.*

Let $i = 1 \ldots m$, then

- $\in (x_i, \varphi_i) \in S(k)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \in (x_i, \varphi_i)] \in S(\epsilon)$,
- $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \in (x_i, \varphi_i)])$.

Proof.

First of all we need to prove the following:

- $x_i \in E(k_i)$,
- $\varphi_i \in E(k_i)$,
- for each $\sigma_i \in \Xi(k_i)$
  - $\#(k_i, \varphi_i, \sigma_i)$ is a set,
  - $\#(k_i, x_i, \sigma_i) \in \#(k_i, \varphi_i, \sigma_i)$.

By lemma 4.2 there exists a positive integer q such that $k_i \in K(q)^+$. So there exist $h \in K(q), \phi \in E_s(q, h), y \in (\mathcal{V} - var(h))$ such that $k_i = h + (y, \phi)$. We have also $k_i = k_{i-1} + (x_i, \varphi_i)$ so

$$x_i = y \in E_a(q+1, k_i) \subseteq E(q+1, k_i) \subseteq E(k_i) \ .$$

For each $\sigma_i = \rho + (y, s) \in \Xi(k_i)$

$$\#(k_i, x_i, \sigma_i) = \#(k_i, y, \sigma_i) = s \in \#(h, \phi, \rho) = \#(k_{i-1}, \varphi_i, \rho) \ .$$

Since $\varphi_i \in E_s(k_{i-1})$ by lemma 4.32 it follows that $\varphi_i \in E(k_i)$

Given $\sigma_i = \rho + (y, s) \in \Xi(k_i)$ we have that $\rho \in \Xi(h) = \Xi(k_{i-1})$, so $\rho \sqsubseteq \sigma_i$. Since $\varphi_i \in E(k_{i-1}) \cap E(k_i)$, $k_{i-1} \sqsubseteq k_i$, $\sigma_i \in \Xi(k_i)$, $\rho \in \Xi(k_{i-1})$, $\rho \sqsubseteq \sigma_i$ we can apply lemma 4.12 and obtain that $\#(k_i, \varphi_i, \sigma_i) = \#(k_{i-1}, \varphi_i, \rho)$ is a set. Moreover, clearly,

$$\#(k_i, x_i, \sigma_i) \in \#(k_i, \varphi_i, \sigma_i) \ .$$

Using lemma 4.32 we also obtain that $x_i \in E(k)$ and $\varphi_i \in E(k)$.

Moreover, given $\sigma \in \Xi(k)$, $\#(k, \varphi_i, \sigma) = \#(k_i, \varphi_i, \sigma_{/dom(k_i)})$ is a set, and

$$\#(k, x_i, \sigma) = \#(k_i, x_i, \sigma_{/dom(k_i)}) \in \#(k_i, \varphi_i, \sigma_{/dom(k_i)}) = \#(k, \varphi_i, \sigma) \ .$$

By lemma 4.27 $\in (x_i, \varphi_i) \in S(k)$ and consequently

$$\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\in)(x_i, \varphi_i)] \in S(\epsilon) \ .$$

Moreover we can rewrite $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, (\in)(x_i, \varphi_i)])$ as follows

$$P_\forall(\{\#(k, (\in)(x_i, \varphi_i), \sigma)| \ \sigma \in \Xi(k)\}) \ ,$$

$$P_\forall(\{P_\in(\#(k, x_i, \sigma), \#(k, \varphi_i, \sigma))| \ \sigma \in \Xi(k)\}) \ .$$

To show this we have to prove that for each $\sigma \in \Xi(k)$ $\#(k, x_i, \sigma)$ belongs to $\#(k, \varphi_i, \sigma)$. But we have just seen this is true. ∎

Lemma 5.4 permits us to create an axiom $A_{5.4}$ which is the set of all sentences $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \in (x_i, \varphi_i)]$ such that

- m is a positive integer, $x_1, \ldots, x_m \in \mathcal{V}$, $x_\alpha \neq x_\beta$ for $\alpha \neq \beta$, $\varphi_1, \ldots, \varphi_m \in E$, $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$,
- $i = 1 \ldots m$.

Lemma 5.5. *Let m be a positive integer. Let $x_1, \ldots, x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_m \in E$ and assume $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$. Define $k = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ and let $\varphi, \psi \in S(k)$.*

Under these assumptions we have

- $\rightarrow (\psi, \varphi) \in S(k)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \varphi] \in S(\epsilon)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\psi, \varphi)] \in S(\epsilon)$.

Moreover if $\# (\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \varphi])$ then $\# (\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\psi, \varphi)])$ also holds.

Proof.

Suppose $\# (\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \varphi])$ holds. It can be rewritten as

$$P_\forall(\{\#(k, \varphi, \sigma)| \ \sigma \in \Xi(k)\}) \ .$$

We can rewrite $\# (\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\psi, \varphi)])$ as

$$P_\forall(\{\#(k, \rightarrow (\psi, \varphi), \sigma)| \ \sigma \in \Xi(k)\}) \ ,$$

$$P_\forall(\{P_\rightarrow(\#(k, \psi, \sigma), \#(k, \varphi, \sigma))| \ \sigma \in \Xi(k)\}) \ .$$

For each $\sigma \in \Xi(k)$ $\#(k, \varphi, \sigma)$ holds, this implies that

$$P_\forall(\{P_\rightarrow(\#(k, \psi, \sigma), \#(k, \varphi, \sigma))| \ \sigma \in \Xi(k)\})$$

holds too and this completes the proof. ∎

Lemma 5.5 allows us to create a rule $R_{5.5}$ which is the set of all pairs

$$(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \varphi], \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \to (\psi, \varphi)])$$

such that

- m is a positive integer, $x_1, \ldots, x_m \in \mathcal{V}$, $x_i \neq x_j$ for $i \neq j$, $\varphi_1, \ldots, \varphi_m \in E$, $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$,
- $\varphi, \psi \in S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$.

LEMMA 5.6. *Let m be a positive integer. Let $x_1, \ldots, x_{m+1} \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_{m+1} \in E$ and assume $H[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]$.*

Define $k = k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]$. Of course $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ also holds, we define $h = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$. Let $\chi \in S(h)$, $t \in E(h)$, $\varphi \in E_s(h)$.

Under these assumptions

- $\in (x_{m+1}, \varphi) \in S(k)$,
- $\forall(\{\}\,(x_{m+1} : \varphi_{m+1}, \in (x_{m+1}, \varphi))) \in S(h)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \to (\chi, \forall(\{\}\,(x_{m+1} : \varphi_{m+1}, \in (x_{m+1}, \varphi))))] \in S(\epsilon)$,
- $\in (t, \varphi_{m+1}) \in S(h)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \to (\chi, \in (t, \varphi_{m+1}))] \in S(\epsilon)$,
- $\in (t, \varphi) \in S(h)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \to (\chi, \in (t, \varphi))] \in S(\epsilon)$.

Moreover if

- $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \to (\chi, \forall(\{\}\,(x_{m+1} : \varphi_{m+1}, \in (x_{m+1}, \varphi))))])$ *and*
- $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \to (\chi, \in (t, \varphi_{m+1}))])$

then $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \to (\chi, \in (t, \varphi))])$.

Proof.

By lemma 4.25 we obtain that $x_{m+1} \in E(k)$.

By lemma 4.32, since $\varphi \in E_s(h)$, we obtain that $\varphi \in E(k)$ and for each $\sigma \in \Xi(k)$ $\sigma_{/dom(h)} \in \Xi(h)$, $\#(k, \varphi, \sigma) = \#(h, \varphi, \sigma_{/dom(h)})$ is a set.

By lemma4.27 we obtain that $\in (x_{m+1}, \varphi) \in S(k)$.

By lemma 4.16 we obtain $\forall(\{\}\,(x_{m+1} : \varphi_{m+1}, \in (x_{m+1}, \varphi))) \in S(h)$.

Clearly this implies that

$$\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \to (\chi, \forall(\{\}\,(x_{m+1} : \varphi_{m+1}, \in (x_{m+1}, \varphi))))] \in S(\epsilon).$$

Furthermore we have $t \in E(h)$, $\varphi_{m+1} \in E_s(h)$, so $\in (t, \varphi_{m+1}) \in S(h)$. It clearly follows that $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \to (\chi, \in (t, \varphi_{m+1}))] \in S(\epsilon)$.

We have also $\varphi \in E_s(h)$, so $\in (t, \varphi) \in S(h)$. It follows that

$$\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \to (\chi, \in (t, \varphi))] \in S(\epsilon).$$

We now assume

- $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\chi, \forall(\{\} (x_{m+1} : \varphi_{m+1}, \in (x_{m+1}, \varphi))))])$ and
- $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\chi, \in (t, \varphi_{m+1}))])$

both hold and we try to prove $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\chi, \in (t, \varphi))])$.

