Exact solution of viscous-plastic flow equations for Glacier dynamics in 2-dimensional case.

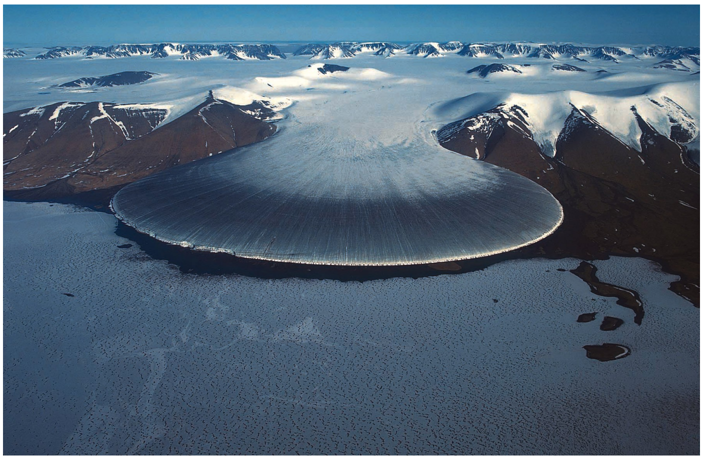
Sergey V. Ershkov
Institute for Time Nature Explorations,
M.V. Lomonosov's Moscow State University,
Leninskie gory, 1-12, Moscow 119991, Russia.

Keywords: Glacier dynamics, exact solution; basal slip, viscous fluid, plastic flow; glacial ice, dynamic viscosity; critical maximal level of stress; shared layer of glacial ice; angle of glacial ice slope; *Riccati's type* of ordinary differential equation.

Here is presented a new exact solution of *Ice dynamics* in Glaciers in terms of viscous-plastic theory of movements, for 2-dimensional case: $x(t) = y(t)$. In general case, 2-D solution of *Ice dynamics* could be classified as *Riccati's type*. Due to a very special character of *Riccati's type* equation, it's general solution is proved to have *a proper gap* of components of such a solution.

It means a possibility of *sudden gradient catastrophe* at definite moment of time-parameter, in regard to the components of solution (2-D profile of Glacier, 2-D components of ice velocity moving).

That's why Glacier seems to be *accelerating* from time to time: it's velocity of moving is *suddenly* rising from few meters to hundreds meters /per day.



A glacier is a massive, slowly moving mass of compacted snow and ice. The action of gravity moves the mass of ice down the slope side: glaciers are being moved from a millimeter to hundreds meters a day. There are two kinds of motion: 1) a slow sliding motion and an avalanche like flow; 2) the internal movement of glacial ice, is a flow similar to plastic flow and viscous flow.

Glaciers move by two mechanisms: basal slip and viscous-plastic flow. In basal slip, the entire glacier slides over bedrock. A glacier also moves by plastic flow, in which it flows as a viscous fluid.

In accordance with [1], 2-dimensional case of glacial ice viscous-plastic flow should be represented in the Cartesian system of coordinates as below (*axis **Ox** coincides to initial direction of glacial ice flow, which is assumed to be a plane-parallel flow*: *z = const*):

$$\rho\cdot\left(\frac{\partial v_x}{\partial t}+v_x\cdot\frac{\partial v_x}{\partial x}+v_y\cdot\frac{\partial v_x}{\partial y}\right)=-\frac{\partial p}{\partial x}+\rho\cdot g\cdot\sin\alpha+\frac{\partial s_{xx}}{\partial x}+\frac{\partial s_{xy}}{\partial y}\ ,$$

$$\rho\cdot\left(\frac{\partial v_y}{\partial t}+v_x\cdot\frac{\partial v_y}{\partial x}+v_y\cdot\frac{\partial v_y}{\partial y}\right)=-\frac{\partial p}{\partial y}+\rho\cdot g\cdot\sin\alpha+\frac{\partial s_{xy}}{\partial x}-\frac{\partial s_{xx}}{\partial y}\ , \qquad (1.1)$$

$$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}=0\,,\quad U=\sqrt{4(\frac{\partial v_x}{\partial x})^2+(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x})^2}\ ,$$

$$s_{xx}=2(\mu+\frac{\tau_s}{U})\cdot\frac{\partial v_x}{\partial x}\,,\ s_{xy}=2(\mu+\frac{\tau_s}{U})\cdot(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x})\ ,$$

- where ρ – is a density of glacial ice; v_x – is the component of ice velocity in the direction x of the Cartesian system x, y; v_y – the component of ice velocity in the direction y; p – is an internal pressure in glacial ice; g – is an acceleration of gravity; α – is a proper *angle* of slope where glacial ice is moving; S_{xx}, S_{xy} – are the appropriate components of stress tensor; μ – is a coefficient of glacial ice dynamic viscosity; τ_s – is a critical maximal level of stress in shared layer of glacial ice (*stage of plastic flow*) when it starts to move as viscous flow.

