The Local-Nonlocal Dichotomy Is but a Relative and Local View Point

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Dedicated to Marie-Louise Nykamp

Abstract

As argued earlier elsewhere, what is the Geometric Straight Line, or in short, the GSL, we shall never know, and instead, we can only deal with various mathematical models of it. The so called standard model, given by the usual linearly ordered field $\mathbb{R}$ of real numbers is essentially based on the ancient Egyptian assumption of the Archimedean Axiom which has no known reasons to be assumed in modern physics. Setting aside this axiom, a variety of linearly ordered fields $\mathbb{F}_{\mathcal{U}}$ becomes available for the mathematical modelling of the GSL. These fields, which are larger than $\mathbb{R}$, have a rich self-similar structure due to the presence of infinitely small and infinitely large numbers. One of the consequences is the obvious relative and local nature of the long ongoing local versus nonlocal dichotomy which still keeps having foundational implications in quantum mechanics.

"History is written with the feet ..."

Ex-Chairman Mao, of the Long March fame ...

Science is not done scientifically, since it is mostly done by non-scientists ...

Anonymous

A "mathematical problem" ?
For sometime by now, American mathematicians have decided to hide their date of birth and not to mention it in their academic CV-s.
Why ?
Amusingly, Hollywood actors and actresses have their birth date easily available on Wikipedia.
Can one, therefore, trust American mathematicians ?
Why are they so blatantly against transparency ?
By the way, Hollywood movies have also for long been hiding the date of their production ...

A bemused non-American mathematician

1. Quanta under the Heavy Shadow of the Local-Nonlocal Dichotomy

The *local-nonlocal dichotomy* has had - and seemingly it still has - its considerable and rather bewildering impact in quantum mechanics, and it has done so at least since the celebrated 1935 EPR paper, and Einstein's negative remark about "spooky action at distance" ...
Not much later, Bohm's deterministic version of quantum mechanics kept being set aside due, among others, its nonlocal aspect ...

Then, with the emergence of quantum information theory during the last two decades or so, it has been noted that the essentially nonlocal phenomenon of entanglement is a fundamental resource, and as such, it is in fact but a particular case of the yet more general nonlocal phenomenon of quantum correlations.

Added to the above, Smolin's recent notable rethinking of quantum mechanics, [4], comes with nothing short of an upfront acceptance of nonlocality.

What is, therefore, possibly going on regarding this long ongoing and foundational dichotomy of local-nonlocal ?

2. Does Archimedes Run the Quanta ?

Quantum physicist, and in fact, physicists at large, are well known to love to indulge themselves minimally in mathematics. Engineers, on the other hand, must sign on projects, and not only on research papers, projects which are then turned soon into real objects or processes, thus become subject to a most obvious public scrutiny as to their effective practical validity. Consequently, although not much more in love with mathematics than physicists, engineers know very well that they must take mathematics seriously enough, and use from it whatever may indeed be relevant in their specific projects ...
And to put that difference more in terms of everyday language, both physicists and engineers feel about mathematics like going to the dentist, except that, unlike physicists, engineers do so more or less as often as necessary, even if with equal reluctance ...

But then, amusingly, even mathematicians have on occasion funny ways when dealing with mathematics ...
Namely, certain basic mathematical structures that have been in use for a longer time tend to be seen as, so to say, THE structures God Himself is using, and does so exclusively, that is, by disregarding all other possible alternative mathematical structures ...

One of such long time used mathematical structures is what can be called the Geometric Straight Line, or GSL, which ever since ancient Egypt and the Euclidean Geometry, is supposed to satisfy among others the Archimedean Axiom, [2,3].
In modern mathematical terms, this means that the GSL is identified

with the usual field $\mathbb{R}$ of real numbers. And as far as mathematicians are concerned, there is here indeed a strong built in uniqueness and exclusiveness argument in favour of $\mathbb{R}$, given by the fact that it is the only field which is linearly ordered, complete, and also satisfies the Archimedean Axiom.