We can rewrite

$$\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\chi, \forall(\{\} (x_{m+1} : \varphi_{m+1}, \in (x_{m+1}, \varphi))))])$$

as

$$P_\forall(\{\# (h, \rightarrow (\chi, \forall (\{\} (x_{m+1} : \varphi_{m+1}, \in (x_{m+1}, \varphi)))) , \rho) \mid \rho \in \Xi(h)\}) \ ,$$
$$P_\forall(\{P_\rightarrow (\# (h, \chi, \rho) , \# (h, \forall (\{\} (x_{m+1} : \varphi_{m+1}, \in (x_{m+1}, \varphi))) , \rho)) \mid \rho \in \Xi(h)\}) \ ,$$
$$P_\forall(\{P_\rightarrow (\# (h, \chi, \rho) , P_\forall (\{\# (k, \in (x_{m+1}, \varphi), \sigma) \mid \sigma \in \Xi(k),\ \rho \sqsubseteq \sigma\})) \mid \rho \in \Xi(h)\}) \ ,$$
$$P_\forall(\{P_\rightarrow (\# (h, \chi, \rho) , P_\forall (\{P_\in (\#(k, x_{m+1}, \sigma), \#(k, \varphi, \sigma)) \mid \sigma \in \Xi(k),\ \rho \sqsubseteq \sigma\})) \mid \rho \in \Xi(h)\}) \ .$$

We can rewrite

$$\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\chi, \in (t, \varphi_{m+1}))])$$

as

$$P_\forall(\{\#(h, \rightarrow (\chi, \in (t, \varphi_{m+1})), \rho)|\ \rho \in \Xi(h)\}) \ ,$$
$$P_\forall(\{P_\rightarrow(\#(h, \chi, \rho), \#(h, \in (t, \varphi_{m+1}), \rho))|\ \rho \in \Xi(h)\}) \ ,$$
$$P_\forall(\{P_\rightarrow(\#(h, \chi, \rho), P_\in(\#(h, t, \rho), \#(h, \varphi_{m+1}, \rho)))|\ \rho \in \Xi(h)\}) \ .$$

We can rewrite

$$\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\chi, \in (t, \varphi))])$$

as

$$P_\forall(\{\#(h, \rightarrow (\chi, \in (t, \varphi)), \rho)|\ \rho \in \Xi(h)\}) \ ,$$
$$P_\forall(\{P_\rightarrow(\#(h, \chi, \rho), \#(h, \in (t, \varphi), \rho))|\ \rho \in \Xi(h)\}) \ ,$$
$$P_\forall(\{P_\rightarrow(\#(h, \chi, \rho), P_\in(\#(h, t, \rho), \#(h, \varphi, \rho)))|\ \rho \in \Xi(h)\}) \ .$$

Let $\rho \in \Xi(h)$ and let $\#(h, \chi, \rho)$. We need to show that $\#(h, t, \rho)$ belongs to $\#(h, \varphi, \rho)$.

There exists a positive integer q such that $k \in K(q)^+$. So there exist $g \in K(q), \phi \in E_s(q, g), y \in (\mathcal{V} - var(g))$ such that $k = g + (y, \phi)$. At the same time

$$k = k_{m+1} = k_m + (x_{m+1}, \varphi_{m+1}) = h + (x_{m+1}, \varphi_{m+1}) \ .$$

Therefore

$$\begin{aligned}\Xi(k) &= \{\delta + (y, s)|\ \delta \in \Xi(g), s \in \#(g, \phi, \delta)\} = \\ &= \{\delta + (x_{m+1}, s)|\ \delta \in \Xi(h), s \in \#(h, \varphi_{m+1}, \delta)\} \, .\end{aligned}$$

We have $\rho \in \Xi(h)$, $\#(h, t, \rho) \in \#(h, \varphi_{m+1}, \rho)$, so $\rho + (x_{m+1}, \#(h, t, \rho)) \in \Xi(k)$.

Let $\sigma = \rho + (x_{m+1}, \#(h, t, \rho)) \in \Xi(k)$, clearly $\rho \sqsubseteq \sigma$, so $\#(k, x_{m+1}, \sigma)$ belongs to $\#(k, \varphi, \sigma)$. And we have also

$$x_{m+1} = y \in E_a(q+1, k) \subseteq E(q+1, k)\,,$$
$$\#(k, x_{m+1}, \sigma) = \#(k, x_{m+1}, \sigma)_{(q+1,k,a)} = \#(h, t, \rho)\,,$$
$$\#(k, \varphi, \sigma) = \#(h, \varphi, \sigma_{/dom(h)}) = \#(h, \varphi, \sigma_{/dom(\rho)}) = \#(h, \varphi, \rho)\,.$$

Finally we obtain $\#(h, t, \rho) = \#(k, x_{m+1}, \sigma)$ belongs to $\#(k, \varphi, \sigma) = \#(h, \varphi, \rho)$. ∎

By virtue of lemma 5.6 we can create a rule $R_{5.6}$ which is the set of all 3-tuples

$$\left(\begin{array}{l} \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\chi, \forall(\{\}\,(x_{m+1} : \varphi_{m+1}, \in (x_{m+1}, \varphi))))], \\ \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\chi, \in (t, \varphi_{m+1}))], \\ \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\chi, \in (t, \varphi))] \end{array}\right)$$

such that

- m is a positive integer, $x_1, \ldots, x_{m+1} \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$, $\varphi_1, \ldots, \varphi_{m+1} \in E$, $H[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]$;
- if we define $k = k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]$ and $h = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ then
  - $\chi \in S(h)$,
  - $t \in E(h)$,
  - $\varphi \in E_s(h)$.

Lemma 5.7. *Let $x_1 \in \mathcal{V}$, $\varphi_1 \in E$ and assume $H[x_1 : \varphi_1]$. Define $k = k[x_1 : \varphi_1]$. Let $\psi \in S(k)$ and $\varphi \in S(k) \cap S(\epsilon)$. Under these assumptions we have*

- $\rightarrow (\psi, \varphi) \in S(k)$,
- $\gamma[x_1 : \varphi_1, \rightarrow (\psi, \varphi)] \in S(\epsilon)$,
- $\exists\,(\{\}(x_1 : \varphi_1, \psi)) \in S(\epsilon)$,
- $\rightarrow (\exists\,(\{\}(x_1 : \varphi_1, \psi))\,, \varphi) \in S(\epsilon)$.

Moreover if $\#(\gamma[x_1 : \varphi_1, \rightarrow (\psi, \varphi)])$ then $\#(\rightarrow (\exists\,(\{\}(x_1 : \varphi_1, \psi))\,, \varphi))$.

Proof.

Suppose $\#(\gamma[x_1 : \varphi_1, (\rightarrow)(\psi, \varphi)])$. By definition we have

$$\#(\forall(\{\}(x_1 : \varphi_1, \rightarrow (\psi, \varphi))))\ ,$$

and then

$$P_\forall(\{\#(k, \rightarrow (\psi, \varphi), \sigma)|\ \sigma \in \Xi(k)\})\ ,$$
$$P_\forall(\{P_\rightarrow(\#(k, \psi, \sigma), \#(k, \varphi, \sigma))|\ \sigma \in \Xi(k)\})\ .$$

In turn $\#(\rightarrow (\exists\,(\{\}(x_1 : \varphi_1, \psi))\,, \varphi))$ can be rewritten as

$$\#(\epsilon, \rightarrow (\exists\,(\{\}(x_1 : \varphi_1, \psi))\,, \varphi)\,, \epsilon)\ ,$$
$$P_\rightarrow(\#(\epsilon, \exists\,(\{\}(x_1 : \varphi_1, \psi))\,, \epsilon), \#(\epsilon, \varphi, \epsilon))\ ,$$
$$P_\rightarrow(\#(\exists\,(\{\}(x_1 : \varphi_1, \psi))), \#(\varphi))\ ,$$

$$P_{\rightarrow}(P_{\exists}(\{\#(k,\psi,\sigma)|\ \sigma\in\Xi(k)\}),\#(\varphi))\ .$$

In order to prove the last statement, we suppose there exists $\sigma \in \Xi(k)$ such that $\#(k,\psi,\sigma)$. This implies $\#(k,\varphi,\sigma)$, but we need to show that $\#(\varphi)$ holds.

