# A Generalization of Stokes Theorem on Combinatorial Manifolds

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**Abstract**: For an integer $m \geq 1$, a combinatorial manifold $\widetilde{M}$ is defined to be a geometrical object $\widetilde{M}$ such that for $\forall p \in \widetilde{M}$, there is a local chart $(U_p, \varphi_p)$ enable $\varphi_p : U_p \to B^{n_{i_1}} \bigcup B^{n_{i_2}} \bigcup \cdots \bigcup B^{n_{i_{s(p)}}}$ with $B^{n_{i_1}} \bigcap B^{n_{i_2}} \bigcap \cdots \bigcap B^{n_{i_{s(p)}}} \neq \emptyset$, where $B^{n_{i_j}}$ is an $n_{i_j}$-ball for integers $1 \leq j \leq s(p) \leq m$. Integral theory on these smoothly combinatorial manifolds are introduced. Some classical results, such as those of *Stokes'* theorem and *Gauss'* theorem are generalized to smoothly combinatorial manifolds in this paper.

**Key Words**: combinatorial manifold, Stokes' theorem, Gauss' theorem.

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## §1. Introduction

As a localized euclidean space, an *n-manifold* $M^n$ is a Hausdorff space $M^n$, i.e., a space that satisfies the $T_2$ separation axiom such that for $\forall p \in M^n$, there is an open neighborhood $U_p, p \in U_p \subset M^n$ and a homeomorphism $\varphi_p : U_p \to \mathbf{R}^n$. These manifolds, particularly, differential manifolds are very important to modern geometries and mechanics. By a notion of mathematical combinatorics, i.e. *mathematics can be reconstructed from or turned into combinatorization*([3]), the conception of combinatorial manifold is introduced in [4], which is a generalization of classical manifolds and can be also endowed with a topological or differential structure as a geometrical object.

Now for an integer $s \geq 1$, let $n_1, n_2, \cdots, n_s$ be an integer sequence with $0 < n_1 < n_2 < \cdots < n_s$. Choose $s$ open unit balls $B_1^{n_1}, B_2^{n_2}, \cdots, B_s^{n_s}$, where $\bigcap\limits_{i=1}^{s} B_i^{n_i} \neq \emptyset$ in $\mathbf{R}^{n_1+2+\cdots n_s}$. A *unit open combinatorial ball of degree* $s$ is a union

$$\widetilde{B}(n_1, n_2, \cdots, n_s) = \bigcup_{i=1}^{s} B_i^{n_i}.$$

Then a combinatorial manifold $\widetilde{M}$ is defined in the next.

**Definition** 1.1 *For a given integer sequence $n_1, n_2, \cdots, n_m, m \geq 1$ with $0 < n_1 < n_2 < \cdots < n_m$, a combinatorial manifold $\widetilde{M}$ is a Hausdorff space such that for any point $p \in \widetilde{M}$, there is a local chart $(U_p, \varphi_p)$ of $p$, i.e., an open neighborhood $U_p$ of $p$ in $\widetilde{M}$ and a homoeomorphism $\varphi_p : U_p \to \widetilde{B}(n_1(p), n_2(p), \cdots, n_{s(p)}(p))$ with*

$\{n_1(p), n_2(p), \cdots, n_{s(p)}(p)\} \subseteq \{n_1, n_2, \cdots, n_m\}$ *and* $\bigcup\limits_{p\in\widetilde{M}} \{n_1(p), n_2(p), \cdots, n_{s(p)}(p)\} =$ $\{n_1, n_2, \cdots, n_m\}$, *denoted by* $\widetilde{M}(n_1, n_2, \cdots, n_m)$ *or* $\widetilde{M}$ *on the context and*

$$\widetilde{\mathcal{A}} = \{(U_p, \varphi_p) | p \in \widetilde{M}(n_1, n_2, \cdots, n_m))\}$$

*an atlas on* $\widetilde{M}(n_1, n_2, \cdots, n_m)$. *The maximum value of* $s(p)$ *and the dimension* $\widehat{s}(p)$ *of* $\bigcap\limits_{i=1}^{s(p)} B_i^{n_i}$ *are called the dimension and the intersectional dimensional of* $\widetilde{M}(n_1, n_2, \cdots, n_m)$ *at the point* $p$, *respectively.*