From (1.1) we obtain the appropriate equalities below:

$$U = \frac{1}{\mu}\cdot\left(\sqrt{s_{xx}^{\ 2} + \frac{s_{xy}^{\ 2}}{4}} - \tau_s\right),$$

$$\frac{\partial v_x}{\partial x} = s_{xx}\Big/ 2(\mu + \frac{\tau_s}{U}) \ .$$

In our modeling we assume that the left part of (1.1) *equals to zero* due to negligible terms for the case of slowly moving glacial ice. Besides, on open air: $p\,(x, y) = const$.

So, system (1.2) should be reduced as below

$$0 = \rho\cdot g\cdot\sin\alpha + \frac{\partial s_{xx}}{\partial x} + \frac{\partial s_{xy}}{\partial y},$$

$$0 = \rho\cdot g\cdot\sin\alpha + \frac{\partial s_{xy}}{\partial x} - \frac{\partial s_{xx}}{\partial y}, \qquad (1.2)$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0, \quad \frac{\partial v_x}{\partial x} = s_{xx}\Big/ 2(\mu + \frac{\tau_s}{U}),$$

$$U = \frac{1}{\mu}\cdot\left(\sqrt{s_{xx}^{\ 2} + \frac{s_{xy}^{\ 2}}{4}} - \tau_s\right).$$

Let's obtain a proper *cross-differentiating* of 1-st & 2-nd equation (1.2) (*in regard to coordinates x & y*), then a proper linear combination:

$$\frac{\partial^2 s_{xy}}{\partial x^2} + \frac{\partial^2 s_{xy}}{\partial y^2} = 0 \ ,$$

- it means that S_{xy} – is the *harmonic function* [2].

According to [*Liouville's theorem*:](Liouville's theorem) "if f is a harmonic function defined on all of $\mathbf{R}^n$ which is bounded above or bounded below, then f is constant" [2].

It is evident that S_{xy}, being the component of stress tensor, is bounded above (*in regard to it's absolute meanings*) - due to general physical sense [3]. So, we have: *S_{xy} is a harmonic function* as well as *it is bounded above* – thus, *Liouville's theorem* lets us conclude that S_{xy} is a constant: S_{xy} = const = 2√C. Then from (1.2) we obtain

$$s_{xx} = \rho \cdot g \cdot \sin\alpha \cdot (y - x) + C_0 \ \ ,$$

- besides

$$\frac{\partial v_x}{\partial x} = \frac{\rho \cdot g \cdot \sin\alpha \cdot (y - x) + C_0}{2(\mu + \dfrac{\tau_s}{U})} \ ,$$

- where ($C_0 = 0$):

$$U = \frac{1}{\mu} \cdot \left(\sqrt{(\rho \cdot g \cdot \sin\alpha \cdot (y - x) + C_0)^2 + C} - \tau_s \right) .$$

Further integral way of calculating leads us to equality below

$$v_x = \frac{1}{2\mu} \left(-\frac{\rho \cdot g \cdot \sin\alpha}{2} \cdot x^2 + (\rho \cdot g \cdot \sin\alpha \cdot y - \tau_s) \cdot x + \tau_s \cdot y \right),$$

- if we choose all the constants of above integrating are equal to zero ≡ 0.