To perform this step we can use lemma 4.12. In fact there exists a positive integer q such that $\epsilon, k \in K(q)$, $\varphi \in E(q,\epsilon) \cap E(q,k)$. Moreover $\epsilon \sqsubseteq k$, $\epsilon \in \Xi(\epsilon)$, $\sigma \in \Xi(k)$, $\epsilon \sqsubseteq \sigma$ so by lemma 4.12 $\#(k,\varphi,\sigma) = \#(\epsilon,\varphi,\epsilon) = \#(\varphi)$. ∎

Lemma 5.7 allows us to create a rule $R_{5.7}$ which is the set of all pairs

$$\left(\begin{array}{l}\gamma[x_1:\varphi_1,\rightarrow(\psi,\varphi)],\\ \rightarrow(\exists\left(\{\}(x_1:\varphi_1,\psi)\right),\varphi)\end{array}\right)$$

such that $x_1 \in \mathcal{V}$, $\varphi_1 \in E$, $H[x_1:\varphi_1]$, $\psi \in S(k[x_1:\varphi_1])$ and $\varphi \in S(k[x_1:\varphi_1]) \cap S(\epsilon)$.

LEMMA 5.8. *Let m be a positive integer. Let $x_1,\ldots,x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1,\ldots,\varphi_m \in E$ and assume $H[x_1:\varphi_1,\ldots,x_m:\varphi_m]$. Define $k = k[x_1:\varphi_1,\ldots,x_m:\varphi_m]$ and let $\varphi,\psi_1,\psi_2 \in S(k)$.*

Under these assumptions we have $\rightarrow(\varphi,\psi_1), \rightarrow(\varphi,\psi_2), \rightarrow(\varphi,\wedge(\psi_1,\psi_2)) \in S(k)$.

Moreover, if

$$\#(\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\rightarrow(\varphi,\psi_1)]),\ \#(\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\rightarrow(\varphi,\psi_2)])$$

then

$$\#(\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\rightarrow(\varphi,\wedge(\psi_1,\psi_2))])\ .$$

Proof.

We need to show

$$\#(\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\rightarrow(\varphi,\wedge(\psi_1,\psi_2))])\ ,$$

that is

$$\begin{aligned}&P_{\forall}(\{\#(k,\rightarrow(\varphi,\wedge(\psi_1,\psi_2)),\sigma)|\ \sigma\in\Xi(k)\})\ ,\\ &P_{\forall}(\{P_{\rightarrow}(\#(k,\varphi,\sigma),\#(k,\wedge(\psi_1,\psi_2),\sigma))|\ \sigma\in\Xi(k)\})\ ,\\ &P_{\forall}(\{P_{\rightarrow}(\#(k,\varphi,\sigma),P_{\wedge}(\#(k,\psi_1,\sigma),\#(k,\psi_2,\sigma)))|\ \sigma\in\Xi(k)\})\ .\end{aligned}\tag{5.0.1}$$

But we have

$$\begin{aligned}&\#(\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\rightarrow(\varphi,\psi_1)])\ ,\\ &P_{\forall}(\{\#(k,\rightarrow(\varphi,\psi_1),\sigma)|\ \sigma\in\Xi(k)\})\ ,\\ &P_{\forall}(\{P_{\rightarrow}(\#(k,\varphi,\sigma),\#(k,\psi_1,\sigma))|\ \sigma\in\Xi(k)\})\ .\end{aligned}$$

And we have

$$\begin{aligned}&\#(\gamma[x_1:\varphi_1,\ldots,x_m:\varphi_m,\rightarrow(\varphi,\psi_2)])\ ,\\ &P_{\forall}(\{\#(k,\rightarrow(\varphi,\psi_2),\sigma)|\ \sigma\in\Xi(k)\})\ ,\\ &P_{\forall}(\{P_{\rightarrow}(\#(k,\varphi,\sigma),\#(k,\psi_2,\sigma))|\ \sigma\in\Xi(k)\})\ .\end{aligned}$$

So for each $\sigma \in \Xi(k)$ if $\#(k, \varphi, \sigma)$ holds true then both $\#(k, \psi_1, \sigma)$ and $\#(k, \psi_2, \sigma)$ hold. This implies 5.0.1 holds true in turn. ∎

Lemma 5.8 allows us to create a rule $R_{5.8}$ which is the set of all 3-tuples

$$\begin{pmatrix} \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\varphi, \psi_1)], \\ \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\varphi, \psi_2)], \\ \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\varphi, \wedge(\psi_1, \psi_2))] \end{pmatrix}$$

such that

- m is a positive integer, $x_1, \ldots, x_m \in \mathcal{V}$, $x_i \neq x_j$ for $i \neq j$, $\varphi_1, \ldots, \varphi_m \in E$, $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$,
- $\varphi, \psi_1, \psi_2 \in S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$.

LEMMA 5.9. *Let m be a positive integer. Let $x_1, \ldots, x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_m \in E$ and assume $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$. Define $k = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ and let $\varphi, \psi \in S(k)$.*

Under these assumptions we have

- $\rightarrow (\varphi, \wedge (\psi, \neg(\psi))), \neg(\varphi) \in S(k)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\varphi, \wedge (\psi, \neg(\psi)))] \in S(\epsilon)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \neg(\varphi)] \in S(\epsilon)$.

Moreover if $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\varphi, \wedge (\psi, \neg(\psi)))])$ then $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \neg(\varphi)])$.

Proof.

We can rewrite $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\varphi, \wedge (\psi, \neg(\psi)))])$ as

$$P_\forall(\{\#(k, \rightarrow (\varphi, \wedge (\psi, \neg(\psi))), \sigma) |\ \sigma \in \Xi(k)\})\ ,$$

$$P_\forall(\{P_\rightarrow (\#(k, \varphi, \sigma), \#(k, \wedge (\psi, \neg(\psi)), \sigma)) \mid \sigma \in \Xi(k)\})\ ,$$

$$P_\forall(\{P_\rightarrow (\#(k, \varphi, \sigma), P_\wedge (\#(k, \psi, \sigma), \#(k, \neg(\psi), \sigma))) \mid \sigma \in \Xi(k)\})\ ,$$

$$P_\forall(\{P_\rightarrow (\#(k, \varphi, \sigma), P_\wedge (\#(k, \psi, \sigma), P_\neg(\#(k, \psi, \sigma)))) \mid \sigma \in \Xi(k)\})\ .$$

This can be expressed as 'for each $\sigma \in \Xi(k)$ either $\#(k, \varphi, \sigma)$ is false or both $\#(k, \psi, \sigma)$ and ($\#(k, \psi, \sigma)$ is false) are true'.

Since $\#(k, \psi, \sigma)$ cannot be both true and false at the same time we have that 'for each $\sigma \in \Xi(k)$ $\#(k, \varphi, \sigma)$ is false'. This is formally expressed as

$$P_\forall(\{P_\neg(\#(k, \varphi, \sigma))|\ \sigma \in \Xi(k)\})\ ,$$

$$P_\forall(\{\#(k, \neg(\varphi), \sigma)|\ \sigma \in \Xi(k)\})\ ,$$

which we can finally rewrite as $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \neg(\varphi)]$. ∎

Lemma 5.9 allows us to create a rule $R_{5.9}$ which is the set of all pairs

$$(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\varphi, \wedge (\psi, \neg(\psi)))], \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \neg(\varphi)])$$

such that

- m is a positive integer, $x_1, \ldots, x_m \in \mathcal{V}$, $x_i \neq x_j$ for $i \neq j$, $\varphi_1, \ldots, \varphi_m \in E$, $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$,
- $\varphi, \psi \in S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$.

LEMMA 5.10. *Let m be a positive integer. Let $x_1, \ldots, x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_m \in E$ and assume $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$. Define $k = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ and let $\varphi, \psi \in S(k)$.*

Under these assumptions we have

- $\neg (\wedge(\varphi, \psi)), \rightarrow (\varphi, \neg(\psi)) \in S(k)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \neg (\wedge(\varphi, \psi))] \in S(\epsilon)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\varphi, \neg(\psi))] \in S(\epsilon)$.

Moreover if $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \neg (\wedge(\varphi, \psi))])$ then $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\varphi, \neg(\psi))])$.

Proof.

We can rewrite $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \neg (\wedge(\varphi, \psi))])$ as

$$P_\forall(\{\#(k, \neg (\wedge(\varphi, \psi)), \sigma) |\ \sigma \in \Xi(k)\}) \ ,$$
$$P_\forall(\{P_\neg(\#(k, \wedge(\varphi, \psi), \sigma)) |\ \sigma \in \Xi(k)\}) \ ,$$
$$P_\forall(\{P_\neg(P_\wedge(\#(k, \varphi, \sigma), \#(k, \psi, \sigma))) |\ \sigma \in \Xi(k)\}) \ .$$

We can rewrite $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\varphi, \neg(\psi))])$ as

$$P_\forall(\{\#(k, \rightarrow (\varphi, \neg(\psi)), \sigma) |\ \sigma \in \Xi(k)\}) \ ,$$
$$P_\forall(\{P_\rightarrow(\#(k, \varphi, \sigma), \#(k, \neg(\psi), \sigma)) |\ \sigma \in \Xi(k)\}) \ ,$$
$$P_\forall(\{P_\rightarrow(\#(k, \varphi, \sigma), P_\neg(\#(k, \psi, \sigma))) |\ \sigma \in \Xi(k)\}) \ .$$

Thus if $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \neg (\wedge(\varphi, \psi))])$ we have that 'for each $\sigma \in \Xi(k)$ it is false that $\#(k, \varphi, \sigma)$ and $\#(k, \psi, \sigma)$ are both true'.