A combinatorial manifold $\widetilde{M}$ is called *finite* if it is just combined by finite manifolds and *smooth* if it can be endowed with a $C^\infty$ differential structure. For a smoothly combinatorial manifold $\widetilde{M}$ and a point $p \in \widetilde{M}$, it has been shown in [4] that $\dim T_p\widetilde{M}(n_1, n_2, \cdots, n_m) = \widehat{s}(p) + \sum\limits_{i=1}^{s(p)} (n_i - \widehat{s}(p))$ and $\dim T_p^*\widetilde{M}(n_1, n_2, \cdots, n_m) = \widehat{s}(p) + \sum\limits_{i=1}^{s(p)} (n_i - \widehat{s}(p))$ with a basis

$$\{\frac{\partial}{\partial x^{hj}}|_p | 1 \le j \le \widehat{s}(p)\} \bigcup (\bigcup_{i=1}^{s(p)} \bigcup_{j=\widehat{s}(p)+1}^{n_i} \{\frac{\partial}{\partial x^{ij}}|_p \mid 1 \le j \le s\})$$

or

$$\{dx^{hj}|_p|\} 1 \le j \le \widehat{s}(p)\} \bigcup (\bigcup_{i=1}^{s(p)} \bigcup_{j=\widehat{s}(p)+1}^{n_i} \{dx^{ij}|_p \mid 1 \le j \le s\}$$

for a given integer $h, 1 \le h \le s(p)$. Denoted all $k$-forms of $\widetilde{M}(n_1, n_2, \cdots, n_m)$ by $\Lambda^k(\widetilde{M})$ and $\Lambda(\widetilde{M}) = \bigoplus\limits_{k=0}^{\widehat{s}(p)-s(p)\widehat{s}(p)+\sum_{i=1}^{s(p)} n_i} \Lambda^k(\widetilde{M})$, then there is a unique exterior differentiation $\widetilde{d} : \Lambda(\widetilde{M}) \to \Lambda(\widetilde{M})$ such that for any integer $k \ge 1$, $\widetilde{d}(\Lambda^k) \subset \Lambda^{k+1}(\widetilde{M})$ with conditions following hold similar to the classical tensor analysis([1]).

*(i)* $\widetilde{d}$ is linear, i.e., for $\forall \varphi, \psi \in \Lambda(\widetilde{M})$, $\lambda \in \mathbf{R}$,

$$\widetilde{d}(\varphi + \lambda\psi) = \widetilde{d}\varphi \wedge \psi + \lambda\widetilde{d}\psi$$

and for $\varphi \in \Lambda^k(\widetilde{M}), \psi \in \Lambda(\widetilde{M})$,

$$\widetilde{d}(\varphi \wedge \psi) = \widetilde{d}\varphi + (-1)^k \varphi \wedge \widetilde{d}\psi.$$

*(ii)* For $f \in \Lambda^0(\widetilde{M})$, $\widetilde{d}f$ is the differentiation of $f$.

*(iii)* $\widetilde{d}^2 = \widetilde{d} \cdot \widetilde{d} = 0$.

*(iv)* $\widetilde{d}$ is a local operator, i.e., if $U \subset V \subset \widetilde{M}$ are open sets and $\alpha \in \Lambda^k(V)$, then $\widetilde{d}(\alpha|_U) = (\widetilde{d}\alpha)|_U$.

Therefore, smoothly combinatorial manifolds poss a local structure analogous smoothly manifolds. But notes that this local structure maybe different for neighborhoods of different points. Whence, geometries on combinatorial manifolds are *Smarandache* geometries([6]-[8]).

There are two well-known theorems in classical tensor analysis, i.e., *Stokes'* and *Gauss'* theorems for the integration of differential $n$-forms on an $n$-manifold $M$, which enables us knowing that

$$\int_M d\omega = \int_{\partial M} \omega$$

for a $\omega \in \Lambda^{n-1}(M)$ with compact supports and

$$\int_M (\mathrm{div} X)\mu = \int_{\partial M} \mathbf{i}_X \mu$$

for a vector field $X$, where $\mathbf{i}_X : \Lambda^{k+1}(M) \to \Lambda^k(M)$ defined by $\mathbf{i}_X \varpi(X_1, X_2, \cdots, X_k) = \varpi(X, X_1, \cdots, X_k)$ for $\varpi \in \Lambda^{k+1}(M)$. The similar local properties for combinatorial manifolds with manifolds natural forwards the following questions: *wether the Stokes' or Gauss' theorem is still valid on smoothly combinatorial manifolds?* or if invalid, *What are their modified forms for smoothly combinatorial manifolds?*.

