Exact solution of viscous-plastic flow equations
for Glacier dynamics in 2-dimensional case.

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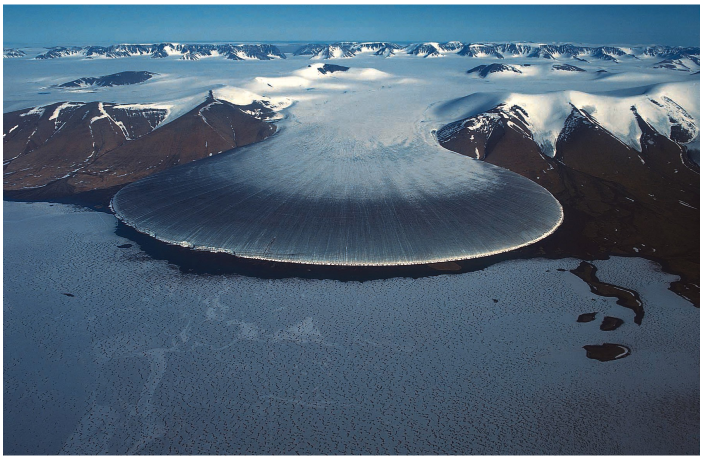
Here is presented a new exact solution of *Ice dynamics* in Glaciers in terms of viscous-plastic theory of movements, for 2-dimensional case. In general case, 2-D solution of *Ice dynamics* could be classified as *Riccati's type*. Due to a very special character of *Riccati's type* equation, it's general solution is proved to have *a proper gap* of components of such a solution.

It means a possibility of *sudden gradient catastrophe* at definite moment of time-parameter, in regard to the components of solution (*2-D profile of Glacier, 2-D components of ice velocity moving*).

That's why surging glacier seems to be *accelerating* from time to time: it's velocity of moving is *suddenly* rising from few meters to hundreds meters /per day.



A glacier is a massive, slowly moving mass of compacted snow and ice. The action of gravity moves the mass of ice down the slope side: glaciers are being moved from a millimeter to hundreds meters a day. There are two kinds of motion: 1) a slow sliding motion and an avalanche like flow; 2) the internal movement of glacial ice, is a flow similar to plastic flow and viscous flow.

Glaciers move by two mechanisms: basal slip and viscous-plastic flow. In basal slip, the entire glacier slides over bedrock. A glacier also moves by plastic flow, in which it flows as a viscous fluid.

In accordance with [1], 2-dimensional case of glacial ice viscous-plastic flow should be represented in the Cartesian system of coordinates as below (*axis* ***Ox*** *coincides to initial direction of glacial ice flow, which is assumed to be a plane-parallel flow*, $z = const$):

$$\rho\cdot\left(\frac{\partial v_x}{\partial t}+v_x\cdot\frac{\partial v_x}{\partial x}+v_y\cdot\frac{\partial v_x}{\partial y}\right)=-\frac{\partial p}{\partial x}+G_x+\frac{\partial s_{xx}}{\partial x}+\frac{\partial s_{xy}}{\partial y},$$

$$\rho\cdot\left(\frac{\partial v_y}{\partial t}+v_x\cdot\frac{\partial v_y}{\partial x}+v_y\cdot\frac{\partial v_y}{\partial y}\right)=-\frac{\partial p}{\partial y}+G_y+\frac{\partial s_{xy}}{\partial x}-\frac{\partial s_{xx}}{\partial y}, \qquad (1.1)$$

$$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}=0, \quad U=\sqrt{4(\frac{\partial v_x}{\partial x})^2+(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x})^2},$$

$$s_{xx}=2(\mu+\frac{\tau_s}{U})\cdot\frac{\partial v_x}{\partial x}, \quad s_{xy}=2(\mu+\frac{\tau_s}{U})\cdot(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}) .$$

- where v_x – is the component of ice velocity in the direction x of Cartesian system x, y; v_y – the component of ice velocity in the direction y; p – is an internal pressure in glacial ice; G_x, G_y – are the appropriate *projections* of gravity (central force) to the chosen initial direction x, y of glacial ice plane-parallel flow; S_{xx}, S_{xy} – are the appropriate components of stress tensor; μ – is a coefficient of glacial ice dynamic viscosity; τ_s – is a critical maximal level of stress in shared layer of glacial ice when it starts to move as viscous flow (*stage of plastic flow*: *if an absolute meaning of stress tensor less than a critical maximal level of stress in shared layer* < τ_s, → *glacial ice does not move*).