In other words for each $\sigma \in \Xi(k)$ ($\#(k, \varphi, \sigma)$ is false) or ($\#(k, \psi, \sigma)$ is false).

In other words for each $\sigma \in \Xi(k)$ $P_\rightarrow(\#(k, \varphi, \sigma), P_\neg(\#(k, \psi, \sigma)))$.

The last condition clearly implies $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\varphi, \neg(\psi))])$. ∎

Lemma 5.10 allows us to create a rule $R_{5.10}$ which is the set of all pairs

$$(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \neg (\wedge(\varphi, \psi))], \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\varphi, \neg(\psi))])$$

such that

- m is a positive integer, $x_1, \ldots, x_m \in \mathcal{V}$, $x_i \neq x_j$ for $i \neq j$, $\varphi_1, \ldots, \varphi_m \in E$, $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$,
- $\varphi, \psi \in S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$.

Lemma 5.11. *Let m be a positive integer. Let $x_1, \ldots, x_{m+1} \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $\varphi_1, \ldots, \varphi_{m+1} \in E$ and assume $H[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]$.*

Define $k = k[x_1 : \varphi_1, \ldots, x_{m+1} : \varphi_{m+1}]$. Of course $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$ also holds, we define $h = k[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$. Let $\chi \in S(h)$, $\varphi \in S(k)$.

Under these assumptions we have

- $\forall(\{\}(x_{m+1} : \varphi_{m+1}, \varphi)) \in S(h)$,
- $\neg\,(\forall(\{\}(x_{m+1} : \varphi_{m+1}, \varphi))) \in S(h)$,
- $\rightarrow(\chi, \neg\,(\forall(\{\}(x_{m+1} : \varphi_{m+1}, \varphi)))) \in S(h)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow(\chi, \neg\,(\forall(\{\}(x_{m+1} : \varphi_{m+1}, \varphi))))] \in S(\epsilon)$,
- $\neg(\varphi) \in S(k)$,
- $\exists(\{\}(x_{m+1} : \varphi_{m+1}, \neg(\varphi))) \in S(h)$,
- $\rightarrow(\chi, \exists(\{\}(x_{m+1} : \varphi_{m+1}, \neg(\varphi)))) \in S(h)$,
- $\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow(\chi, \exists(\{\}(x_{m+1} : \varphi_{m+1}, \neg(\varphi))))] \in S(\epsilon)$.

Moreover if $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow(\chi, \neg\,(\forall(\{\}(x_{m+1} : \varphi_{m+1}, \varphi))))])$ then

$$\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow(\chi, \exists(\{\}(x_{m+1} : \varphi_{m+1}, \neg(\varphi))))]) \; .$$

Proof.

We can rewrite $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow(\chi, \neg\,(\forall(\{\}(x_{m+1} : \varphi_{m+1}, \varphi))))])$ as

$$P_\forall(\{\#(h, \rightarrow(\chi, \neg\,(\forall(\{\}(x_{m+1} : \varphi_{m+1}, \varphi)))), \rho)|\; \rho \in \Xi(h)\}) \; ,$$

$$P_\forall(\{P_\rightarrow(\#(h, \chi, \rho), \#(h, \neg\,(\forall(\{\}(x_{m+1} : \varphi_{m+1}, \varphi))), \rho)) \mid \rho \in \Xi(h)\}) \; ,$$

$$P_\forall(\{P_\rightarrow(\#(h, \chi, \rho), P_\neg(\#(h, \forall(\{\}(x_{m+1} : \varphi_{m+1}, \varphi)), \rho))) \mid \rho \in \Xi(h)\}) \; ,$$

$$P_\forall(\{P_\rightarrow(\#(h, \chi, \rho), P_\neg(P_\forall(\{\#(k, \varphi, \sigma)|\; \sigma \in \Xi(k),\; \rho \sqsubseteq \sigma\}))) \mid \rho \in \Xi(h)\}) \; .$$

We can furtherly express this as

'for each $\rho \in \Xi(h)$ $P_\rightarrow(\#(h, \chi, \rho), P_\neg(P_\forall(\{\#(k, \varphi, \sigma)|\; \sigma \in \Xi(k),\; \rho \sqsubseteq \sigma\})))$',

'for each $\rho \in \Xi(h)$ if $\#(h, \chi, \rho)$ then it is false that $P_\forall(\{\#(k, \varphi, \sigma)|\; \sigma \in \Xi(k),\; \rho \sqsubseteq \sigma\})$',

'for each $\rho \in \Xi(h)$ if $\#(h, \chi, \rho)$ then it is false that (for each $\sigma \in \Xi(k)$ such that $\rho \sqsubseteq \sigma$ $\#(k, \varphi, \sigma)$ holds)',

'for each $\rho \in \Xi(h)$ if $\#(h, \chi, \rho)$ then (there exists $\sigma \in \Xi(k)$ such that $\rho \sqsubseteq \sigma$ and $\#(k, \varphi, \sigma)$ is false)'.

We can rewrite $\#(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow(\chi, \exists(\{\}(x_{m+1} : \varphi_{m+1}, \neg(\varphi))))])$ as

$$P_\forall(\{\#(h, \rightarrow(\chi, \exists(\{\}(x_{m+1} : \varphi_{m+1}, \neg(\varphi)))), \rho)|\; \rho \in \Xi(h)\}) \; ,$$

$$P_\forall(\{P_\rightarrow(\#(h, \chi, \rho), \#(h, \exists(\{\}(x_{m+1} : \varphi_{m+1}, \neg(\varphi))), \rho)) \mid \rho \in \Xi(h)\}) \; ,$$

$$P_\forall(\{P_\rightarrow(\#(h, \chi, \rho), P_\exists(\{\#(k, \neg(\varphi), \sigma)|\; \sigma \in \Xi(k),\; \rho \sqsubseteq \sigma\})) \mid \rho \in \Xi(h)\}) \; ,$$

$$P_\forall(\{P_\to(\#(h,\chi,\rho), P_\exists(\{P_\neg(\#(k,\varphi,\sigma))|\ \sigma\in\Xi(k),\ \rho\sqsubseteq\sigma\}))\ |\ \rho\in\Xi(h)\})\ .$$

This can be furtherly rewritten as

'for each $\rho\in\Xi(h)$ $P_\to(\#(h,\chi,\rho), P_\exists(\{P_\neg(\#(k,\varphi,\sigma))|\ \sigma\in\Xi(k),\ \rho\sqsubseteq\sigma\}))$ ',
'for each $\rho\in\Xi(h)$ if $\#(h,\chi,\rho)$ then $P_\exists(\{P_\neg(\#(k,\varphi,\sigma))|\ \sigma\in\Xi(k),\ \rho\sqsubseteq\sigma\})$ ',
'for each $\rho\in\Xi(h)$ if $\#(h,\chi,\rho)$ then (there exists $\sigma\in\Xi(k)$ such that $\rho\sqsubseteq\sigma$ and $\#(k,\varphi,\sigma)$ is false)'.

The last condition is clearly ensured by our hypothesis. ■

Lemma 5.11 allows us to create a rule $R_{5.11}$ which is the set of all pairs

$$\begin{pmatrix} \gamma[x_1:\varphi_1,\dots,x_m:\varphi_m,\to(\chi,\neg(\forall(\{\}(x_{m+1}:\varphi_{m+1},\varphi))))], \\ \gamma[x_1:\varphi_1,\dots,x_m:\varphi_m,\to(\chi,\exists(\{\}(x_{m+1}:\varphi_{m+1},\neg(\varphi))))] \end{pmatrix}$$

such that

- m is a positive integer, $x_1,\dots,x_{m+1}\in\mathcal{V}$, with $x_i\neq x_j$ for $i\neq j$, $\varphi_1,\dots,\varphi_{m+1}\in E$, $H[x_1:\varphi_1,\dots,x_{m+1}:\varphi_{m+1}]$;
- if we define $k=k[x_1:\varphi_1,\dots,x_{m+1}:\varphi_{m+1}]$ and $h=k[x_1:\varphi_1,\dots,x_m:\varphi_m]$ then $\chi\in S(h)$, $\varphi\in S(k)$.