The main purpose of this paper is to find the revised Stokes' or Gauss' theorem for combinatorial manifolds, namely, the Stokes' or Gauss' theorem is still valid for $n$-forms on smoothly combinatorial manifolds $\widetilde{M}$ if $n \in \mathscr{H}_{\widetilde{M}}$, where $\mathscr{H}_{\widetilde{M}}$ is an integer set determined by the smoothly combinatorial manifold $\widetilde{M}$. For this objective, we consider a particular case of combinatorial manifolds, i.e., the combinatorial Euclidean spaces in the next section, then generalize the definition of integration on manifolds to combinatorial manifolds in Section 3. The generalized form for Stokes' or Gauss' theorem can be found in Section 4. Terminologies and notations used in this paper are standard and can be found in [1] – [2] or [4] for those of manifolds and combinatorial manifolds respectively.

## §2. **Combinatorially Euclidean Spaces**

As a simplest case of combinatorial manifolds, we characterize combinatorially euclidean spaces of finite and generalize some results in eucildean spaces in this section.

**Definition** 2.1 *For a given integer sequence $n_1, n_2, \cdots, n_m, m \geq 1$ with $0 < n_1 < n_2 < \cdots < n_m$, a combinatorially eucildean space $\widetilde{\mathbf{R}}(n_1, \cdots, n_m)$ is a union of finitely euclidean spaces $\bigcup\limits_{i=1}^{m} \mathbf{R}^{n_i}$ such that for $\forall p \in \widetilde{\mathbf{R}}(n_1, \cdots, n_m)$, $p \in \bigcap\limits_{i=1}^{m} \mathbf{R}^{n_i}$ with $\widehat{m} = \dim(\bigcap\limits_{i=1}^{m} \mathbf{R}^{n_i})$ a constant.*

By definition, we can express a point $p$ of $\widetilde{\mathbf{R}}$ by an $m \times n_m$ coordinate matrix $[\overline{x}]$ following with $x^{il} = \frac{x^l}{m}$ for $1 \leq i \leq m, 1 \leq l \leq \widehat{m}$.

$$[\overline{x}] = \begin{bmatrix} x^{11} & \cdots & x^{1\widehat{m}} & x^{1(\widehat{m})+1)} & \cdots & x^{1n_1} & \cdots & 0 \\ x^{21} & \cdots & x^{2\widehat{m}} & x^{2(\widehat{m}+1)} & \cdots & x^{2n_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ x^{m1} & \cdots & x^{m\widehat{m}} & x^{m(\widehat{m}+1)} & \cdots & \cdots & x^{mn_m-1} & x^{mn_m} \end{bmatrix}$$

For making a combinatorially Euclidean space to be a metric space, we introduce *inner product of matrixes* similar to that of vectors in the next.

**Definition** 2.2 *Let $(A) = (a_{ij})_{m\times n}$ and $(B) = (b_{ij})_{m\times n}$ be two matrixes. The inner product $\langle (A), (B) \rangle$ of $(A)$ and $(B)$ is defined by*

$$\langle (A), (B) \rangle = \sum_{i,j} a_{ij} b_{ij}.$$

**Theorem** 2.1 *Let $(A), (B), (C)$ be $m \times n$ matrixes and $\alpha$ a constant. Then*

(1) $\langle A, B \rangle = \langle B, A \rangle$;
(2) $\langle A + B, C \rangle = \langle A, C \rangle + \langle B, C \rangle$;
(3) $\langle \alpha A, B \rangle = \alpha \langle B, A \rangle$;
(4) $\langle A, A \rangle \geq 0$ with equality hold if and only if $(A) = O_{m\times n}$.

