A Lagrangian for the Lambda CDM Model.

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The Lambda CDM model or "concordance model" is the standard model of modern cosmology. This model contains a number of separate theories with different mathematical formulations. The subject of this paper is a proposed Lagrangian which would provide a unified mathematical framework for the concordance model of cosmology. In the process new insight has been gained into the nature of dark matter and dark energy which the separate formulations do not provide. This single Lagrangian contains General Relativity and fields for both scalar and vector inflation. When these inflation fields are treated as quantum fields it becomes clear each will have a particle associated with it which would be a candidate for dark matter. If this Lagrangian is validated by observations it could serve as a unified mathematical basis for LCDM which prior to this exist as several disconnected elements of theory.

I. INTRODUCTION

The ΛCDM model or "concordance model" is the standard model of modern cosmology. This model contains a number of separate theories with different mathematical formulations. The subject of this paper is a proposed Lagrangian which would provide a unified mathematical framework for the concordance model of cosmology. In the process new insight can be gained into the nature of dark matter and dark energy which the separate formulations do not provide.

Much like the state of particle physics in about the mid 1970's cosmologies standard model exist as a number of elements of theory without a unified description. In particle physics there is now one Lagrangian which can describe all the physics of the weak force, strong force, and electromagnetism. Each of these have Lagrangian's which can be written separately however the unified description is now the preferred one. Having such a unified mathematical model has benefited theoretical and experimental particle physics immensely. My motivation for writing this paper is to provide a similar mathematical basis for Lambda CDM. Simply by writing these together new physical insight can be gained if it is done correctly.

There are certain mysteries to the standard model of cosmology. It contains vast amounts of matter and energy of an unknown energy. Matter which we cannot detect in spite of massive efforts such as the cryogenic dark matter search II (CDMS II)[3]. Energy which we can only detect by it's effect on the acceleration of the expansion of the universe.... then model with a constant Λ. This simple model makes very good predictions however coming from a background of theoretical particle physics it is mathematically unsatisfying. There has to be a mathematically more elegant formulation than the current collection of no less than three very different parts (depending on how one counts) which make up the current model. The following outlines an attempt at such a model. Only time, observation, and experimentation can validate or falsify it.

II. THE LAGRANGIAN

We have not observed any dark matter particles on Earth to date. The best result we have is a signal with much noise[3]. One way to explain this would be to have dark matter decay as the Ricci curvature increases. *This means that the dark matter-energy fields should grow weak as R increases and strong as R decreases.* The fields precise behavior will depend on which metric and hence which R is in effect. In the case of a galaxy the Schwarzchild R would be used, in the case of the universe the FLRW R would be used. This effect would naturally explain *why* a spherical halo of dark matter would be expected at a characteristic distance from a galaxy as R in the Schwarzchild metric depends on radial distance r. In that regime a higher r, means a lower R which would mean a longer life for dark matter particles at cosmological distances from concentrations of mass.

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Reviewing the published literature it contains Lagrangian's for inflation, dark matter, dark energy, alternative gravity, etc etc. The standard formulation of Lambda CDM would consist of Einsteins field equations, a Lagrangian for inflation, another one for dark matter, and another one for dark energy [5].

Based on the above I propose the following deceptively simple looking Lagrangian for the Lambda CDM model.

$$s=\int\sqrt{-g}\left(-\frac{R}{16\pi}-k(\phi)\nabla^{\mu}\phi\nabla_{\mu}\phi-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}-\frac{1}{2}\left(m_{\phi}^{2}\phi^{2}+m_{A}^{2}A^{\mu}A_{\mu}\right)-\frac{R}{6}\left(\phi^{2}+A^{\mu}A_{\mu}\right)\right)d^{4}x \tag{1}$$

$$A^{\mu}=A^{\mu}\left(R\right),\phi=\phi\left(R\right) \tag{2}$$

Where R is the Ricci curvature. Using functions of the Ricci curvature has done in a program known as f of R gravity. Here I have identified the functions f with the scalar and vector fields of inflation and assumed that said fields were massive.

This Lagrangian should look familiar if one looks at it as a combination of the Lagrangian for a massive scalar field in curved space-time[2], and a massive vector field also in curved space time (much like the Proca Lagrangian for a massive photon). These fields are found in theories of inflation in which they drive the rapid expansion[1, 4, 6–10]. The details of how these fields would drive inflation, their slow roll parameters, etc, are given in those citations. Even though these may look the same, they are not, ϕ and A are not merely fields of inflation in this model. These fields are with us today in the form of dark matter and dark energy. Whereas in models of inflation they decay away during reheating and totally disappear.

III. EQUATIONS OF MOTION.

Following all the elementary steps of classical field theory the Euler-Lagrange equations for this theory can be derived. One of those equations is for R itself. That is none other than the Einstein field equation. Then there are two more equations one for the scalar and one for the vector fields. One more constraint is desirable. The Stress energy tensor of this field must be proportional to the cosmological constant. This ensures agreement with known observations. The result is a set of four equations, derived from the above Lagrangian encode all the physics of the Lambda CDM model.

$$\left\{\begin{array}{c}R^{\mu\nu}-Rg^{\mu\nu}=\frac{8\pi G}{c^{2}}T^{\mu\nu}\\ \nabla_{\alpha}F^{\alpha\mu}-\left(\frac{m_{A}^{2}}{2}+\frac{R}{6}\right)A^{\mu}=0\\ \nabla_{\mu}\nabla^{\mu}\phi-\left(\frac{m_{\phi}^{2}}{2}+\frac{R}{6}\right)\phi=0\\ T^{\mu\nu}=\lambda g^{\mu\nu}\Lambda\end{array}\right\} \tag{3}$$