LEMMA 5.12. *Let m be a positive integer. Let $x_1,\dots,x_m\in\mathcal{V}$, with $x_i\neq x_j$ for $i\neq j$. Let $\varphi_1,\dots,\varphi_m\in E$ and assume $H[x_1:\varphi_1,\dots,x_m:\varphi_m]$. Define $k=k[x_1:\varphi_1,\dots,x_m:\varphi_m]$ and let $\varphi,\psi,\chi\in S(k)$.*

Under these assumptions we have

- $\to(\wedge(\varphi,\psi),\chi), \to(\varphi,\to(\psi,\chi))\in S(k)$,
- $\gamma[x_1:\varphi_1,\dots,x_m:\varphi_m,\to(\wedge(\varphi,\psi),\chi)]\in S(\epsilon)$,
- $\gamma[x_1:\varphi_1,\dots,x_m:\varphi_m,\to(\varphi,\to(\psi,\chi))]\in S(\epsilon)$.

Moreover if $\#(\gamma[x_1:\varphi_1,\dots,x_m:\varphi_m,\to(\wedge(\varphi,\psi),\chi)])$ then $\#(\gamma[x_1:\varphi_1,\dots,x_m:\varphi_m,\to(\varphi,\to(\psi,\chi))])$.

Proof.

We assume $\#(\gamma[x_1:\varphi_1,\dots,x_m:\varphi_m,\to(\wedge(\varphi,\psi),\chi)])$ which can be rewritten

$$P_\forall(\{\#(k,\to(\wedge(\varphi,\psi),\chi),\sigma)|\ \sigma\in\Xi(k)\})$$

$$P_\forall(\{P_\to(\#(k,\wedge(\varphi,\psi),\sigma),\#(k,\chi,\sigma))|\ \sigma\in\Xi(k)\})$$

$$P_\forall(\{P_\to(P_\wedge(\#(k,\varphi,\sigma),\#(k,\psi,\sigma)),\#(k,\chi,\sigma))|\ \sigma\in\Xi(k)\})\ .$$

Of course we now try to show $\#(\gamma[x_1:\varphi_1,\dots,x_m:\varphi_m,\to(\varphi,\to(\psi,\chi))])$ which in turn can be rewritten

$$P_\forall(\{\#(k,\to(\varphi,\to(\psi,\chi)),\sigma)|\ \sigma\in\Xi(k)\})$$

$$P_\forall(\{P_\to(\#(k,\varphi,\sigma),\#(k,\to(\psi,\chi),\sigma))|\ \sigma\in\Xi(k)\})$$

$$P_\forall(\{P_\to(\#(k,\varphi,\sigma),P_\to(\#(k,\psi,\sigma),\#(k,\chi,\sigma)))|\ \sigma\in\Xi(k)\})\ .$$

Let $\sigma \in \Xi(k)$, suppose $\#(k, \varphi, \sigma)$ and $\#(k, \psi, \sigma)$, then we have $\#(k, \chi, \sigma)$ and this completes the proof. ■

Lemma 5.12 allows us to create a rule $R_{5.12}$ which is the set of all pairs

$(\gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\wedge(\varphi, \psi), \chi)], \gamma[x_1 : \varphi_1, \ldots, x_m : \varphi_m, \rightarrow (\varphi, \rightarrow (\psi, \chi))])$

such that

- m is a positive integer, $x_1, \ldots, x_m \in \mathcal{V}$, $x_i \neq x_j$ for $i \neq j$, $\varphi_1, \ldots, \varphi_m \in E$, $H[x_1 : \varphi_1, \ldots, x_m : \varphi_m]$,
- $\varphi, \psi, \chi \in S(k[x_1 : \varphi_1, \ldots, x_m : \varphi_m])$.

Lemma 5.13. *Let $\varphi, \psi, \chi \in S(\epsilon)$. We have*

- $\rightarrow (\varphi, \rightarrow (\psi, \chi)) \in S(\epsilon)$,
- $\rightarrow (\wedge(\varphi, \psi), \chi) \in S(\epsilon)$.

Moreover if $\#(\rightarrow (\varphi, \rightarrow (\psi, \chi)))$ then $\#(\rightarrow (\wedge(\varphi, \psi), \chi))$.

Proof.

Suppose $\#(\rightarrow (\varphi, \rightarrow (\psi, \chi)))$ holds. It can be rewritten

$$P_{\rightarrow}(\#(\varphi), \#(\rightarrow (\psi, \chi)))\ ,$$
$$P_{\rightarrow}(\#(\varphi), P_{\rightarrow}(\#(\psi), \#(\chi)))\ .$$

In turn, $\#(\rightarrow (\wedge(\varphi, \psi), \chi))$ can be rewritten

$$P_{\rightarrow}(\#(\wedge(\varphi, \psi)), \#(\chi))\ ,$$
$$P_{\rightarrow}(P_{\wedge}(\#(\varphi), \#(\psi)), \#(\chi))\ .$$

Suppose $\#(\varphi)$ and $\#(\psi)$ both hold, we need to show that $\#(\chi)$ holds. This is granted by

$$P_{\rightarrow}(\#(\varphi), P_{\rightarrow}(\#(\psi), \#(\chi)))\ .$$

■

Lemma 5.13 allows us to create a rule $R_{5.13}$ which is the set of all pairs

$$\begin{pmatrix} \rightarrow (\varphi, \rightarrow (\psi, \chi)), \\ \rightarrow (\wedge(\varphi, \psi), \chi) \end{pmatrix}$$

such that $\varphi, \psi, \chi \in S(\epsilon)$.

6. Example of a proof

As an example of proof, we want to prove a form of the Bocardo syllogism. In Ferreirós' referenced paper ([3]), on paragraph 3.1, the syllogism is expressed as follows:

Some A are not B. All C are B. Therefore, some A are not C.

Suppose A, B and C represent sets, the statement we actually want to prove is the following:

If ((there exists $x \in A$ such that $x \notin B$) and (for each $y \in C$ $y \in B$)) then (there exists $z \in A$ such that $z \notin C$).

In order to formalize this, our language must be as follows

$$\mathcal{C} = \{A, B, C\},$$

$$\mathcal{F} = \{\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \forall, \exists, \in, =\},$$

$$\mathcal{V} = \{x, y, z\},$$

where A, B, C are constants each representing a set.

At this point we suppose we can formalize the statement as

$$\rightarrow \left(\wedge \left(\begin{array}{l} \exists\left(\{\}\left(x : A, \neg\left(\in(x, B)\right)\right)\right), \\ \forall\left(\{\}\left(y : C, \in(y, B)\right)\right) \end{array} \right), \exists\left(\{\}\left(z : A, \neg\left(\in(z, C)\right)\right)\right) \right) . \qquad (Th_1)$$

We'll soon see a proof of this statement and within the proof we'll also prove Th_1 is a sentence in our language.

First of all we need the following lemma, that can be applied to any language which includes all the symbols $\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \forall, \exists, \in$ in the set $\mathcal{F}$, and therefore it can also be applied to our current language.

Lemma 6.1. *Let m be a positive integer, $x_1, \ldots, x_m \in \mathcal{V}$, with $x_i \neq x_j$ for $i \neq j$. Let $A_1, \ldots, A_m \in \mathcal{C}$ such that for each $i = 1 \ldots m$ $\#(A_i)$ is a set. Let $D \in \mathcal{C}$ such that $\#(D)$ is a set. We have $H[x_1 : A_1, \ldots, x_m : A_m]$. If we define $k = k[x_1 : A_1, \ldots, x_m : A_m]$ then for each $i = 1 \ldots m$*

- $\in(x_i, D) \in S(k)$,
- *for each $\sigma \in \Xi(k)$ $\#(k, \in(x_i, D), \sigma) = P_{\in}(\#(k, x_i, \sigma), \#(D))$.*

Proof.

We first consider that $A_1 \in E(\epsilon)$ and $\#(A_1)$ is a set, so $A_1 \in E_s(\epsilon)$ and $H[x_1 : A_1]$. Let $k_1 = k[x_1 : A_1]$.

If $m > 1$ then for each $i = 1 \ldots m-1$ we suppose $H[x_1 : A_1, \ldots, x_i : A_i]$ holds and we define $k_i = k[x_1 : A_1, \ldots, x_i : A_i]$.
Clearly by lemma 4.24 $A_{i+1} \in E(k_i)$ and for each $\rho \in \Xi(k_i)$ $\#(k_i, A_{i+1}, \rho) = \#(A_{i+1})$ is a set.
So $A_{i+1} \in E_s(k_i)$, which implies $H[x_1 : A_1, \ldots, x_{i+1} : A_{i+1}]$ (and we can define $k_{i+1} = k[x_1 : A_1, \ldots, x_{i+1} : A_{i+1}]$).