Well, if mathematicians, and especially those of the more pure mathematical persuasion, do not ask themselves what may indeed be the relevance of the Archimedean Axiom in realms of modern physics, such as quanta or relativity, for instance, it is quite strange, and also regrettable, that physicists, especially those involved with quanta or relativity, among others, do not much bother about that question ...
Indeed, so far it seems that there is no reason whatsoever in modern physics why one should still hold to the Archimedean Axiom, even if in ancient times it proved to be particularly useful when remeasuring land in Egypt each year after the blessings of the flood of the Nile ...

But to be somewhat more brief and to the point, and for the sake of all those involved in using or building mathematical models, let us mention three statements which may elicit strong - and quite likely wrong - reactions, [3] :

(2.1) We do not - and can never ever - know what *geometry* is !

(2.2) We do not - and can never ever - know what *numbers* are !

(2.3) We do not - and can never ever - know what the *geometric straight line*, or in short, the GSL is !

Indeed, it often happens among those who use mathematics that they fail to recognize the *fundamental difference* between *abstract ideas*, and on the other hand, one or another of their *mathematical model* which happens to be chosen upon specific reasons, or rather, upon mere historical circumstance.

However, as a kind of excuse for those not aware of the mentioned fundamental difference, it should be recalled that less than two centuries ago, following a philosophy of the calibre of that of Kant, the abstract

idea of geometry was seen as being reduced to, and perfectly identical with, one single mathematical model, namely, that of Euclidean geometry. And as if to aggravate the error in such a view, Kant considered Euclidean geometry as an a priori concept.
As it happened nevertheless in the 1820s, non-Euclidean geometry was introduced by Lobachevski and Bolyai, followed not much later by Riemannian geometry, and more near to our times, by the more or less explicit recognition of the immense difference between the abstract idea of geometry, and on the other hand, any of its mathematical models.
And still, the errors in approaching the concept of geometry were to continue for a while longer. The "Erlangen Program", for instance, published by Felix Klein in 1872, saw geometry as reduced to the rather narrow framework of the study of properties invariant under certain group transformations ...

In this regard, one may simply say that one does not - and in fact, can never ever - know what geometry, or for that matter, numbers are. And all one can know instead are merely various mathematical models of geometry, or of numbers.
And needless to say, the very same of course goes for the GSL ...

Fortunately, since the emergence of special relativity, and even more so of general relativity, there is an awareness among physicists that geometry is, so to say, a *collective noun*, that is, there is *no* such a thing like THE geometry ...
In fact, general relativity introduced the physical concept of *background independent* theory. And several of the presently ongoing attempts at bringing the quanta together with general relativity consider it important to set up the respective theories precisely in such a background independent manner.

And yet, holding to the Archimedean Axiom is a rather universal and seemingly quite unshakable tendency ...

3. What Changes If At Last We Retire Archimedes ?

What changes ?

Well, an immensely large variety of *brave new worlds* opens up for, among others, the mathematical modelling the GSL ...

And what is even more important, many of the new mathematical models lead to linearly ordered fields $\mathbb{F}_{\mathcal{U}}$ which are larger and far more rich in structure than the usual field $\mathbb{R}$ of real numbers.

Furthermore, in such fields $\mathbb{F}_{\mathcal{U}}$ one can quite easily find out that the dichotomy local-nonlocal which has for long troubled the foundations of quantum mechanics is relegated to a mere *relative*, and in fact, *local* status ...

For those who may, nevertheless, be quite concerned about having to try to deal with yet another ... mathematical phantasy ... the following should be somewhat easing the possible worries :

The mentioned fields $\mathbb{F}_{\mathcal{U}}$ have been known, dealt with, and used in mathematics, as well as in physics, for more than four decades by now. Indeed, they were first introduced in a rigorous manner back in 1966, by Abraham Robinson, in his Nonstandard Analysis.

What happened, however, was that due to Robinson's insistence on having the benefit of the so called Transfer Principle when dealing with such fields $\mathbb{F}_{\mathcal{U}}$, the mathematics involved was so technically complicated that it did keep away even most of mathematicians.
And added to that disadvantage came the further one that, upon a cost-return consideration over some decades, it turned out that the benefits of having the Transfer Principle were not compensating enough for the price which had to be paid in the technically complicated mathematics.

