Why the Colombeau Algebras Cannot Formulate, Let Alone Prove the Global Cauchy-Kovalevskaia Theorem ?

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Dedicated to Marie-Louise Nykamp

Abstract

It is briefly shown that, due to the growth conditions in their definition, the Colombeau algebras cannot handle arbitrary analytic nonlinear PDEs, and in particular, cannot allow the formulation, let alone, give the proof of the global Cauchy-Kovalevskaia theorem.

"History is written with the feet ..."

Ex-Chairman Mao, of the Long March fame ...

Science is not done scientifically, since it is mostly done by non-scientists ...

Anonymous

1. The Globalization of the Cauchy-Kovalevskaia Theorem

We consider on a nonvoid open set $\Omega \subseteq \mathbb{R}^n$ the *general nonlinear analytic* partial differential operator

$$(1.1) \qquad T(x, D)U(x) \;=\; D_t^m U(t, y) - G(t, y, ..., D_t^p D_y^q U(t, y), ...)$$

where $U : \Omega \longrightarrow \mathbb{C}$ is the unknown function, while $x = (t, y) \in \Omega,\ t \in \mathbb{R},\ y \in \mathbb{R}^{n-1},\ p \in \mathbb{N},\ 0 \leq p < m,\ q \in \mathbb{N}^{n-1},\ p + |q| \leq m$, and G is arbitrary analytic in all of its variables.

Together with the analytic nonlinear PDE

$$(1.2) \qquad T(x, D)U(x) \;=\; 0, \quad x \in \Omega$$

we consider the non-characteristic analytic hypersurface

$$(1.3) \qquad S \;=\; \{\ x \;=\; (t, y) \in X \ \mid\ t \;=\; t_0\ \}$$

for any given $t_0 \in \mathbb{R}$, and on it, we consider the analytic initial value problem

$$(1.4) \qquad D_t^p U(t_0, y) \;=\; g_p(y), \quad 0 \leq p < m, \quad (t_0, y) \in S$$

Obviously, the analytic nonlinear partial differential operator $T(x, D)$ in (1.2) generates a mapping

$$(1.5) \qquad T(x, D) : \mathcal{C}^\infty(\Omega) \;\longrightarrow\; \mathcal{C}^\infty(\Omega)$$

also, in view of [4-12], it generates a mapping

$$(1.6) \qquad T(x, D) : \mathbb{A}(\Omega) \;\longrightarrow\; \mathbb{A}(\Omega)$$

and the mappings (1.5), (1.6) form a commutative diagram

(1.7)
$$
\begin{tikzcd}
\mathcal{C}^\infty(\Omega) \arrow[r, "T(x,D)"] \arrow[d] & \mathcal{C}^\infty(\Omega) \arrow[d] \\
\mathbb{A}(\Omega) \arrow[r, "T(x,D)"'] & \mathbb{A}(\Omega)
\end{tikzcd}
$$

In this way, [4,5,9,11], one could obtain the *global existence* result :

Theorem G C-K

The analytic nonlinear PDE in (1.2), with the analytic non-characteristic initial value problem (1.3), (1.4), has *global* generalized solutions

(1.8) $\quad U \in \mathbb{A}(\Omega)$

defined on the whole of Ω. These solutions U are *analytic* functions

(1.9) $\quad \psi : \Omega \setminus \Sigma \longrightarrow \mathbb{C}$

when restricted to the *open dense* subsets $\Omega \setminus \Sigma$, where the singularity subsets

(1.10) $\quad \Sigma \subset \Omega, \quad \Sigma$ closed, nowhere dense in Ω

can be suitably chosen. Furthermore, one can choose Σ to have zero Lebesgue measure, namely

(1.11) $\quad \text{mes } \Sigma = 0$

$\square$

As it turns out, the above kind of *global* version of the Cauchy-Kovalevskaia Theorem cannot even be formulated, let alone proved in the Colombeau algebras. The reason for that failure, as seen next, is in the restrictive polynomial type *growth conditions* which are essential in the definition of the Colombeau algebras.