Where the stress energy tensor is as follows.

$$\begin{array}{c}T^{\mu\nu}=-2k(\phi)\nabla^{\mu}\phi\nabla^{\nu}\phi-F^{\mu\nu}g_{\lambda\delta}F^{\lambda\delta}-\left(m_{A}^{2}+\frac{R}{3}\right)A^{\mu}A^{\nu}\\ -g^{\mu\nu}\left(k(\phi)\nabla_{\mu}\phi\nabla^{\mu}\phi+F^{\mu\nu}F_{\mu\nu}-\frac{1}{2}\left(m_{\phi}^{2}\phi^{2}+m_{A}A^{\mu}A_{\nu}\right)-\frac{R}{6}\left(\phi^{2}+A^{\mu}A_{\mu}\right)\right)\end{array} \tag{4}$$

The next task is to solve these equations for the scalar and vector field. First the scalar field's solution.

$$\phi\left(R\right)=\phi_{0}Exp\left(\int_{1}^{R}\frac{\left(\frac{m_{\phi}^{2}}{2}+\frac{R'}{6}\right)}{\Box R'}dR'\right) \tag{5}$$

For the vector field the A^{0} component must be zero in order to satisfy the equation of motion. The derivatives which make up F^{00} work out that way just as one would expect for an electromagnetism like field. In the process of solving for A^{0} the mass of the A field can be calculated.

$$\frac{m_{A}^{2}}{2}+\frac{R}{6}=0\rightarrow m_{A}=\sqrt{\frac{-R}{3}} \tag{6}$$

That could be a problem however in the universe we observe R is either zero, for a flat space time, or negative for a negatively curved space time. This means that in a totally flat space time the mass of this field vanishes. The universe is never the less not perfectly flat, it has a slight negative curvature which makes the quantity under the square root positive and gives this field a small but real mass. The negativity of R ensures the physical behavior of the fields which matches our observations of dark matter particles in particle physics experiments (or the lack of our observations in said experiments to be precise)[3].

For the space like components the solution is almost identical to that for the scalar field.

$$A^{\mu} = \left(0,\; A_0^i Exp\left(\int_1^R \frac{\left(\frac{m_A^2}{2} + \frac{R'}{6}\right)}{\Box R'} dR'\right)\right) \tag{7}$$

Wherein $i \in \{1, 2, 3\}$.

The masses of these fields are fixed theoretically by the constraint that the stress energy tensor $T^{\mu\nu}$ needs to be proportional to the cosmological constant. It is possible to determine the mass m_ϕ from that constraint. To find an expression for this mass note that the T^{00} component of the stress energy tensor will be of a simple form. Terms which depend on the vector field drop out as it's zero in that component. Terms which depend on the velocity $\nabla^0\phi$ can be set to zero to ensure the resulting mass is the rest mass of the particle. The resulting equation is

$$T^{00} = g^{00}\left(\frac{1}{2}m_\phi^2 + \frac{R}{6}\right)\phi^2 = \lambda g^{00}\Lambda \tag{8}$$

Which simplifies to...

$$m_\phi = \sqrt{\frac{6\lambda\Lambda - R\phi}{3\phi}} \tag{9}$$

Unlike the mass of the vector field the mass of this scalar field cannot be zero, ever. Further note that for the observed universe R will either be negative or zero which means the mass can only be real. Never the less measured values for the cosmological constant and the density and hence curvature of the universe are needed to arrive at numbers for these masses.

Elevating this model to a quantum field theory in curved space time would follow a straight forward recipe laid out in[2, 11]. The Lagrangian has already been written in a manifestly covariant form which will be valid on any manifold of any curvature. The result will be that there is not one unique vacuum state with which to define the number of particles. One could only speak of particles existing at a certain point in space time with a particular value for R. The results of quantizing a field and counting particles are well known as the Unruh effect. So while a detector buried deep within the Earth would detect hardly any particles, another detector floating in space (as impractical as it would be for reasons of cost and noise) would have a better chance of detecting the elusive dark matter. That is not to say that experiments such as CDMS II are a total waste of time but that dark matter may prove even harder to detect than previously thought.

IV. CONCLUSIONS

The proposed Lagrangian contains all the physics needed to represent the Lambda CDM model. There is a source of dark matter, dark energy, and inflation. The behavior of the fields is in agreement with our overall observations . This Lagrangian also provides a minimal explanation for why dark matter has been so hard to observe in experiments such as CDMS II. The dark matter simply decays into dark energy when the curvature R is too high. Thus there are not "particles" to detect in a region of high space time curvature, like on Earth. This would provide an explanation for why it would be harder than expected to detect these particles in a ground based experiment.

The dark matter mass in this theory is simply the mass of the fields and their associated bosonic particles. There may well be other fermionic and super symmetric types of dark matter. There certainly numerous particles which will be discovered at accelerator laboratories in the future which may or may not be dark matter candidates. I have no hypothesis about such dark matter, or how the hypothesized particles could be produced via accelerator based experiments in this model at this time. Needless to say their interactions with ordinary matter could only be mediated by gravity and the weak force. Such is the very definition of dark matter. These particles would also decay very rapidly in any experiment where they were produced.

This theory also provides a more mathematically satisfying formulation for the dark energy. Instead of a mere constant the dark energy is represented by a proper stress energy tensor. The advantage of this formulation is that the physics of the model can be studied in different regimes of curvature. One can now ask how the model would behave when the curvature was very high, as in near the big bang, or when the curvature will be very negative as in the distant future.

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