This proves that $H[x_1 : A_1, \ldots, x_m : A_m]$ holds.

Let $i = 1 \ldots m$. Using lemma 4.25 we obtain that $x_i \in E(k)$.

Moreover $D \in E(k)$ and for each $\sigma \in \Xi(k)$ $\#(k, D, \sigma) = \#(D)$ is a set. By lemma 4.27 we have

- $\in (x_i, D) \in S(k)$,
- for each $\sigma \in \Xi(k)$ $\#(k, \in (x_i, D), \sigma) = P_{\in}(\#(k, x_i, \sigma), \#(D))$.

■

To provide a proof of statement Th_1 we'll make use of a deductive system which includes all the axioms and rules listed in chapter 5.

Using the former lemma we can derive $H[x : A]$ and we can define $h = k[x : A]$. Moreover $\in (x, B) \in S(h)$, so $\neg(\in (x, B)) \in S(h)$.

We also have $H[x : A, y : C]$ and we define $k_y = k[x : A, y : C]$.
We have $\in (y, B) \in S(k_y)$ and by lemma 4.16 $\forall(\{\}(y : C, \in (y, B))) \in S(h)$.

Thus $\wedge(\neg(\in (x, B)), \forall(\{\}(y : C, \in (y, B))))$ also belongs to $S(h)$.

Moreover $H[x : A, z : A]$ and we define $k_z = k[x : A, z : A]$.
We have $\in (z, C) \in S(k_z)$ and by lemma 4.16 $\forall(\{\}(z : A, \in (z, C))) \in S(h)$.

The first sentence in our proof is an instance of axiom $A_{5.2}$.

$$\gamma \left[x : A, \to \left(\wedge \left(\begin{array}{l} \wedge \left(\begin{array}{l} \neg(\in (x, B)), \\ \forall(\{\}(y : C, \in (y, B))) \end{array} \right), \\ \forall(\{\}(z : A, \in (z, C))) \end{array} \right), \wedge \left(\begin{array}{l} \neg(\in (x, B)), \\ \forall(\{\}(y : C, \in (y, B))) \end{array} \right) \right) \right]. \tag{6.0.1}$$

By $A_{5.2}$ we also obtain

$$\gamma \left[x : A, \to \left(\wedge \left(\begin{array}{l} \neg(\in (x, B)), \\ \forall(\{\}(y : C, \in (y, B))) \end{array} \right), \neg(\in (x, B)) \right) \right]. \tag{6.0.2}$$

By 6.0.1, 6.0.2 and rule $R_{5.3}$

$$\gamma \left[x : A, \to \left(\wedge \left(\begin{array}{l} \wedge \left(\begin{array}{l} \neg(\in (x, B)), \\ \forall(\{\}(y : C, \in (y, B))) \end{array} \right), \\ \forall(\{\}(z : A, \in (z, C))) \end{array} \right), \neg(\in (x, B)) \right) \right]. \tag{6.0.3}$$

Another instance of $A_{5.2}$ is the following

$$\gamma \left[x : A, \to \left(\wedge \left(\begin{array}{l} \wedge \left(\begin{array}{l} \neg(\in (x, B)), \\ \forall(\{\}(y : C, \in (y, B))) \end{array} \right), \\ \forall(\{\}(z : A, \in (z, C))) \end{array} \right), \forall(\{\}(z : A, \in (z, C))) \right) \right]. \tag{6.0.4}$$

By axiom $A_{5.4}$ we obtain

$$\gamma[x : A, \in (x, A)]. \tag{6.0.5}$$

By 6.0.5 and rule $R_{5.5}$ we also get

$$\gamma\left[x : A, \rightarrow\left(\wedge\left(\begin{array}{l}\wedge\left(\begin{array}{l}\neg(\in (x, B)),\\ \forall(\{\}(y : C, \in (y, B)))\end{array}\right),\\ \forall(\{\}(z : A, \in (z, C)))\end{array}\right), \in (x, A)\right)\right]. \tag{6.0.6}$$

Since $x \in E(h)$, $C \in E_s(h)$ etc. we can apply rule $R_{5.6}$ to 6.0.4 and 6.0.6 and obtain

$$\gamma\left[x : A, \rightarrow\left(\wedge\left(\begin{array}{l}\wedge\left(\begin{array}{l}\neg(\in (x, B)),\\ \forall(\{\}(y : C, \in (y, B)))\end{array}\right),\\ \forall(\{\}(z : A, \in (z, C)))\end{array}\right), \in (x, C)\right)\right]. \tag{6.0.7}$$

By axiom $A_{5.2}$

$$\gamma\left[x : A, \rightarrow\left(\wedge\left(\begin{array}{l}\neg(\in (x, B)),\\ \forall(\{\}(y : C, \in (y, B)))\end{array}\right), \forall(\{\}(y : C, \in (y, B)))\right)\right]. \tag{6.0.8}$$

By 6.0.1, 6.0.8 and rule $R_{5.3}$

$$\gamma\left[x : A, \rightarrow\left(\wedge\left(\begin{array}{l}\wedge\left(\begin{array}{l}\neg(\in (x, B)),\\ \forall(\{\}(y : C, \in (y, B)))\end{array}\right),\\ \forall(\{\}(z : A, \in (z, C)))\end{array}\right), \forall(\{\}(y : C, \in (y, B)))\right)\right]. \tag{6.0.9}$$

Since $x \in E(h)$, $B \in E_s(h)$ etc. we can apply rule $R_{5.6}$ to 6.0.7 and 6.0.9 and obtain

$$\gamma\left[x : A, \rightarrow\left(\wedge\left(\begin{array}{l}\wedge\left(\begin{array}{l}\neg(\in (x, B)),\\ \forall(\{\}(y : C, \in (y, B)))\end{array}\right),\\ \forall(\{\}(z : A, \in (z, C)))\end{array}\right), \in (x, B)\right)\right]. \tag{6.0.10}$$

By 6.0.10, 6.0.3 and $R_{5.8}$

$$\gamma\left[x : A, \rightarrow\left(\wedge\left(\begin{array}{l}\wedge\left(\begin{array}{l}\neg(\in (x, B)),\\ \forall(\{\}(y : C, \in (y, B)))\end{array}\right),\\ \forall(\{\}(z : A, \in (z, C)))\end{array}\right), \wedge\left(\begin{array}{l}\in (x, B),\\ \neg(\in (x, B))\end{array}\right)\right)\right]. \tag{6.0.11}$$

By $R_{5.9}$

$$\gamma\left[x : A, \neg\left(\wedge\left(\begin{array}{l}\wedge\left(\begin{array}{l}\neg(\in (x, B)),\\ \forall(\{\}(y : C, \in (y, B)))\end{array}\right),\\ \forall(\{\}(z : A, \in (z, C)))\end{array}\right)\right)\right]. \tag{6.0.12}$$

By $R_{5.10}$

$$\gamma\left[x : A, \rightarrow\left(\wedge\left(\begin{array}{l}\neg(\in (x, B)),\\ \forall(\{\}(y : C, \in (y, B)))\end{array}\right), \neg\left(\forall(\{\}(z : A, \in (z, C)))\right)\right)\right]. \tag{6.0.13}$$

By $R_{5.11}$

$$\gamma\left[x : A, \rightarrow\left(\wedge\left(\begin{array}{l}\neg(\in (x, B)),\\ \forall(\{\}(y : C, \in (y, B)))\end{array}\right), \exists(\{\}(z : A, \neg(\in (z, C))))\right)\right]. \tag{6.0.14}$$

Since $\exists(\{\}(z : A, \neg(\in (z, C)))) \in S(h)$ we can apply $R_{5.12}$ and obtain

$$\gamma\left[x : A, \rightarrow \left(\neg(\in (x, B)), \rightarrow \begin{pmatrix} \forall(\{\}(y : C, \in (y, B))), \\ \exists(\{\}(z : A, \neg(\in (z, C)))) \end{pmatrix}\right)\right]. \tag{6.0.15}$$

Using lemma 6.1 we obtain that $\in (y, B) \in S(k[y : C])$ and $\in (z, C) \in S(k[z : A])$.

By lemma 4.16 we obtain that $\forall(\{\}(y : C, \in (y, B))) \in S(\epsilon)$ and similarly $\exists(\{\}(z : A, \neg(\in (z, C)))) \in S(\epsilon)$.