And then, it was noted that, in fact, one can simply construct directly such fields $\mathbb{F}_{\mathcal{U}}$, and do so based on no more than "Algebra 101". As for the loss of the Transfer Principle, it is fully compensated by the comparative simplicity of mathematics needed for the necessary developments. Details in thins regard can be found in [2] and the references cited there.

3. A Rich Self-Similar Structure of the GSL

The rich self-similar structure of the mentioned fields $\mathbb{F}_{\mathcal{U}}$ comes from the presence in them of *infinitesimal* and *infinitely large* numbers, [2]. And it is precisely this presence which gives the fields $\mathbb{F}_{\mathcal{U}}$ their rich self-similar structure.

The elements $t \in \mathbb{F}_{\mathcal{U}}$ are of *three* kind, namely, *infinitesimal, finite*, and *infinitely large*, as defined by the following respective conditions

(3.1) $\quad \forall \;\; r \in \mathbb{R}, \; r > 0 \; : \; t \in (-r, r)$

(3.2) $\quad \exists \;\; r \in \mathbb{R}, \; r > 0 \; : \; t \in (-r, r)$

(3.3) $\quad \forall \;\; r \in \mathbb{R}, \; r > 0 \; : \; t \notin (-r, r)$

where for $a, b \in \mathbb{F}_{\mathcal{U}}$, we denote as usual $(a, b) = \{s \in \mathbb{F}_{\mathcal{U}} \mid a < s < b\}$. Now, following Leibniz, one denotes

(3.4) $\quad monad(0) = \{ \; t \in \mathbb{F}_{\mathcal{U}} \mid \; t \;$ is infinitesimal $\}$

and calls it the *monad* of $0 \in \mathbb{F}_{\mathcal{U}}$, while one denotes

(3.5) $\quad Gal(0) = \{ \; t \in \mathbb{F}_{\mathcal{U}} \mid \; t \;$ is finite $\}$

and calls it the *Galaxy* of $0 \in \mathbb{F}_{\mathcal{U}}$.

It is easy to see that

(3.6) $\quad Gal(0) = \bigcup_{r \in \mathbb{R}} monad(r)$

where for $t \in \mathbb{F}_{\mathcal{U}}$, we denote

(3.7) $\quad monad(t) = t + monad(0)$

Finally

(3.8) $\quad \mathbb{F}_{\mathcal{U}} \setminus Gal(0)$

is the set of infinitely large elements in the field $\mathbb{F}_{\mathcal{U}}$.

In this way, all the elements of $\mathbb{F}_{\mathcal{U}}$, be they infinitesimal, finite, or infinitely large, have been expressed respectively in (3.4) by the monad of $0 \in \mathbb{F}_{\mathcal{U}}$, in (3.6) by the Galaxy of $0 \in \mathbb{F}_{\mathcal{U}}$, and in (3.8). And as one notes, all these sets can in fact be expressed in terms of the monad of $0 \in \mathbb{F}_{\mathcal{U}}$ alone.

Now in order to grasp more easily the *rich self-similar* structure of the field $\mathbb{F}_{\mathcal{U}}$ let us start by first recalling the much simpler self-similar structure of the usual field $\mathbb{R}$ of real numbers. In this regard, we have the *self-similarity* property given by the following *bijective, order reversing* mapping

(3.9) $\quad \mathbb{R} \setminus (-1, 1) \ni r \longmapsto 1/r \in [-1, 1] \setminus \{0\}$

thus the unbounded set

$$\mathbb{R} \setminus (-1, 1) = (-\infty, -1] \cup [1, \infty)$$

has through the mapping (3.9) the inverse linear order structure of the bounded set

$$[-1, 1] \setminus \{0\} = [-1, 0) \cup (0, 1]$$

Now by translation and scaling, we obtain the family of self-similarities of the usual field $\mathbb{R}$ of real numbers, given by the *bijective, order reversing* mappings

(3.10) $\quad \mathbb{R} \setminus (-a, a) \ni r \longmapsto (1/r) + r_0 \in [r_0 - \frac{1}{a}, r_0 + \frac{1}{a}] \setminus \{r_0\}$

where $r_0, a \in \mathbb{R}, a > 0$.

