Abstraction Theory Central

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Abstract:

Making use of the laws of physical transactions, we study symmetrical many-points systems. Relation of group-theory to physical transactions in such symmetrical systems is dealt with. Studying perturbations in the stability states in the attractor-maps for transactions, approximate values of the observables are to be predicted for such systems. Further, Abstraction Theory is typified with respect to studying the properties of irreducible representations, if any, inside a given such group.

Introduction:

In previous papers , the laws of physical transactions, in the light of Theory of Abstraction have been formulated. The mother equation (F= $\omega \frac{\lambda ST}{DR}$) describes the physical transactions taking place between two given points or between a given set of points in the vicinity of a concerned environment. A given point is influenced by its environment. On the other, it influences its concerned environment. A given point has some intrinsic properties. A group of such points form a field of extrinsic properties. The field of extrinsic properties, in turn, may influence the intrinsic characters of each of the individual points.

A set of points with same or of a similar-set of properties may be considered to belong to a given same group. For any given system, there can be one or a number of stability-states or symmetries. Further, each of such symmetries may have perturbations, affecting the average value of observable quantities. Measures of such perturbations are a useful way of finding approximate functions for systems when we know the exact transaction-functions for similar systems.

Average value of observables:

Let, f (λ, D) be a transaction function for a system, where λ is the difference in concentrations of a given observable quantity between two given points of transaction and D the distance between the points. Let, f_1,f_2, $f_{3,}$....,f_n be the complete orthonormal set of eigenfunctions for an operator Ô corresponding to some observable quantity in the system. f can be expanded such that:

$$f \;=\; j_1 f_1 \;+\; j_2 f_2 \;+\; j_3 f_3 + \ldots + j_n f_n \qquad \ldots(1)$$

where $j_1, j_2, j_3, \ldots, j_n$ are constants.

Operated with Ô from the left on both sides yield,

$$\hat{O}f = j_1\hat{O}f_1 + j_2\hat{O}f_2 + j_3\hat{O}f_3 + \ldots + j_n\hat{O}f_n$$

$f_1, f_2, f_3, \ldots, f_n$ being eigenfunctions of Ô, we can write

$$\hat{O}f = j_1\mathrm{k}_1 f_1 + j_2\mathrm{k}_2 f_2 + j_3\mathrm{k}_3 f_3 + \ldots + j_n\mathrm{k}_\mathrm{n} f_n$$

where $k_1, k_2, k_3, \ldots, k_n$ are the eigenvalues corresponding to the eigenfunctions.

Considering the complex conjugate of the transaction-function *f* in equation (1), we have,

$$f^* = j_1{}^* f_1{}^* + j_2{}^* f_2{}^* + j_3^* f_3{}^* + \ldots + j_n{}^* f_n{}^* \qquad \ldots(2)$$

Using these equations, after integrating over all co-ordinate space, we get:

$$\int f^*\hat{O}f\,dt = j_1{}^* j_1 \mathrm{k}_1 \int f_1{}^* f_1\,dt + j_2{}^* j_2 \mathrm{k}_2 \int f_2{}^* f_2\,dt + \ldots + j_n{}^* j_n \mathrm{k}_\mathrm{n} \int f_n{}^* f_n dt \qquad \ldots(3)$$

In this equation, we have got rid of all the terms of the type $j_a^* j_b \hat{O}_b \int f_a^* f_b\,dt$, as these are all zero because of the orthogonality of the eigenfunctions. Only when $a = b$, are all the terms non-zero and these are the ones we have retained. The integrals on the right side of equation (3) are each equal to one because of the normality condition. Therefore, we write,

$$\int f^*\hat{O}f\,dt = j_1{}^* j_1 \mathrm{k}_1 + j_2{}^* j_2 \mathrm{k}_2 + \ldots + j_n{}^* j_n \mathrm{k}_\mathrm{n}$$