We can apply rule $R_{5.7}$ to 6.0.15 and obtain

$$\rightarrow \left(\exists\left(\{\}(x : A, \neg(\in (x, B)))\right), \rightarrow \begin{pmatrix} \forall(\{\}(y : C, \in (y, B))), \\ \exists(\{\}(z : A, \neg(\in (z, C)))) \end{pmatrix}\right) \tag{6.0.16}$$

Finally, by $R_{5.13}$, we obtain

$$\rightarrow \left(\wedge \begin{pmatrix} \exists\left(\{\}(x : A, \neg(\in (x, B)))\right), \\ \forall(\{\}(y : C, \in (y, B))) \end{pmatrix}, \exists(\{\}(z : A, \neg(\in (z, C))))\right) \tag{6.0.17}$$

■

We have proved statement Th_1, this also means that Th_1 is a sentence in our language. It seems quite obvious that the statement's meaning is as expected, anyway to complete the argument we also want to prove this.

We need the following lemma, that can be applied to any language which includes all the symbols $\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \forall, \exists, \in$ in the set $\mathcal{F}$, and therefore it can also be applied to our current language.

LEMMA 6.2. *Let $u \in \mathcal{V}$, $D \in \mathcal{C}$ such that $\#(D)$ is a set. We have $H[u : D]$ and we can define $h = k[u : D]$. Then $u \in E(h)$ and for each $\sigma \in \Xi(h)$ $\#(h, u, \sigma) \in \#(D)$. Moreover, for each $\alpha \in \#(D)$, if we define $\sigma = \epsilon + (u, \alpha)$ then $\sigma \in \Xi(h)$ and $\#(h, u, \sigma) = \alpha$. This clearly implies $\{\#(h, u, \sigma) |\ \sigma \in \Xi(h)\} = \#(D)$.*

Proof.

We have $D \in E(1, \epsilon)$ and $\#(\epsilon, D, \epsilon)$ is a set, so $D \in E_s(1, \epsilon)$, $h = \epsilon + (u, D) \in K(1)^+$ and

$$\Xi(h) = \{\epsilon + (u, s) |\ s \in \#(\epsilon, D, \epsilon)\} = \{\epsilon + (u, s) |\ s \in \#(D)\}\,.$$

We have also $E_a(2, h) = \{u\}$ and for each $\sigma = \epsilon + (u, s) \in \Xi(h)$

$$\#(h, u, \sigma)_{(2,h,a)} = s \in \#(D)\ .$$

Since $h \in K(1)^+$, $E(2, h) = E_a(2, h)$, so $u \in E(2, h)$, for each $\sigma = \epsilon + (u, s) \in \Xi(h)$

$$\#(h, u, \sigma) = \#(h, u, \sigma)_{(2,h,a)} = s \in \#(D)\ .$$

Given $\alpha \in \#(D)$, if we define $\sigma = \epsilon + (u, \alpha)$ then $\sigma \in \Xi(h)$ and $\#(h, u, \sigma) = \alpha$. ■

We now examine the meaning of $\exists\,(\{\}(x:A,\neg(\in(x,B))))$.

We can rewrite $\#(\exists\,(\{\}(x:A,\neg(\in(x,B)))))$ as

$$P_\exists(\{\#(k[x:A],\neg(\in(x,B)),\sigma)|\ \sigma\in\Xi(k[x:A])\})\ ,$$

$$P_\exists(\{P_\neg(\#(k[x:A],\in(x,B),\sigma))|\ \sigma\in\Xi(k[x:A])\})\ ,$$

$$P_\exists(\{P_\neg(P_\in(\#(k[x:A],x,\sigma),\#(B)))|\ \sigma\in\Xi(k[x:A])\})\ .$$

This can be furtherly expressed as

'there exists $\sigma\in\Xi(k[x:A])$ such that $P_\neg(P_\in(\#(k[x:A],x,\sigma),\#(B)))$',

which is the same as

'there exists $\alpha_x\in\{\#(k[x:A],x,\sigma)|\,\sigma\in\Xi(k[x:A])\}$ such that $P_\neg(P_\in(\alpha_x,\#(B)))$',
'there exists $\alpha_x\in\#(A)$ such that $P_\neg(P_\in(\alpha_x,\#(B)))$',
'there exists $\alpha_x\in\#(A)$ such that α_x doesn't belong to $\#(B)$'.

Similarly we can rewrite $\#(\forall(\{\}(y:C,\in(y,B))))$ as

$$P_\forall(\{\#(k[y:C],\in(y,B),\sigma)|\ \sigma\in\Xi(k[y:C])\})$$

$$P_\forall(\{P_\in(\#(k[y:C],y,\sigma),\#(B))|\ \sigma\in\Xi(k[y:C])\})$$

This can be furtherly expressed as

'for each $\sigma\in\Xi(k[y:C])$ $P_\in(\#(k[y:C],y,\sigma),\#(B))$',

which is the same as

'for each $\alpha_y\in\{\#(k[y:C],y,\sigma)|\,\sigma\in\Xi(k[y:C])\}$ $P_\in(\alpha_y,\#(B))$',
'for each $\alpha_y\in\#(C)$ $P_\in(\alpha_y,\#(B))$',
'for each $\alpha_y\in\#(C)$ α_y belongs to $\#(B)$'.

Similarly we can also rewrite $\#(\exists\,(\{\}(z:A,\neg(\in(z,C)))))$ as

$$P_\exists(\{\#(k[z:A],\neg(\in(z,C)),\sigma)|\ \sigma\in\Xi(k[z:A])\})\ ,$$

$$P_\exists(\{P_\neg(\#(k[z:A],\in(z,C),\sigma))|\ \sigma\in\Xi(k[z:A])\})\ ,$$

$$P_\exists(\{P_\neg(P_\in(\#(k[z:A],z,\sigma),\#(C)))|\ \sigma\in\Xi(k[z:A])\})\ .$$

This can be furtherly expressed as

'there exists $\sigma\in\Xi(k[z:A])$ such that $P_\neg(P_\in(\#(k[z:A],z,\sigma),\#(C)))$',

which is the same as

'there exists $\alpha_z\in\{\#(k[z:A],z,\sigma)|\,\sigma\in\Xi(k[z:A])\}$ such that $P_\neg(P_\in(\alpha_z,\#(C)))$',
'there exists $\alpha_z\in\#(A)$ such that $P_\neg(P_\in(\alpha_z,\#(C)))$',
'there exists $\alpha_z\in\#(A)$ such that α_z doesn't belong to $\#(C)$'.

At this point we can rewrite

$$\#\left(\to\left(\wedge\left(\begin{array}{l}\exists\,(\{\}(x:A,\neg(\in(x,B))))\,,\\ \forall(\{\}(y:C,\in(y,B)))\end{array}\right),\exists(\{\}(z:A,\neg(\in(z,C))))\right)\right)$$

as

$$P_{\to}\left(\#\left(\wedge\left(\begin{array}{l}\exists\,(\{\}(x:A,\neg(\in(x,B))))\,,\\ \forall(\{\}(y:C,\in(y,B)))\end{array}\right)\right),\#\,(\exists(\{\}(z:A,\neg(\in(z,C)))))\right)$$

and then

$$P_{\to}\left(P_{\wedge}\left(\begin{array}{l}\#\,(\exists\,(\{\}(x:A,\neg(\in(x,B)))))\,,\\ \#\,(\forall(\{\}(y:C,\in(y,B))))\end{array}\right),\#\,(\exists(\{\}(z:A,\neg(\in(z,C)))))\right)$$

This can be furtherly expressed as:

‘if (there exists $\alpha_x \in \#(A)$ such that α_x doesn’t belong to $\#(B)$) and (for each $\alpha_y \in \#(C)$ α_y belongs to $\#(B)$) then (there exists $\alpha_z \in \#(A)$ such that α_z doesn’t belong to $\#(C)$)’.

So the statement which we have proved has the expected meaning.

7. Consistency, paradoxes and further study

We have proved that a deductive system is sound, i.e. if we can derive a sentence φ in our system then $\#(\varphi)$ holds. We now discuss the consistency of a deductive system.

A deductive system $\mathcal{D} = (\mathcal{A}, \mathcal{R})$ is said to be *consistent* if and only if for each φ sentence in $\mathcal{L}$ $(\vdash_{\mathcal{D}} \varphi)$ and $(\vdash_{\mathcal{D}} \neg(\varphi))$ aren't both true.

LEMMA 7.1. *Let $\mathcal{D} = (\mathcal{A}, \mathcal{R})$ be a deductive system in $\mathcal{L}$. Then $\mathcal{D}$ is consistent.*

Proof.