Here we can note that none of the self-similarities (3.10) refers to the structure of $\mathbb{R}$ itself at any given point $r_0 \in \mathbb{R}$, but only to the structure of the sets

$$(3.11) \qquad [r_0 - a, r_0 + a] \setminus \{r_0\} = [r_0 - a, r_0) \cup (r_0, r_0 + a], \qquad a > 0$$

around the point $r_0 \in \mathbb{R}$, sets which are whole neighbourhoods of r_0 from which, however, the point r_0 itself has been taken out. This is obviously inevitable, since each point $r_0 \in \mathbb{R}$ is at a finite strictly positive - thus *not* infinitesimal - distance from any other point in $\mathbb{R}$.

In addition, we also have the self-similarities

$$(3.12) \qquad \mathbb{R} \xrightarrow{f} (a, b)$$

where $-\infty \leq a < b \leq \infty$, while f can be any bijective order preserving, or for that matter, order reversing, continuous mapping.

On the other hand, the field $\mathbb{F}_{\mathcal{U}}$ has a far more *rich self-similar* structures due to the presence of the infinitesimal, and thus as well, of the infinitely large elements. Indeed, this time, the self-similarities can also refer to the whole *monad* of each point, except for the point itself.

Let us start with a self-similarity of any field $\mathbb{F}_{\mathcal{U}}$ which does *not* exist in the case of the usual real line $\mathbb{R}$. Namely, it is easy to see that we have the *order reversing bijective* mapping

$$(3.13) \qquad (\mathbb{F}_{\mathcal{U}} \setminus Gal(0)) \ni t \longmapsto 1/t \in (monad(0) \setminus \{0\})$$

which means that the set of all infinitely large elements in $\mathbb{F}_{\mathcal{U}}$ has the inverse order structure of the set of infinitesimal elements from which one excludes 0.

This shows the *important* fact that the infinitesimally local structure, and on the other hand, the global structure of $\mathbb{F}_{\mathcal{U}}$ do in fact *mirror* one another, a property which has no correspondence in the case of the usual field $\mathbb{R}$ of real numbers.

Also, through translation and scaling, we have, for each $t_0, u \in \mathbb{F}_{\mathcal{U}}, u > 0$, the *order reversing bijective* mapping

$$(3.14) \quad \mathbb{F}_{\mathcal{U}} \setminus (-u, u) \ni t \longmapsto (1/t) + t_0 \in [t_0 - \tfrac{1}{u}, t_0 + \tfrac{1}{u}] \setminus \{t_0\}$$

where $\mathbb{F}_{\mathcal{U}} \setminus (-u, u)$ will always contain infinitely large elements.

These again are self-similarities not present in the case of the usual real line $\mathbb{R}$.

Furthermore, in (3.14) we have a far more rich possibility for translations and scalings than in the usual case of the real line $\mathbb{R}$. Indeed, in addition to translations and scalings with non-zero finite elements $r_0, a \in \mathbb{R}, a > 0$, as in (3.11), we can now also translate and scale with all $t_0, u \in \mathbb{F}_{\mathcal{U}}, u > 0$, thus with all infinitely large elements, as well as with all infinitesimal elements, except for scaling with $0 \in \mathbb{F}_{\mathcal{U}}$.

Let us consider the above in some detail by listing the different possibilities for the sets

$$(3.15) \quad [t_0 - \tfrac{1}{u}, t_0 + \tfrac{1}{u}] \setminus \{t_0\}$$

in (3.14).

First of all, these sets are no longer mere subsets in $\mathbb{R}$, but instead, they are subsets in $\mathbb{F}_{\mathcal{U}}$, and will always contain infinitesimals, since they contain nonvoid intervals. Furthermore, as seen below, they may also contain infinitely large elements.

Also, $t_0, u \in \mathbb{F}_{\mathcal{U}}, u > 0$ in (3.15) can independently be finite, infinitesimal, or infinitely large, thus resulting in 9 possible combinations and 6 distinct outcomes regarding the set (3.15), which we list below. This is in sharp contradistinction with the case in (3.11) which applies to the real line $\mathbb{R}$. Indeed :

1) Let us start the listing of these 9 different cases and 6 distinct outcomes with both t_0 and u being finite. Then obviously (3.15) is a

subset of $Gal(0)$, and it has the finite, non-infinitesimal length $2u$.