*Proof* (1)-(3) can be gotten immediately by definition. Now calculation shows that

$$\langle A, A \rangle = \sum_{i,j} a_{ij}^2 \geq 0$$

and with equality hold if and only if $a_{ij} = 0$ for any integers $i, j, 1 \leq i \leq m, 1 \leq j \leq n$, namely, $(A) = O_{m\times n}$ ♮

**Theorem** 2.2 *$(A), (B)$ be $m \times n$ matrixes. Then*

$$\langle (A), (B) \rangle^2 \leq \langle (A), (A) \rangle \langle (B), (B) \rangle$$

*and with equality hold only if $(A) = \lambda(B)$, where $\lambda$ is a constant.*

*Proof* If $(A) = \lambda(B)$, then $\langle A, B \rangle^2 = \lambda^2 \langle B, B \rangle^2 = \langle A, A \rangle \langle B, B \rangle$. Now if there are no constant $\lambda$ enabling $(A) = \lambda(B)$, then $(A) - \lambda(B) \neq O_{m\times n}$ for any real number $\lambda$. According to Theorem 2.1, we know that

$$\langle (A) - \lambda(B), (A) - \lambda(B) \rangle > 0,$$

i.e.,

$$\langle (A), (A) \rangle - 2\lambda \langle (A), (B) \rangle + \lambda^2 \langle (B), (B) \rangle > 0.$$

Therefore, we find that

$$\Delta = (-2\lambda)^2 - 4\langle (B), (B)\rangle \geq 0,$$

namely,

$$\langle (A), (B)\rangle^2 \leq \langle (A), (A)\rangle \langle (B), (B)\rangle . \quad \natural$$

**Corollary** 2.1 *For given real numbers $a_{ij}, b_{ij}$, $1 \leq i \leq m, 1 \leq j \leq n$,*

$$(\sum_{i,j} a_{ij}b_{ij})^2 \leq (\sum_{i,j} a_{ij}^2)(\sum_{i,j} b_{ij}^2).$$

Let $\widetilde{O}$ be the origin of $\widetilde{\mathbf{R}}(n_1, \cdots, n_m)$. Then $[O] = O_{m\times n_m}$. For $\forall p, q \in \widetilde{\mathbf{R}}(n_1, \cdots, n_m)$, we also call $\overrightarrow{Op}$ the vector correspondent to the point $p$ similar to classical euclidean space, Then $\overrightarrow{pq} = \overrightarrow{Oq} - \overrightarrow{Op}$. Theorem 2.2 enables us to introduce an angle between two vectors $\overrightarrow{pq}$ and $\overrightarrow{uv}$ for points $p, q, u, v \in \widetilde{\mathbf{R}}(n_1, \cdots, n_m)$.

**Definition** 2.3 *Let $p, q, u, v \in \widetilde{\mathbf{R}}(n_1, \cdots, n_m)$. Then the angle $\theta$ between vectors $\overrightarrow{pq}$ and $\overrightarrow{uv}$ is determined by*

$$\cos\theta = \frac{\langle [p]-[q], [u]-[v]\rangle}{\sqrt{\langle [p]-[q], [p]-[q]\rangle \langle [u]-[v], [u]-[v]\rangle}}$$

*with the condition $0 \leq \theta \leq \pi$.*

**Corollary** 2.2 *The conception of angle between two vectors is well defined.*

*Proof* Notice that

$$\langle [p]-[q], [u]-[v]\rangle^2 \leq \langle [p]-[q], [p]-[q]\rangle \langle [u]-[v], [u]-[v]\rangle$$

by Theorem 2.2. Thereby, we know that

$$-1 \leq \frac{\langle [p]-[q], [u]-[v]\rangle}{\sqrt{\langle [p]-[q], [p]-[q]\rangle \langle [u]-[v], [u]-[v]\rangle}} \leq 1.$$

Therefore there is a unique angle $\theta$ with $0 \leq \theta \leq \pi$ enabling Definition 2.3 hold. $\natural$

For two points $p, q$ in $\widetilde{\mathbf{R}}(n_1, \cdots, n_m)$, the *distance* $d(p, q)$ between points $p$ and $q$ is defined to be $\sqrt{\langle [p]-[q], [p]-[q]\rangle}$. We get the following result.