Taking also into consideration *the continuity equality* (see (1.2)):

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \ ,$$

- we obtain, by the same calculating in regard to the component of ice velocity v_y:

$$v_y \ = \ \frac{1}{2\mu}\left(-\frac{\rho \cdot g \cdot \sin\alpha}{2} \cdot y^2 + (\rho \cdot g \cdot \sin\alpha \cdot x + \tau_s) \cdot y \ - \tau_s \cdot x \right).$$

Thus, the initial system (1.1) has been reduced to representation below:

$$\frac{dx}{dt} \ = \ \frac{1}{2\mu}\left(-\frac{\rho \cdot g \cdot \sin\alpha}{2} \cdot x^2 + (\rho \cdot g \cdot \sin\alpha \cdot y - \tau_s) \cdot x \ + \tau_s \cdot y \right),$$

$$\frac{dy}{dt} \ = \ \frac{1}{2\mu}\left(-\frac{\rho \cdot g \cdot \sin\alpha}{2} \cdot y^2 + (\rho \cdot g \cdot \sin\alpha \cdot x + \tau_s) \cdot y \ - \tau_s \cdot x \right) \ ,$$

- or

$$\begin{aligned} \frac{dx}{dt} \ &= \ \frac{1}{2\mu}\left(-\frac{\rho g \sin\alpha}{2} \cdot x^2 \ + \ (\rho g \sin\alpha) \cdot yx \ + \tau_s \cdot (y - x) \right), \\ \frac{dy}{dt} \ &= \ \frac{1}{2\mu}\left(-\frac{\rho g \sin\alpha}{2} \cdot y^2 \ + \ (\rho g \sin\alpha) \cdot xy \ + \tau_s \cdot (y - x) \right) \ . \end{aligned} \qquad (1.3)$$

Subtracting from 1-st the 2-nd equation (1.2), we obtain:

$$\frac{dx}{dt} + \frac{\rho g \sin\alpha}{4\mu} \cdot x^2 \ = \ \frac{dy}{dt} + \frac{\rho g \sin\alpha}{4\mu} \cdot y^2 \ ,$$

- a *Riccati's type* equations [3-4], the evident solution of which is $x = y$. In this case, system (1.3) could be reduced to a proper solution below ($t \geq 0$):

$$x = y, \quad \frac{dx}{dt} \ = \ \frac{\rho g \sin\alpha}{4\mu} \cdot x^2, \quad \Rightarrow \ x(t) \ = -\left(x_0^{-1} + \frac{\rho g \sin\alpha}{4\mu} \cdot t \right)^{-1},$$

- it means *the inverse way of* coordinates x, y *to depend* on time-parameter t (*see* Fig.1).

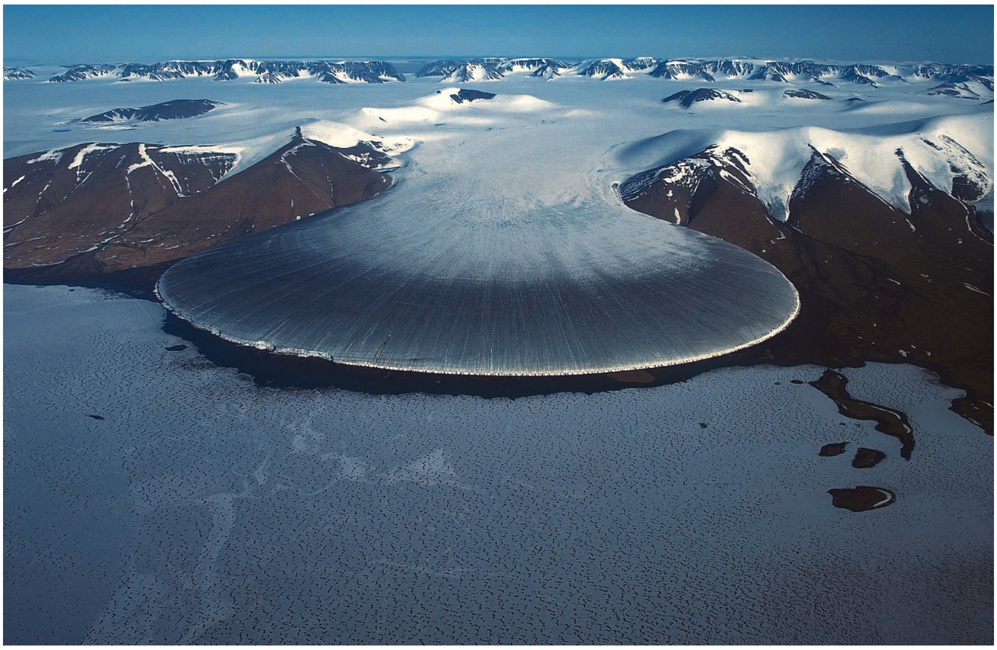


Fig.1. An example of exact solution: $x\ (t) = y\ (t)$.

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