2) When t_0 is finite and u is infinitesimal, then the set (3.15) is infinitely large, and is no longer contained in $Gal(0)$, however, it contains $Gal(0) \setminus \{t_0\}$.

3) If t_0 is finite, but u is infinitely large, then (3.15) is again a subset of $Gal(0)$, and in fact, it has the infinitesimal length $2u$, which means that it is a subset of $monad(t_0)$.

4) Let us now assume that t_0 is infinitesimal and u finite. Then regarding the set (3.9), we are back to case 1) above.

5) If both t_0 and u are infinitesimal then the set (3.15) is as in 2) above.

6) When t_0 is infinitesimal and u is infinitely large, the set (3.15) is as in 3) above.

7) Let us now take t_0 infinitely large and u finite. Then the set (3.15) is disjoint from $Gal(0)$, and it has the finite, non-infinitesimal length $2u$.

8) When t_0 infinitely large and u infinitesimal, then the set (3.15) is again not contained in $Gal(0)$, and it has the infinitely large length $2u$. Furthermore, depending on the relationship between $|t_0|$ and $1/u$, it may, or it may not intersect $Gal(0)$.

9) Finally, when both t_0 and u are infinitely large, then the set (3.15) is disjoint from $Gal(0)$, and it has the infinitesimal length $2u$.

We conclude that he *local* structure of $\mathbb{F}_{\mathcal{U}}$ is given by

(3.16) $\quad Gal(0) = \bigcup_{r \in \mathbb{R}} (r + monad(0))$

while the *global* structure of $\mathbb{F}_{\mathcal{U}}$ is given by

(3.17) $\quad \mathbb{F}_{\mathcal{U}} = \left(\bigcup_{\lambda \in \Lambda} Gal(-s_\lambda) \right) \bigcup Gal(0) \bigcup \left(\bigcup_{\lambda \in \Lambda} Gal(s_\lambda) \right)$

where Λ is an uncountable set of indices, while $s_\lambda \in \mathbb{F}_\mathcal{U}$ are positive infinite, and such that $s_\mu - s_\lambda$ is infinite, for $\lambda, \mu \in \Lambda, \lambda \neq \mu$.

Here we can point to a self-similar aspect of the interrelation between the local and global structure of $\mathbb{F}_\mathcal{U}$ which may remind us of a typical feature of *fractals*. Indeed, similar with (3.16), the relation (3.17) can also be expressed in terms monads, namely

$$\begin{aligned}(3.18) \quad \mathbb{F}_\mathcal{U} = (\ \textstyle\bigcup_{r \in \mathbb{R},\ \lambda \in \Lambda} (r - s_\lambda + monad(0))\ ) \ \bigcup \\ (\ \textstyle\bigcup_{r \in \mathbb{R}} (r + monad(0))\ ) \\ \textstyle\bigcup\ (\ \bigcup_{r \in \mathbb{R},\ \lambda \in \Lambda} (r + s_\lambda + monad(0))\ )\end{aligned}$$

In this way, in view of (3.13), we obtain the self-similar order reversing bijection, which is now expressed solely in terms of $mon(0)$, namely

$$\begin{aligned}(3.19) \quad [\ (\ \textstyle\bigcup_{r \in \mathbb{R},\ \lambda \in \Lambda} (r - s_\lambda + monad(0))\ ) \\ \textstyle\bigcup\ (\ \bigcup_{r \in \mathbb{R},\ \lambda \in \Lambda} (r + s_\lambda + monad(0))\ )\ ] \ni t \longmapsto \\ \longmapsto 1/t \in [\ monad(0) \setminus \{0\}\ ]\end{aligned}$$

and conversely

$$\begin{aligned}(3.20) \quad [\ monad(0) \setminus \{0\}\ ] \ni t \longmapsto \\ \longmapsto 1/t \in [\ (\ \textstyle\bigcup_{r \in \mathbb{R},\ \lambda \in \Lambda} (r - s_\lambda + monad(0))\ ) \\ \textstyle\bigcup\ (\ \bigcup_{r \in \mathbb{R},\ \lambda \in \Lambda} (r + s_\lambda + monad(0))\ )\ ]\end{aligned}$$