**Theorem** 2.3 *For a given integer sequence $n_1, n_2, \cdots, n_m, m \geq 1$ with $0 < n_1 < n_2 < \cdots < n_m$, $(\widetilde{\mathbf{R}}(n_1, \cdots, n_m); d)$ is a metric space.*

*Proof* We only need to verify each condition for a metric space is hold in $(\widetilde{\mathbf{R}}(n_1, \cdots, n_m); d)$. For two point $p, q \in \widetilde{\mathbf{R}}(n_1, \cdots, n_m)$, by definition we know that

$$d(p,q) = \sqrt{\langle [p]-[q],[p]-[q]\rangle} \geq 0$$

with equality hold if and only if $[p]=[q]$, namely, $p=q$ and

$$d(p,q) = \sqrt{\langle [p]-[q],[p]-[q]\rangle} = \sqrt{\langle [q]-[p],[q]-[p]\rangle} = d(q,p).$$

Now let $u \in \widetilde{\mathbf{R}}(n_1,\cdots,n_m)$. Then by Theorem 2.2, we find that

$$\begin{aligned}
& (d(p,u)+d(u,p))^2 \\
& = \langle [p]-[u],[p]-[u]\rangle + 2\sqrt{\langle [p]-[u],[p]-[u]\rangle \langle [u]-[q],[u]-[q]\rangle} \\
+ \quad & \langle [u]-[q],[u]-[q]\rangle \\
\geq \quad & \langle [p]-[u],[p]-[u]\rangle + \langle [p]-[u],[u]-[q]\rangle + \langle [u]-[q],[u]-[q]\rangle \\
= \quad & \langle [p]-[q],[p]-[q]\rangle = d^2(p,q).
\end{aligned}$$

Whence, $d(p,u)+d(u,p) \geq d(p,q)$ and $(\widetilde{\mathbf{R}}(n_1,\cdots,n_m);d)$ is a metric space. ♮

## §3. Integration on combinatorial manifolds

We generalize the integration on manifolds to combinatorial manifolds and show it is independent on the choice of local charts and partition of unity in this section.

### 3.1 Partition of unity

**Definition** 3.1 *Let $\widetilde{M}$ be a smoothly combinatorial manifold and $\omega \in \Lambda(\widetilde{M})$. A support set Supp$\omega$ of $\omega$ is defined by*

$$\mathrm{Supp}\omega = \overline{\{p \in \widetilde{M}; \omega(p) \neq 0\}}$$

*and say $\omega$ has compact support if* Supp$\omega$ *is compact in $\widetilde{M}$. A collection of subsets $\{C_i | i \in \widetilde{I}\}$ of $\widetilde{M}$ is called locally finite if for each $p \in \widetilde{M}$, there is a neighborhood $U_p$ of $p$ such that $U_p \cap C_i = \emptyset$ except for finitely many indices $i$.*

A *partition of unity* on a combinatorial manifold $\widetilde{M}$ is defined in the next.

**Definition** 3.2 *A partition of unity on a combinatorial manifold $\widetilde{M}$ is a collection $\{(U_i, g_i) | i \in \widetilde{I}\}$, where*

(1) $\{U_i | i \in \widetilde{I}\}$ is a locally finite open covering of $\widetilde{M}$;
(2) $g_i \in \mathscr{X}(\widetilde{M})$, $g_i(p) \geq 0$ for $\forall p \in \widetilde{M}$ and supp$g_i \in U_i$ for $i \in \widetilde{I}$;
(3) For $p \in \widetilde{M}$, $\sum\limits_i g_i(p) = 1$.

We get the next result for the partition of unity on smoothly combinatorial manifolds.

**Theorem** 3.1 *Let $\widetilde{M}$ be a smoothly combinatorial manifold. Then $\widetilde{M}$ admits partitions of unity.*

*Proof* For $\forall M \in V(G[\widetilde{M}])$, since $\widetilde{M}$ is smooth we know that $M$ is a smoothly submanifold of $\widetilde{M}$. As a byproduct, there is a partition of unity $\{(U_M^\alpha, g_M^\alpha)|\alpha \in I_M\}$ on $M$ with conditions following hold.