When the system is in the state *f,* the average value (ā) of the observable *k* is given by the right-hand side of the previous equation, such that,

$$\bar{a} = \int f^*\hat{O}f\,dt \qquad \ldots(4)$$

<u>Relation of group theory to physical transactions in symmetrical systems:</u>

Say a given dynamic system has a given set of symmetries or stability points. For all points having similar intrinsic properties within such a system, the probability densities of occurrence are equal and must remain unaltered, being all in a similar environment. Thus the energy and Hamiltonian for the system must not change. If E_i is the energy corresponding to the eigenfunction f_i, we may write:

$$\hat{H}f_i = E_i f_i$$

If a symmetry operation ($\hat{x}$) is performed on the system, we have,

$$\hat{x}\,\hat{H}f_i = \hat{x}\,E_i f_i$$

But since $\hat{x}$ does not affect $\hat{H}$ or E, we may write,

$$\hat{H}\,(\hat{x}f_i) = E_i\,(\hat{x}f_i)$$

The function $\hat{x}\, f_i$ is therefore an eigenfunction of $\hat{\mathrm{H}}$ with the same eigenvalues as f_i . We can therefore conclude, if the state is non-degenerate, for normalized functions,

$$\hat{x} f_i \ = \pm\, f_i \qquad \ldots (5)$$

If a state (f_{in}) is degenerate with two or more eigenfunctions corresponding to a given energy, the energy can remain the same under the symmetry operation provided the original eigenfunction is transformed into a linear combination of the degenerate functions. For a l-fold degenerate state,

$$\hat{x}\, f_{in} = \sum_{m=1}^{l} r_{mn}\, f_{in} \qquad \ldots (6)$$

where r_{mn} are the coefficients of the linear combination. They form representation matrices expressing the effect of the symmetry operations on the set of degenerate eigenfunctions f_{in} . The representation is irreducible.

This result is important as it relates the eigenfunctions of a system to its symmetry. It limits the forms of the eigenfunctions a symmetrical system can have.

Properties of Irreducible Representations of A Group:

The sum of the squares of the dimensions of the irreducible representations of a group is equal to the order of the group, in accordance with group-theory. The characters of the irreducible representations of a group behave as orthogonal vectors in an h-dimensional space; h being the order of the concerned group. If we label the character of the x^{th} symmetry operation in the i^{th} irreducible representation $y_i(x)$, this means that,

$$\sum_x y_i(x)\; y_j(x) = 0\,;\text{if }\ i \neq j \qquad \ldots (7)$$

If we are dealing with complex characters, this equation would read

$$\sum_x [y_i(x)]^*\; y_j(x) = 0$$

If $\ i = j$, then,

$$\sum_x [y_i(x)]^2 \ = h \qquad \ldots (8)$$

Combining equations (7) and (8), we get,

$$\sum_x y_i(x)\; y_j(x) = h\delta_{ij} \qquad \ldots (9)$$

where δ_{ij} is the Kronecker delta.

Equation (9) expresses a necessary and sufficient condition that a representation is irreducible.

As the character of a given matrix remains the same after a similarity operation, for a particular similarity operation x, the sum of the characters of all the irreducible representations, we obtain from a reducible representation, is equal to the character of the reducible representation, such that,

$$y(x) = \sum_i a_i y_i(x)$$

where a_i is the number of times the i^{th} irreducible representation occurs in the reducible representation.

Multiplying this equation for $y_i(x)$ and summing over all operations, we get,

$$\sum_x y(x) y_j(x) = \sum_i \sum_x a_i y_i(x) y_j(x) \qquad \ldots (10)$$

Substituting equation (9) into the right-side of equation (10), we have:

$$\sum_x y(x) y_j(x) = h a_j$$

a_j being the only remaining coefficient because the right-side of equation (10) is zero if $i \neq j$,

$$a_j = \frac{1}{h} \sum_x y(x) y_j(x) \qquad \ldots (11)$$

A reducible representation can be broken down into irreducible representations. Equation (11) gives a method of finding the number of times each irreducible representation occurs in a reducible representation.