2. The Growth Conditions in Colombeau Algebras Cannot Handle Picard Singularities

Let us briefly recall the way *growth conditions* are essential in defining the Colomebau algebras [1-5]. For simplicity, we shall consider the case of the domains $\Omega = \mathbb{R}^n$, and on them, of the general Colombeau algebras first introduced in [1], see also [2-5]. Their construction starts with the auxiliary family of smooth functions, defined for each $m \in \mathbb{N}$, namely

$$(2.1) \quad \Phi_m(\Omega) = \left\{ \phi \in \mathcal{D}(\Omega) \;\middle|\; \begin{array}{l} (i) \quad \int_\Omega \phi(x)dx = 1 \\ \\ (ii) \;\; \int_\Omega x^p \phi(x)dx = 0, \quad p \in \mathbb{N}^n, 1 \leq |p| \leq m \end{array} \right\}$$

Further, for $\epsilon > 0$ and $\phi \in \mathcal{D}(\Omega)$, we define $\phi_\epsilon \in \mathcal{D}(\Omega)$ by

$$(2.2) \qquad \phi_\epsilon(x) = \phi(x/\epsilon)/\epsilon^n, \qquad x \in \Omega$$

Now, the basic space of functions for defining the Colombeau algebras will be

$$(2.2) \qquad \mathcal{E}(\Omega) = (\mathcal{C}^\infty(\Omega))^{\Phi(\Omega)}$$

which is obviously a *differential algebra* with the term-wise operations.

Then the general Colombeau algebra on $\Omega = \mathbb{R}^n$ is constructed in three steps.

First, one considers the *differential subalgebra* $\mathcal{A}(\Omega)$ in $\mathcal{E}(\Omega)$, given by all the functions $f \in \mathcal{E}(\Omega)$ which satisfy the *growth condition*

$$
(2.3)\quad
\begin{array}{l}
\forall \;\; \text{compact } K \subseteq \Omega, \;\; p \in \mathbb{N}^n \;\; : \\[2ex]
\exists \;\; m \in \mathbb{N}, \;\; m \geq 1 \;\; : \\[2ex]
\forall \;\; \phi \in \Phi_m(\Omega) \;\; : \\[2ex]
\exists \;\; \eta, c > 0 \;\; : \\[2ex]
\forall \;\; x \in K, \;\; \epsilon \in (0, \eta) \;\; : \\[2ex]
\qquad | \, D^p f(\phi_\epsilon, x) \, | \leq c/\epsilon^m
\end{array}
$$

Second, one considers in the algebra $\mathcal{A}(\Omega)$ the *ideal* $\mathcal{I}(\Omega)$ given by by all the functions $f \in \mathcal{A}(\Omega)$ which satisfy the *growth condition*

$$
(2.4)\quad
\begin{array}{l}
\forall \;\; \text{compact } K \subseteq \Omega, \;\; p \in \mathbb{N}^n \;\; : \\[2ex]
\exists \;\; k \in \mathbb{N}, \;\; k \geq 1, \;\; \beta \in B \;\; : \\[2ex]
\forall \;\; m \in \mathbb{N}, \;\; m \geq k, \;\; \phi \in \Phi_m(\Omega) \;\; : \\[2ex]
\exists \;\; \eta, c > 0 \;\; : \\[2ex]
\forall \;\; x \in K, \;\; \epsilon \in (0, \eta) \;\; : \\[2ex]
\qquad | \, D^p f(\phi_\epsilon, x) \, | \leq c \, \epsilon^{\beta(m) - k}
\end{array}
$$

where

$$
(2.5)\qquad B = \left\{ \beta \in (0, \infty)^{\mathbb{N}} \;\middle|\;
\begin{array}{ll}
(i) & \beta \text{ is non-decreasing} \\[2ex]
(ii) & \lim_{m \to \infty} \beta(m) = \infty
\end{array}
\right\}
$$

Third, and finally, the general Colombeau algebra of generalized functions on $\Omega = \mathbb{R}^n$ is the *quotient algebra*

$$
(2.6)\qquad \mathcal{G}(\Omega) = \mathcal{A}(\Omega) / \mathcal{I}(\Omega)
$$