Suppose there exists a sentence φ such that $\vdash_{\mathcal{D}} \varphi$ and $\vdash_{\mathcal{D}} \neg(\varphi)$ both hold. By the soundness property we have $\#(\varphi)$ and $\#(\neg(\varphi))$. Clearly

$$\#(\neg(\varphi)) = \#(\epsilon, \neg(\varphi), \epsilon) = P_{\neg}(\#(\varphi)) = \#(\varphi) \text{ is false.}$$

So $\#(\varphi)$ would be true and false at the same time, a plain contradiction. ∎

We have proved the soundness of our system, then what can we say about the converse property, i.e. the completeness? A deductive system $\mathcal{D} = (\mathcal{A}, \mathcal{R})$ is said to be *complete* if and only if for each φ sentence in $\mathcal{L}$ if $\#(\varphi)$ holds then $\vdash_{\mathcal{D}} \varphi$. It was easy to prove the soundness of our system, unfortunately the topic of completeness is not as easy, and in general there is no reason to expect that completeness holds. For instance Cutland's book [1] has interesting material in this regard, in chapter 8. Actually Cutland introduces a notion of 'recursively axiomatised formal system' and what he names a 'simplified version of Gödel incompleteness theorem'. This theorem states that, given a recursively axiomatised formal system in which all provable statements are true, in this system there is a statement which is true but not provable (and so this system is not complete). The proof of this theorem is based on the fact that the set $\mathcal{P}$ of the provable statements of the system is recursively enumerable (r.e.) while the set $\mathcal{T}$ of the true statements of the system is not r.e.. So if we could prove that our system is a recursively axiomatised formal system we would have proved that our system is not complete. If we aren't able to prove our system is a 'recursively axiomatised formal system' we can still try to show that in our system (under certain further assumptions, perhaps) it still holds that $\mathcal{P}$ is r.e. and $\mathcal{T}$ is not r.e., so the system is not complete. Anyway I'll leave this topic just at most as a conjecture, I don't want to prove or even claim anything in this regard.

Let's talk a bit about paradoxes. A paradox is usually a situation in which a contradiction or inconsistency occurs, in other words a paradox arises when we can build a sentence φ such that both φ and $\neg(\varphi)$ can be derived. Since our system is consistent it shouldn't be possible to have true paradoxes in it, anyway it seems appropriate to discuss how our system relates with some of the most famous paradoxical arguments.

We begin with Russell's paradox. Assume we can build the set A of all those sets X such that X is not a member of X. Clearly, if $A \in A$ then $A \notin A$ and conversely if $A \notin A$

then $A \in A$. We have proved both $A \in A$ and its negation, and this is the Russell's paradox.
It seems in our system we cannot generate this paradox since building a set is permitted only if you rely on already defined sets. When trying to build set A in our language we could obtain something like this:

$$\{\}(\neg(\in(X,X)),X)\ .$$

However it is clear this isn't a legal expression in our language, since in our language if you want to build a context-independent expression using a variable X, then you have to assign a domain to X.

We now turn to Cantor's paradox. Often the wording of this paradox involves the theory of cardinal numbers (see e.g. Mendelson's book [4]), but here we use a simpler wording.

First of all we prove that for each set A there doesn't exist a surjective function with domain A and codomain $\mathcal{P}(A)$ (where $\mathcal{P}(A)$ is the power set of A).

This clearly holds when $A = \emptyset$. In fact in this case $\mathcal{P}(A) = \{\emptyset\}$ and for each function $f : \emptyset \to \{\emptyset\}$ there doesn't exist $y \in \emptyset$ such that $\emptyset = f(y)$.

Let $A \neq \emptyset$ and let f be a function from A to $\mathcal{P}(A)$. Let $B = \{x \in A | x \notin f(x)\}$.
Suppose there exists $y \in A$ such that $B = f(y)$. If $y \in B$ then $y \notin f(y) = B$, and conversely if $y \notin B = f(y)$ then $y \in B$. So there isn't $y \in A$ such that $B = f(y)$ and therefore f is not surjective.

At this point, suppose there exists a set Ω such that any member of Ω is a set and any set is a member of Ω. Clearly Ω and all of its subsets belong to Ω, so we can define a function f from Ω to $\mathcal{P}(\Omega)$ such that for each $X \subseteq \Omega$ $f(X) = X$. Obviously this is a surjective function, and we have a contradiction.

The contradiction is due to having assumed the existence of Ω. In this case too in our language we cannot build an expression with such meaning. One expression like the following:

$$\{\}(set(X),X)$$

is not a valid expression in our language.

Finally we want to examine the liar paradox. Let's consider how the paradox is stated in Mendelson's book.

A man says, 'I am lying'. If he is lying, then what he says is true, so he is not lying. If he is not lying, then what he says is false, so he is lying. In any case, he is lying and he is not lying.

Mendelson classifies this paradox as a 'semantic paradox' because it makes use of concepts which need not occur within our standard mathematical language. I agree that, in his formulation, the paradox has some step which seems not mathematically rigorous.

We'll try to provide a more rigorous wording of the paradox.

Let A be a set, and let δ be the condition 'for each x in A x is false'. Suppose δ is the only member of A. In this case if δ is true then it is false; if on the contrary δ is false then it is true.

The explanation of the paradox is the following: simply δ cannot be the only item in set A. In fact, suppose A has only one element, and let's call it φ. This implies δ is equivalent to 'φ is false' so it seems acceptable that δ is not φ.

Another approach to the explanation is the following.

If δ is true then for each x in A x is false, so δ is not in A. By contraposition if δ is in A then δ is false.

Moreover if δ is false and the uniqueness condition 'for each x in A $x = \delta$' is true then δ is true, thus if δ is false then 'for each x in A $x = \delta$' is false too. By contraposition if 'for each x in A $x = \delta$' then δ is true.

Therefore if δ is the only element in A then δ is true and false at the same time. This implies δ cannot be the only item in A.

On the basis of this argument I consider the liar paradox as an apparent paradox that actually has an explanation. What is the relation between our approach to logic and the liar paradox?

Standard logic isn't very suitable to express this paradox. In fact first-order logic is not designed to construct a condition like our condition δ (= 'for each x in A x is false'), and moreover, it is clearly not designed to say 'δ belongs to set A'. These conditions aren't plainly leading to inconsistency, so it is desirable they can be expressed in a general approach to logic. And our system permits to express them. The paradox isn't ought to simply using these conditions, it is due to an assumption that is clearly false, and the so-called paradox is simply the proof of its falseness.

Related to the liar paradox is the Cretan 'paradox', which is actually not a proper paradox, but is perhaps even more 'unsettling' and we quote again Mendelson in this regard: ([4]).

> The Cretan "paradox", known in antiquity, is similar to the Liar Paradox. The Cretan philosopher Epimenides said, "All Cretans are liars". If what he said is true, then, since Epimenides is a Cretan, it must be false. Hence, what he said is false. Thus, there must be some Cretan who is not a liar. This is not logically impossible, so we do not have a genuine paradox. However, the fact that the utterance by Epimenides of that false sentence could imply the existence of some Cretan who is not a liar is rather unsettling.

If we try to put this argument in a more formal statement, it still refers to a sentence δ of the type 'for each x in A x is false', where this time A is the set of all the statements made by a Cretan and δ is a member of A. Here if δ is true then it is false, so we have to conclude that δ is false, hence there exists $x \in A$ such that x is true. As noticed by Mendelson, it can be unsettling to accept this just because δ is a member of A.

We can still use an argument we have shown above with respect to the liar paradox: If δ is true then for each x in A x is false, so δ is not in A. By contraposition if δ is in A then δ is false.

Is it still disturbing that δ is false? If it is, then we can consider that the problem could be conceiving a set A where a statement δ states that all the statements in A are false and we expect δ belongs to A. Conceiving a set like this seems not a cleaner and more founded idea than conceiving the set of all sets or the set of all sets that aren't members of themselves.

As a conclusion, with respect to paradoxes, our system is not specifically designed to prevent any possible form of paradox, it doesn't prevent anyone to conceive something which is unsettling or contradictory. Anyway the system, as far as I can evaluate, is not significantly affected by paradoxes, in other words I suppose it is not more vulnerable to paradoxes than other accepted systems.

Of course, further investigations about our approach to logic can be performed, both with respect to paradoxes and other topics. We have mentioned the topic on the completeness or incompleteness of the system. Another interesting (and not extremely easy) topic is about comparing the expressive power of our system with the one of standard logic systems.

Another topic to consider is substitution. First-order logic features the notion of 'substitution' (see e.g. Enderton's book [2]). Under appropriate assumptions, we can apply substitution to a formula φ and obtain a new formula φ^x_t, by replacing the free occurrences of the variable x by the term t. In our approach we could be able to define a similar notion, with the difference that for us t could be a generic expression. I have introduced general mechanisms of substitution in former versions of this paper (e.g. [5]), so I am rather confident they could be successfully integrated in this new version. Anyway

I decided not to try to include them in this version because it would have been complex, it would have required much time. After all I suppose the introduction of general substitution mechanisms could be considered as not being properly a core topic about this approach, since for instance we can use simplified substitution mechanisms.

References

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