Stability states in The Attractor-Maps For A Many-Points System:

Let a given system have (N) stability states or symmetries. (N) equals the number of types of intrinsic properties inside the system, i.e., N equals the number of groups inside the system. One possibility for the product-function for transactions may be written as,

$$f' = f_1(1)\, f_2(2)\, f_3(3), \ldots, f_n(N);$$

where f_1, f_2, f_3,..., f_n are the intrinsic transaction-functions for the groups 1, 2, 3,...,N, respectively.

As each of the stability states are otherwise indistinguishable (all being stability states), exchange of the co-ordinates of the stability states 1 and 2 amongst the intrinsic property groups will yield an equally good function,

$$f'' = f_1(2)\, f_2(1)\, f_3(3), \ldots, f_N(N).$$

The number of functions of this type that can be written is N!, allowing for all possible exchanges of the co-ordinates of the stability states. The Slater determinant for an N-stability system is,

$$f = \frac{1}{\sqrt{N!}} \begin{vmatrix} f_1(1) & f_2(1) \dots & f_N(1) \\ f_1(2) & f_2(2) \dots & f_N(2) \\ \dots \dots & \dots \dots & \dots \dots \\ f_1(N) & f_2(N) \dots & f_n(N) \end{vmatrix} \qquad \dots (12)$$

where $\frac{1}{\sqrt{N!}}$ is a normalization factor.

Time Independent Perturbations:

Perturbation may be a useful way of predicting approximate functions for systems when we know the exact transaction-functions for similar systems. Let us consider a system for which we know the transaction-function f_i° and corresponding energy E_i°. These functions satisfy the equation,

$$\hat{\mathrm{H}}^\circ f_i^\circ = E_i^\circ f_i^\circ$$

Let, there be a small perturbation that changes the functions to f_i ; the energies change to E_i. Let, the Hamiltonian for the perturbed system be $\hat{\mathrm{H}}$. We can then write:

$$\hat{\mathrm{H}} f_i = E_i f_i \qquad \dots (13)$$

As the perturbation tends to zero, f_i tends to f_i°. We can therefore write,

$$f_i = f_i^\circ + \sum_{j \neq i} a_{ij} f_j^\circ \qquad \dots (14)$$

where a_{ij} are constants.

Substituting equations (14) into equation (13), and rearranging, we get,

$$\left(\hat{\mathrm{H}} - E_i\right) f_i^\circ + \sum_{j \neq i} a_{ij} \left(\hat{\mathrm{H}} - E_i\right) f_j^\circ = 0$$

We write the Hamiltonian as the sum of two parts, the unperturbed Hamiltonian $\hat{\mathrm{H}}^\circ$ and a perturbation term $\hat{\mathrm{H}}'$; i.e., $\hat{\mathrm{H}} = \hat{\mathrm{H}}^\circ + \hat{\mathrm{H}}'$. Substituting this relation into the last equation, we get,

$$\left(\hat{\mathrm{H}}^\circ + \hat{\mathrm{H}}' - E_i\right) f_i^\circ + \sum_{j \neq i} a_{ij} \left(\hat{\mathrm{H}}^\circ + \hat{\mathrm{H}}' - E_i\right) f_j^\circ = 0$$

As, $\hat{\mathrm{H}}^\circ f_i^\circ = E_i^\circ f_i^\circ$ and $\hat{\mathrm{H}}^\circ f_j^\circ = E_j^\circ f_i^\circ$, the last equation becomes,

$$\left(E_i^\circ + \hat{\mathrm{H}}' - E_i\right) f_i^\circ + \sum_{j \neq i} a_{ij} \left(E_j^\circ + \hat{\mathrm{H}}' - E_i\right) f_j^\circ = 0$$

Multiplying by $f_i^{\circ *}$ from the left and integrating over all space, we get,

$$E_i^\circ \int f_i^{\circ *} f_i^\circ dt + \int f_i^{\circ *} \hat{\mathrm{H}}' f_i^\circ dt - E_i \int f_i^{\circ *} f_i^\circ dt + \sum_{j \neq i} E_j^\circ a_{ij} \int f_i^{\circ *} f_j^\circ dt + \sum_{j \neq i} a_{ij} \int f_i^{\circ *} \hat{\mathrm{H}}' f_j^\circ dt$$
$$- \sum_{j \neq i} E_i a_{ij} \int f_i^{\circ *} f_j^\circ dt = 0$$