As we can note, the above bijections in (3.19), (3.20) are given by the very simple algebraic, explicit, and order reversing mapping $s \longmapsto 1/s$, which involves what is essentially a *field* operation, namely, division. And these two bijections take the place of the much simpler order reversing bijections in the case of the usual real line $\mathbb{R}$, namely

$$(3.21) \quad (\mathbb{R} \setminus (-1, 1)) \ni r \longmapsto 1/r \in ([-1, 1] \setminus \{0\})$$

(3.22) $\quad ([-1,1] \setminus \{0\}) \ni r \longmapsto 1/r \in (\mathbb{R} \setminus (-1,1))$

The considerable difference between (3.19), (3.20), and on the other hand, (3.21), (3.22) is obvious. Indeed, in the former two, which describe the self-similar structure of $\mathbb{F}_{\mathcal{U}}$, the order reversing bijections represent the set

$$mon(0) \setminus \{0\}$$

through the set

$$[\ (\ \bigcup_{r \in \mathbb{R},\ \lambda \in \Lambda} (r - s_\lambda + monad(0))\ )$$
$$\bigcup\ (\ \bigcup_{r \in \mathbb{R},\ \lambda \in \Lambda} (r + s_\lambda + monad(0))\ )\ ]$$

which contains *uncountably* many translates of the set $mon(0)$. And it is precisely this manifestly *rich* self-similarity of the set $mon(0)$ of monads which is the novelty in the *non-Archimedean* structure of $\mathbb{F}_{\mathcal{U}}$, when compared with the much simpler Archimedean structure of $\mathbb{R}$. This novelty is remarkable since it makes $mon(0)$ have the very same *complexity* with the whole of

$$\mathbb{F}_{\mathcal{U}} \setminus Gal(0) = [\ (\ \bigcup_{r \in \mathbb{R},\ \lambda \in \Lambda} (r - s_\lambda + monad(0))\ )$$
$$\bigcup\ (\ \bigcup_{r \in \mathbb{R},\ \lambda \in \Lambda} (r + s_\lambda + monad(0))\ )\ ]$$

In this way $mon(0)$, which is but the set of infinitesimals, thus it cannot be represented in terms of the usual field $\mathbb{R}$ of real numbers, turns out to have the very *same* complexity as the set $\mathbb{F}_{\mathcal{U}} \setminus Gal(0)$ of all infinitely large numbers, which again cannot be represented in terms of the usual field $\mathbb{R}$ of real numbers.

4. One's Chosen or Given Scale

When as customary, one uses for the GSL the mathematical model given by the usual field $\mathbb{R}$ of real numbers, one can choose, or be given, a *scale* by specifying as an arbitrary *unit* any finite number $u \in \mathbb{R},\ u > 0$.

Thus in such a situation, any finite interval $(a, b) \subseteq \mathbb{R}$ has a length $b-a$ which, when compared to u, is of similar size, considerably smaller, or significantly larger. However, in each of these situations both of the numbers

$$(4.1) \qquad (b-a)/u, \qquad u/(b-a)$$

will be *finite*.

It follows that with the above mathematical modelling of the GSL, any possible difference between local and nonlocal is *limited* exclusively to finite proportions.

On the other hand, when the GSL is modelled by any of the fields $\mathbb{F}_{\mathcal{U}}$, the scale can again be defined by specifying any unit $U \in \mathbb{F}_{\mathcal{U}},\ U > 0$, even if in this case U can not only be finite, but can also be infinitely small, or infinitely large.

And then, the difference between local and nonlocal acquires *two new* possibilities. Namely, given an interval $(A, B) \subseteq \mathbb{F}_{\mathcal{U}}$, its length $B - A$ compared to the unit U can be not only finite, but it can also be infinitely large, or on the contrary, infinitely small. More precisely, the number

$$(4.2) \qquad (B-A)/U$$

can be finite, infinitely small, or infinitely large. And the same can happen with the number

$$(4.3) \qquad U/(B-A)$$

The consequent novelty compared to (4.1) is that one can have the simultaneous situation when

(4.4) $\quad (B-A)/U$ is infinitely small, and $U/(B-A)$ is infinitely large or the other way round, when

(4.5) $\quad (B-A)/U$ is infinitely large, and $U/(B-A)$ is infinitely small

In this way, the dichotomy local versus nonlocal acquires two new meanings, namely

(4.6) $\quad A$ and B are *infinitely local* with respect to one another

when (4.4) holds, or alternatively

(4.7) $\quad A$ and B are *infinitely nonlocal* with respect to one another

in the case of (4.5).