From (1.1) we obtain the appropriate equalities below:

$$U = \frac{1}{\mu} \cdot \left(\sqrt{s_{xx}^{2} + \frac{s_{xy}^{2}}{4}} - \tau_s \right),$$

$$\frac{\partial v_x}{\partial x} = s_{xx} \Big/ 2(\mu + \frac{\tau_s}{U}) \; .$$

Let's assume in our modeling that the left part of (1.1) *equals to zero* due to negligible terms for the case of *slowly moving* glacial ice. But for the case of *slow* glacial ice flow system (1.1) could be reduced as below

$$0 = -\frac{\partial p}{\partial x} + G_x + \frac{\partial s_{xx}}{\partial x} + \frac{\partial s_{xy}}{\partial y} \, ,$$

$$0 = -\frac{\partial p}{\partial y} + G_y + \frac{\partial s_{xy}}{\partial x} - \frac{\partial s_{xx}}{\partial y} \, , \qquad (1.2)$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \, , \;\; \frac{\partial v_x}{\partial x} = s_{xx} \Big/ 2(\mu + \frac{\tau_s}{U}) \, ,$$

$$U = \frac{1}{\mu} \cdot \left(\sqrt{s_{xx}^{2} + \frac{s_{xy}^{2}}{4}} - \tau_s \right) .$$

Then for finding a solution, we should cross-differentiate 1-st & 2-nd equation (1.2) in regard to x & y, as well as we should combine it by a proper linear way (*besides, on open air* $p\,(x, y) = const$); in result, we obtain:

$$\frac{\partial^2 s_{xy}}{\partial x^2} + \frac{\partial^2 s_{xy}}{\partial y^2} = 0 \ ,$$

- it means that S_{xy} – is the *harmonic function* [2].

According to *Liouville's theorem:* "if f is a harmonic function defined on all of $\mathbf{R}^n$ which is bounded above or bounded below, then f is constant" [2].

It is evident that S_{xy}, being the component of stress tensor, is bounded above - *in regard to it's absolute meanings* - due to general physical sense [3].

So, we have: 1) S_{xy} *is a harmonic function*, 2) S_{xy} *is bounded above*. Thus, in accordance with *Liouville's theorem*, S_{xy} is a constant: S_{xy} = const = 2C. Then from (1.2) we obtain $S_{xx} = -\,G_x \cdot x + G_y \cdot y + C_0$ (C_0 = const $\neq 0$), but:

$$U = \frac{1}{\mu} \cdot \left(\sqrt{(-G_x \cdot x + G_y \cdot y + C_0)^2 + C^2} - \tau_s \right) \ ,$$

$$\frac{\partial v_x}{\partial x} = \frac{s_{xx}}{2\mu} \left(1 - \frac{\tau_s}{\sqrt{s_{xx}{}^2 + \frac{s_{xy}{}^2}{4}}}\right) ,$$

- hence, we obtain in result:

Let's choose $C = 0$, then above equality could be simplified to the form below

$$\frac{\partial v_x}{\partial x} = \frac{1}{2\mu} \left\{ G_y \cdot y - G_x \cdot x + (C_0 - \tau_s) \right\} \ ,$$

If we take also into consideration *the continuity equation* (*see* (1.2)):

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \ ,$$

- we obtain that initial system (1.1) is reduced to representation below

$$\frac{\partial v_x}{\partial x} = \frac{1}{2\mu}\left\{ G_y \cdot y - G_x \cdot x + (C_0 - \tau_s) \right\} \ ,$$

$$\frac{\partial v_y}{\partial y} = -\frac{1}{2\mu}\left\{ G_y \cdot y - G_x \cdot x + (C_0 - \tau_s) \right\} \ . \qquad (1.3)$$

The system above could be easily solved if $G_x = 0$ or $G_y = 0$. Indeed, let's choose for example $G_y = 0$, $G_x \neq 0$ in (1.3), then we obtain below ($C_1 = \text{const} \neq 0$):

$$v_x \equiv \frac{\partial x}{\partial t} = \frac{1}{2\mu}\left\{ - G_x \cdot \frac{x^2}{2} + (C_0 - \tau_s)\cdot x + C_1 \right\}, \quad \Rightarrow$$

$$\Rightarrow \ \int \frac{d\,x}{\left\{ - G_x \cdot \frac{x^2}{2} + (C_0 - \tau_s)\cdot x + C_1 \right\}} = \frac{t}{2\mu} \ ,$$