Because of the orthonormality of the f_i°,we can write,

$$\int f_i^{\circ *} f_i^{\circ} dt = 1 , \qquad \int f_i^{\circ *} f_j^{\circ} dt = 0,$$

which transforms the last equation into,

$$E_i^{\circ} - E_i + \mathrm{H}'_{ii} + \sum_{j \neq i} a_{ij}\, \mathrm{H}'_{ij} = 0 \qquad \text{...} (15)$$

where, $\mathrm{H}'_{ij} = \int f_i^{\circ *}\, \hat{\mathrm{H}}' f_j^{\circ} dt$

Multiplying from the left by $f_k^{\circ *}$; where $k \neq i$ and after manipulation, we get,

$$a_{ik}\left(E_k^{\circ} - E_i\right) + \mathrm{H}'_{ki} + a_{ik}\mathrm{H}'_{kk} + \sum_{j \neq i,k} a_{ij}\, \mathrm{H}'_{kj} = 0$$

We now consider that the energies for the perturbed system, E_i and the coefficients a_{ij} can be written in terms of series in which successive terms become smaller, such that,

$$E_i = E_i^{\circ} + E_i' + E_i'' + \cdots$$

$$a_{ij} = a_{ij}' + a_{ij}'' + \cdots ;$$

where a single prime denotes a first-order term, a double prime denotes a second order term and so on.

Placing these last conditions into equation (15) and taking out the first-order terms (as two first-order terms multiplied together constitute a second-order term and so on), we get,

$$E_i' = \mathrm{H}'_{ij} = \int f_i^{\circ *}\, \hat{\mathrm{H}}' f_j^{\circ} dt \qquad \text{...} (16)$$

<u>Formation of Poles:</u>

Each of the intrinsic properties contained inside a system will, for obvious reasons (in accordance with the Theory of Abstraction) transact in such a manner so as to be distributed as much as possible. This will give rise to two sets of transactions:

1. The transaction of the points themselves in every direction, including towards each other. This gives rise to an additional effect in the direction between two given points as compared to all the other directions. This is due to the fact that in the direction between the points, there is an effect due to both the points, while in all other directions (considering a two-point system) the effect is due to one point only as shown in fig.1

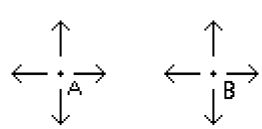

fig.1: effect between points

This gives rise to an ‘attraction‘ between the points.

2. As a given intrinsic property transports in such a manner so as to be distributed as much as possible, the effect of ‘repulsion’ is generated. Let us consider two clusters of charges, either both positive or both negative. Not only the charge tends to be distributed in all directions, but also the type of charge. Presence of same type of charges will thus give rise to an effect of a field of repulsion in the vicinity of the clusters.

   Thus we see that formation of poles and attraction between unlike poles and repulsion between like ones may be explained using the Theory of Abstraction. Here, we have examined only the qualitative aspects, while the quantitative aspects may be determined as detailed in my earlier paper , in view of the laws of physical transactions.

Conclusion:

Abstraction Theory is made use of to study symmetrical many-points systems. The intrinsic properties of each of the constituents of the system need to be treated in accordance. We may relate each of the concerned points inside the given system to various groups, as per their intrinsic properties. A group of such points i.e., a system having individual intrinsic properties inside it may be regarded as a point with its own intrinsic properties (a function of its constituent intrinsic properties) when it is considered as a part of some larger system. This way, the intrinsic properties of each of the smallest points may influence a system or even a group of systems. A group of such individual points may give rise to a field of extrinsic properties, affecting each of the individual points.

Considering the symmetries or the stability states inside a given group or inside a given system as being similar in the fact that they are all basically states of stability, the irreducible representations, if any, inside a given system may be studied. Perturbations regarding a given average observable value may be predicted in a similar way.