5. The Local-Nonlocal Dichotomy Is Merely Relative and Local ...

Let us look at the above when comparing two intervals in the GSL.

In the case of the usual field $\mathbb{R}$ of real numbers we have the following obvious property. Given any $-\infty < a < b < \infty$ and $-\infty < c < d < \infty$, we have the *strictly increasing bijection* $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by the *linear* mapping

(5.1) $\quad f(x) = ((d-c)x + bc - ad)/(b-a), \quad x \in \mathbb{R}$

such that

(5.2) $\quad f(a) = c, \quad f(b) = d$

with the induced *strictly increasing linear bijection* between the two intervals

(5.3) $\quad f : (a, b) \longrightarrow (c, d)$

Now, since each $\mathbb{F}_{\mathcal{U}}$ is a linearly ordered field, the operations in (5.1) can again be performed. Thus for every

$$(5.4) \qquad A, B, C, D \in \mathbb{F}_{\mathcal{U}}, \quad A < B, \quad C < D$$

we have the *strictly increasing bijection*

$$(5.5) \qquad F : \mathbb{F}_{\mathcal{U}} \longrightarrow \mathbb{F}_{\mathcal{U}}$$

given by the *linear* mapping

$$(5.6) \qquad F(t) = ((D - C)t + BC - AD)/(B - A), \qquad t \in \mathbb{F}_{\mathcal{U}}$$

for which we have

$$(5.7) \qquad F(A) = C, \quad F(B) = D$$

with the induced *strictly increasing linear bijection* between the two intervals

$$(5.8) \qquad F : (A, B) \longrightarrow (C, D)$$

Clearly, such a mapping F in (5.4) - (5.8) is nothing else but a *translation* followed by a *scaling*, for instance, A is translated to C, and the interval (A, B) is scaled to the interval (C, D).

And here the *remarkable* and *nontrivial* fact - one that is *not* possible within the usual field $\mathbb{R}$ of real numbers - is that any of the numbers A, B, C, D in (5.4) - (5.8) can be infinitesimal, finite, or infinitely large.
Consequently, any of the intervals (A, B) or (C, D) in (5.8) can be infinitesimal, finite, or infinitely large.

This is, therefore, precisely the way the *relative* and *local* nature of the local-nonlocal dichotomy becomes further apparent in the sense mentioned at (4.6), (4.7).

Indeed, let, for instance, have two finite intervals (A, B) and (C, D)

such that the second is large enough compared with the first one, in order for C and D to be considered nonlocal relative to one another, while A and B are considered local with respect to one another.

If we now take two infinitesimal numbers $\gamma, \delta \in \mathbb{F}_{\mathcal{U}},\ \gamma < \delta$, and take the corresponding interval (γ, δ) as our *local* reference, then the mapping (5.5) corresponding to γ, δ, A, B will make both intervals (A, B) and (C, D) be relatively infinitely large, thus certainly *nonlocal*.
And obviously the same happens with the mapping (5.5) corresponding to γ, δ, C, D.

On the other hand, if we take two infinitely large numbers $\Gamma, \Delta \in \mathbb{F}_{\mathcal{U}},\ \Gamma < \Delta$, with $\Delta - \Gamma$ being also infinitely large, then the mapping (5.5) corresponding to Γ, Δ, A, B will make both intervals (A, B) and (C, D) be relatively infinitely small, thus certainly *local*.
And obviously the same happens with the mapping (5.5) corresponding to Γ, Δ, C, D.

What makes the above of special concern is the following.