- where [4]:

$$\int \frac{d\,x}{\left\{ - G_x \cdot \frac{x^2}{2} + (C_0 - \tau_s)\cdot x + C_1 \right\}} \equiv \begin{cases} 1) \ \dfrac{2}{\sqrt{\Delta}}\, arctg\, \dfrac{- G_x \cdot x + (C_0 - \tau_s)}{\sqrt{\Delta}} \\ (\ \Delta > 0, \ \Delta = -2\, G_x \cdot C_1 - (C_0 - \tau_s)^2\ ) \\ 2) \ \dfrac{1}{\sqrt{-\Delta}} \ln \dfrac{- G_x \cdot x + (C_0 - \tau_s) - \sqrt{-\Delta}}{- G_x \cdot x + (C_0 - \tau_s) + \sqrt{-\Delta}} \, . \\ (\ \Delta < 0\ ) \end{cases}$$

Let's choose in above equalities $C_0 = \tau_s$ (*for the aim of clear presentation of final solution*); in such a case the equalities above are simplified then we could obtain a final solution:

$$1)\quad x = -\frac{\sqrt{\Delta}}{G_x}\cdot tg\,\frac{\sqrt{\Delta}}{4\mu}\,t\,; \qquad 2)\quad x = \frac{\sqrt{-\Delta}}{G_x}\cdot\frac{1+\exp\left(-\frac{\sqrt{-\Delta}}{2\mu}t\right)}{1-\exp\left(-\frac{\sqrt{-\Delta}}{2\mu}t\right)} \qquad (1.4)$$

$$(\Delta = -2\,G_x\cdot C_1\,,\Rightarrow\; C_1 < 0\,) \qquad (\,\Delta < 0,\;\Rightarrow\; C_1 > 0\,)$$

First type of solutions (1.4) could be associated with *pulsating glaciers* or *surging glaciers*, which are characterized by periodic movements of glacial ice.

As for coordinate $y = y\,(t)$, we could obtain from (1.3):

$$\frac{\partial v_y}{\partial y} \equiv \frac{\partial v_y}{\partial t}\cdot\frac{\partial t}{\partial y} \equiv \ddot{y}\cdot\left(\dot{y}\right)^{-1},\quad \Rightarrow$$

$$\Rightarrow\; \ddot{y} - \left(\frac{G_x\cdot x}{2\mu}\right)\cdot\dot{y} = 0\;,$$

- *Bernoulli's type* ordinary differential equation, which has a proper regular solution [4].

But in general case, if G_x, $G_y \neq 0$, equations (1.3) could be classified as *Riccati's type*. Due to a very special character of *Riccati's type* equation, it's general solution is proved to have *a proper gap* of components of such a solution [3-4].

It means a possibility of *sudden gradient catastrophe* [5] at definite moment of time-parameter, in regard to the components of solution (*2-D profile of Glacier, 2-D components of ice velocity moving*). That's why Glacier seems to be *accelerating* from time to time: it's velocity of moving is *suddenly* rising from few meters to hundreds meters /per day.

Let's also explore the case $C_0 = \tau_s$, $C_1 = 0$ (*we choose all new constants below are equal to zero*):

$$\frac{\partial v_x}{\partial x} = -\frac{1}{2\mu} G_x \cdot x\,, \;\Rightarrow\; v_x = \dot{x} = -\frac{1}{4\mu} G_x \cdot x^2\,, \;\Rightarrow\; x = \left(\frac{4\mu}{G_x}\right) \cdot t^{-1}\,,$$

$$\frac{\partial v_y}{\partial y} \equiv \frac{\partial v_y}{\partial t} \cdot \frac{\partial t}{\partial y} \equiv \ddot{y} \cdot (\dot{y})^{-1}, \;\Rightarrow\; \ddot{y} \cdot (\dot{y})^{-1} = 2t^{-1}, \;\Rightarrow\; \ddot{y} - 2t^{-1} \cdot \dot{y} = 0\,,$$

- here the last equation is also the *Bernoulli's type* of ODE in regard to component *y (t)*, which has a proper regular solution [4].

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