Exact solution of 2D plane-parallel flow for Glacier dynamics.

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A new exact solution of Glacier dynamics is presented here in terms of viscous-plastic theory of 2-dimensional movements within \((x, y)\)-plane. Most of researchers used to investigate preferably 2D solution of Glacier dynamics in \((x, z)\)-plane: -by using the continuity equation, they obtain \((x, z)\)-profile of Glacier in dependence on initial conditions of ice flow.
We proved that \((x, y)\)-component of stress-tensor is a harmonic function, it let us execute the conditions of Liouville's theorem which allow to obtain the exact expression for the components of velocity of Glacier ice flux in \((x, y)\)-directions.
1. Introduction.

A glacier is a massive, slowly moving mass of compacted snow and ice. The action of gravity moves the mass of ice down the slope side: glaciers are being moved from a millimeter to hundreds meters a day. There are two kinds of motion: 1) a slow sliding motion and an avalanche like flow; 2) the internal movement of glacial ice, is a flow similar to plastic flow and viscous flow [1-3].

Glaciers move by two mechanisms [4-6]: basal slip and viscous-plastic flow. In basal slip, the entire glacier slides over bedrock. A glacier also moves by plastic flow, in which it flows as a viscous fluid.

According to [1], 2D case of glacial ice viscous-plastic flow should be represented in the Cartesian coordinates as below (we consider a plane-parallel flow: \( z = \text{const} \)):

\[
\begin{align*}
\rho \left( \frac{\partial v_x}{\partial t} + v_x \cdot \frac{\partial v_x}{\partial x} + v_y \cdot \frac{\partial v_x}{\partial y} \right) &= -\frac{\partial p}{\partial x} + \rho \cdot g \cdot \sin \alpha + \frac{\partial s_{xx}}{\partial x} + \frac{\partial s_{xy}}{\partial y}, \\
\rho \left( \frac{\partial v_y}{\partial t} + v_x \cdot \frac{\partial v_y}{\partial x} + v_y \cdot \frac{\partial v_y}{\partial y} \right) &= -\frac{\partial p}{\partial y} + \rho \cdot g \cdot \sin \alpha + \frac{\partial s_{xy}}{\partial x} - \frac{\partial s_{xx}}{\partial y}, \quad (1.1)
\end{align*}
\]

\[
\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0, \quad U = \sqrt{4 \left( \frac{\partial v_x}{\partial x} \right)^2 + \left( \frac{\partial v_y}{\partial y} + \frac{\partial v_y}{\partial x} \right)^2},
\]

\[
s_{xx} = 2(\mu + \frac{\tau_s}{U}) \frac{\partial v_x}{\partial x}, \quad s_{xy} = 2(\mu + \frac{\tau_s}{U}) \left( \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right),
\]

\[
\Rightarrow \quad U = \frac{1}{\mu} \left( \sqrt{s_{xx}^2 + s_{xy}^2} - \tau_s \right),
\]

\[
\frac{\partial v_x}{\partial x} = s_{xx}/2(\mu + \frac{\tau_s}{U}). \quad (*)
\]
- where $\rho$ – is a density of glacial ice; $v_x$ – is the component of ice velocity in the $x$-direction; $v_y$ – the component of ice velocity in the direction $y$; $p$ – is an internal pressure in glacial ice; $g$ – is an acceleration of gravity; $\alpha$ – is a proper angle of slope where glacial ice is moving; $s_{xx}, s_{xy}$ – are the appropriate components of stress tensor; $\mu$ – is the coefficient of glacial ice dynamic viscosity; $\tau_s$ – is a critical maximal level of stress in the shared layer of glacial ice when it begin to move as viscous flow.

2. Exact solution.

Let us assume that the left part of (1.1) equals to zero due to negligible terms for slowly moving glacial ice. So, we obtain from (*) and (1.1):

$$
0 = -\frac{\partial p}{\partial x} + \rho \cdot g \cdot \sin \alpha + \frac{\partial s_{xx}}{\partial x} + \frac{\partial s_{xy}}{\partial y},
$$

$$
0 = -\frac{\partial p}{\partial y} + \rho \cdot g \cdot \sin \alpha + \frac{\partial s_{xy}}{\partial x} - \frac{\partial s_{xx}}{\partial y},
$$

(2.1)

$$
\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0, \quad \frac{\partial v_x}{\partial x} = s_{xx}/2(\mu + \frac{\tau_s}{U}),
$$

$$
U = \frac{1}{\mu} \left( \sqrt{s_{xx}^2 + \frac{s_{xy}^2}{4}} - \tau_s \right).
$$

The cross-differentiating of the 1-st & 2-nd equation of (2.1) in regard to the coordinates $x, y$ yields $(p(x, y) = const$ in a nature):

$$
\frac{\partial^2 s_{xy}}{\partial x^2} + \frac{\partial^2 s_{xy}}{\partial y^2} = 0,
$$

- it means that $s_{xy}$ is a harmonic function [7].

Let us note that according to the Liouville’s theorem [7-8]: “if $f$ is a harmonic function defined on all of $\mathbb{R}^n$ which is bounded above or bounded below, then $f$ is constant”.
It is evident that $s_{xy}$ (the component of stress tensor) is bounded above due to the general physical sense [9]. Thus, according to the Liouville's theorem we should conclude that $s_{xy}$ equals to the constant: $s_{xy} = \text{const} = 2\sqrt{C}$. Then from (2.1) we obtain ($C_0 = 0$):

$$s_{xx} = \rho \cdot g \cdot \sin \alpha \cdot (y - x) + C_0 ,$$

- but

$$\frac{\partial v_x}{\partial x} = \frac{\rho \cdot g \cdot \sin \alpha \cdot (y - x)}{2 (\mu + \frac{\tau_s}{U})} ,$$

- where:

$$U = \frac{1}{\mu} \cdot \left( \sqrt{(\rho \cdot g \cdot \sin \alpha \cdot (y - x))^2 + C - \tau_s} \right).$$

So, we should obtain the expression for component $v_x$ as shown below (Fig.1):

$$v_x = \frac{1}{2\mu} \left( -\frac{\rho \cdot g \cdot \sin \alpha}{2} \cdot x^2 + (\rho \cdot g \cdot \sin \alpha \cdot y - \tau_s) \cdot x + \tau_s \cdot y \right).$$

Taking also into consideration the continuity equation and (2.1):

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 ,$$

- we also obtain the expression for component $v_y$ (Fig.2):

$$v_y = \frac{1}{2\mu} \left( -\frac{\rho \cdot g \cdot \sin \alpha}{2} \cdot y^2 + (\rho \cdot g \cdot \sin \alpha \cdot x + \tau_s) \cdot y - \tau_s \cdot x \right).$$
Fig. 1. A *schematic* plot of the function $v_x$,
here we designate the range: $x \in (0, 10)$, $y \in (0, 10)$.

Fig. 2. A *schematic* plot of the function $v_y$,
here we designate the range: $x \in (0, 10)$, $y \in (0, 10)$. 

Fig.3. "Elephant Foot Glacier" in Greenland.

References:

   See also, in Russian (branch 2.2.2): http://www.ipmnet.ru/~petrov/files/vpdchp.pdf.