In case the GSL is modelled by the usual field $\mathbb{R}$ of real numbers, the choice of the *unit* is arbitrary in the sense tat every number $u \in \mathbb{R},\ u > 0$ can be chosen as a unit, and we have the corresponding strictly increasing linear bijection

(5.9) $\quad \mathbb{R} \ni x \longmapsto ux \in \mathbb{R}$

which changes the unit given by 1 into that given by u. The converse change is, of course, by the strictly increasing linear bijection

(5.10) $\quad \mathbb{R} \ni x \longmapsto x/u \in \mathbb{R}$

Here also, a certain choice of $u \in \mathbb{R},\ u > 0$ may turn two finite intervals (a, b) and (c, d) in two relatively local ones, or relatively nonlocal ones, depending on the size of u.

However, it is not possible to turn both of such intervals into relatively infinitesimally small, or into relatively infinitely large ones.

Returning to the fields $\mathbb{F}_{\mathcal{U}}$, their rich self-similar structure makes it a far more relevant fact that the unit $U \in \mathbb{F}_{\mathcal{U}},\ U > 0$, which one chooses is indeed arbitrary. Namely, as seen above, relative to the choice of such a unit, any two finite intervals relative to some earlier given alternative unit may now turn into the irrelevance of both having a mere infinitesimal size, or on the contrary, of becoming all but inaccessible in their far removed immensity.

So much for the still ongoing ... quantum battles ... related to the assumed foundational dichotomy between local versus nonlocal properties ...

6. What Is There To Underlie the Dichotomy Local-Nonlocal ?

Certainly, from the point of view of our thought processes related to the dichotomy local versus nonlocal, or for that matter, from the point of view of the world at large, the world in which we happen to be while concerning ourselves with that dichotomy, there is a most obvious underlying unifying realm, a realm from where the two sides of this dichotomy do happen to arise, a realm where no matter how far from each other these two sides may turn out to be conceived as being, they nevertheless are and must be most close to one another, and thus quite local as well. Indeed, this unifying realm is simply that of our respective thought processes ...

As it happens, however, in present day science, such thought processes are not supposed to be included into the scientific enquiry, and instead, they are only there to implement such an enquiry. In other words, it is much like when one drives across a field in a car, but has no concern about the ways of the inner workings of that car, let alone of the interactions between the terrain and the car, and instead, simply takes for granted the car and its proper functioning ...

And yet, strange things happen which such an approach cannot clarify satisfactorily ...

For instance, [1], we can easily think simultaneously about two stars which are many light years apart from one another, and from us as well. And then clearly, such a thinking - which in some ways happens to relate those two stars to one another, and also us with them - cannot take place as a usual physical phenomenon, and certainly not, as long as we accept the limitation on the speed of propagation of all physical phenomena imposed by special and general relativity ...

So then, where and how does such a thinking happen ?

Do we really have to try to go back to some version of that much misunderstood and derided Cartesian distinction between "res extensa", where the limitation imposed by relativity rules, and on the other hand, "'res cogitans", where quite likely no such limitation exists ?

By the way, the standard accusation and consequent dismissal of the above Cartesian differentiation between those two realms is that it is but a mere dualism. And the essential fact is missed that Descartes, like all major Western thinkers of his time, was a genuinely religious Christian man who firmly believed in God. And any such person can simply be definition not be a dualist, since he or she sees the world as nothing else but the miraculous unity of the Creation of God ...
Furthermore, even is someone may honestly believe to be a dualist, this is simply not really possible. Indeed, the very thought of dualism is but inevitably one thought, one single, one unique, one coherent thought, a thought which by definition underlies that assumed duality. And precisely to the extent that it underlies it, it severely limits its validity and realms of existence ...

But let us keep here to the present day limitations imposed upon what is supposed to be a valid scientific enquiry ...

And then, within such terms, is there some deeper unity which may underlie the duality local versus nonlocal ?

Well, it is easy to note that, indeed, there is ...
And in fact, we can suggest at least two such candidates ...

The first one can be given by the GSL, more precisely, by the fact that we shall never know what the GSL is, and instead, we can only know its various mathematical models.
Indeed, in this sense the dichotomy local versus nonlocal simply does not arise on the level of the GSL, but only on that of its various mathematical models.

Second, the mathematical models of the GSL given, for instance, by various fields $\mathbb{F}_{\mathcal{U}}$ show, as seen in sections 4 and 5, the relativity and locality of the dichotomy local versus nonlocal.

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