The reason for the failure of the Colombeau algebras (2.6) in allowing the formulation, let alone, the proof of a global version of the Cauchy-Kovalevskaia theorem becomes now easily obvious. Namely, the algebras $\mathcal{A}(\Omega)$ in (2.3) which are used in the definition (2.6) of the Colombeau algebras do *not* allow arbitrary smooth, and not even arbitrary analytic operations on Colombeau generalized functions. And this is obviously due to the specific *growth conditions* in the definition (2.3) of these algebras $\mathcal{A}(\Omega)$. Indeed, let us consider the following set of *slowly increasing* smooth functions

$$(2.7) \qquad \mathcal{O}(\mathbb{R}^r) = \left\{ \alpha \in \mathcal{C}^\infty(\mathbb{R}^r) \;\middle|\; \begin{array}{l} \forall \; p \in \mathbb{N}^r \;: \\ \\ \quad D^p\alpha \;\text{ is slowly increasing} \end{array} \right\}$$

where a function $\beta \in \mathcal{C}^\infty(\mathbb{R}^r)$ is called *slowly increasing*, if and only if there exist $K, c > 0$, such that

$$(2.8) \qquad |\, \beta(\xi) \,| \leq K(\, 1 + |\, \xi \,|\,)^c, \qquad \xi \in \mathbb{R}^r$$

The mentioned limitation regarding smooth operations on Colombeau generalized functions is described by the following result, [1,4], :

Given Colombeau generalized functions $T_1, \ldots, T_m \in \mathcal{G}(\Omega)$ and $\alpha \in \mathcal{O}(\mathbb{R}^{2n})$, then there exists a Colombeau generalized function $\alpha(T_1, \ldots, T_m) \in \mathcal{G}(\Omega)$, and it is defined by

$$(2.9) \qquad \alpha(T_1, \ldots, T_m) = \alpha(f_1, \ldots, f_m) + \mathcal{I}(\Omega) \in \mathcal{G}(\Omega)$$

where

$$(2.10) \qquad T_i = f_i + \mathcal{I}(\Omega) \in \mathcal{G}(\Omega), \qquad 1 \leq i \leq m$$

The problem here clearly is in the fact that, as soon as a smooth nonlinear operation α is no longer in $\mathcal{O}(\mathbb{R}^{2n})$, one cannot in general obtain the growth condition (2.3) being satisfied by $\alpha(f_1, \ldots, f_m)$ in (2.9) for all Colombeau generalized functions $T_1, \ldots, T_m \in \mathcal{G}(\Omega)$.

And clearly, arbitrary analytic functions G in (1.1) which are involved in the definition (1.2) of analytic PDEs, need not belong to a space $\mathcal{O}(\mathbb{R}^r)$.

Thus the impossibility to formulate, let alone, to prove a global version of the Cauchy-Kovalevskaia Theorem within the confines of the Colombeau algebras.

In view of the above it is obvious that the Colombeau algebras cannot deal with a variety of large classes of nonlinear PDEs.

3. The Inevitable Infinite Branching in the Multiplication of Singularities

The rather amusing fact, after decades of studies in the nonlinear algebraic theory of generalized functions, see subject 46F30 in the AMS classification, is what appears to be the inability on the part of not a few specialists involved to realize and understand that multiplication of generalized functions does quite *inevitably branches* when faced with dealing with singularities, [4-12]. And as seen easily, this branching has most simple algebraic, more precisely, ring theoretic reasons.

As it happens, however, realizing the presence and importance of that branching seems not to be so easy, since it has so far eluded several notable mathematicians, as mentioned for instance in [12].

The immediate and most obvious consequence of the mentioned inevitable *infinite* branching is that a *variety* of differential algebras of generalized functions should be considered when, for instance, solving nonlinear PDEs. After all, such an approach is in no way a novelty, as for more than seven decades by now a large variety of Sobolev spaces have been used for such a purpose.

As for the Colombeau algebras, they obviously have a number of convenient properties. Moreover, as stressed in [4,5], their construction has a rather important *natural* feature which, however, is seldom mentioned, let alone used in the literature.

However, as with all mathematical constructs, so with the Colombeau algebras, they manifest clear limitations in certain important situations.

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