(1) $\{U_M^\alpha|\alpha \in I_M\}$ is a locally finite open covering of $M$;

(2) $g_M^\alpha(p) \geq 0$ for $\forall p \in M$ and $\mathrm{supp} g_M^\alpha \in U_M^\alpha$ for $\alpha \in I_M$;

(3) For $p \in M$, $\sum\limits_i g_M^i(p) = 1$.

By definition, for $\forall p \in \widetilde{M}$, there is a local chart $(U_p, [\varphi_p])$ enable $\varphi_p : U_p \to B^{n_{i_1}} \bigcup B^{n_{i_2}} \bigcup \cdots \bigcup B^{n_{i_{s(p)}}}$ with $B^{n_{i_1}} \bigcap B^{n_{i_2}} \bigcap \cdots \bigcap B^{n_{i_{s(p)}}} \neq \emptyset$. Now let $U^\alpha_{M_{i_1}}$, $U^\alpha_{M_{i_2}}$, $\cdots$, $U^\alpha_{M_{i_{s(p)}}}$ be $s(p)$ open sets on manifolds $M, M \in V(G[\widetilde{M}])$ such that

$$p \in U_p^\alpha = \bigcup_{h=1}^{s(p)} U^\alpha_{M_{i_h}}. \quad (3.1)$$

We define

$$\widetilde{S}(p) = \{U_p^\alpha| \text{ all integers } \alpha \text{ enabling (3.1) hold}\}.$$

Then

$$\widetilde{\mathcal{A}} = \bigcup_{p \in \widetilde{M}} \widetilde{S}(p) = \{U_p^\alpha | \alpha \in \widetilde{I}(p)\}$$

is locally finite covering of the combinatorial manifold $\widetilde{M}$ by properties $(1)-(3)$. For $\forall U_p^\alpha \in \widetilde{S}(p)$, define

$$\sigma_{U_p^\alpha} = \sum_{s \geq 1} \sum_{\{i_1, i_2, \cdots, i_s\} \subset \{1, 2, \cdots, s(p)\}} (\prod_{h=1}^{s} g_{M^\varsigma_{i_h}})$$

and

$$g_{U_p^\alpha} = \frac{\sigma_{U_p^\alpha}}{\sum\limits_{\widetilde{V} \in \widetilde{S}(p)} \sigma_{\widetilde{V}}}.$$

Then it can be checked immediately that $\{(U_p^\alpha, g_{U_p^\alpha})|p \in \widetilde{M}, \alpha \in \widetilde{I}(p)\}$ is a partition of unity on $\widetilde{M}$ by properties (1)-(3) on $g_M^\alpha$ and the definition of $g_{U_p^\alpha}$. $\natural$

**Corollary** 3.1 *Let $\widetilde{M}$ be a smoothly combinatorial manifold with an atlas $\widetilde{\mathcal{A}} = \{(V_\alpha, [\varphi_\alpha])|\alpha \in \widetilde{I}\}$ and $t_\alpha$ be a $C^k$ tensor field, $k \geq 1$, of field type $(r, s)$ defined on $V_\alpha$ for each $\alpha$, and assume that there exists a partition of unity $\{(U_i, g_i)|i \in J\}$*

*subordinate to* $\widetilde{\mathcal{A}}$*, i.e., for* $forall i \in J$*, there exists* $\alpha(i)$ *such that* $U_i \subset V_{\alpha(i)}$*. Then for* $\forall p \in \widetilde{M}$*,*

$$t(p) = \sum_i g_i t_{\alpha(i)}$$

*is a* $C^k$ *tensor field of type* $(r, s)$ *on* $\widetilde{M}$

*Proof* Since $\{U_i | i \in J\}$ is locally finite, the sum at each point $p$ is a finite sum and $t(p)$ is a type $(r, s)$ for every $p \in \widetilde{M}$. Notice that $t$ is $C^k$ since the local form of $t$ in a local chart $(V_{\alpha(i)}, [\varphi_{\alpha(i)}])$ is

$$\sum_j g_i t_{\alpha(j)},$$

where the summation taken over all indices $j$ such that $V_{\alpha(i)} \bigcap V_{\alpha(j)} \neq \emptyset$. Those number $j$ is finite by the local finiteness. $\natural$

3.2 **Integration on combinatorial manifolds**

First, we introduce integration on combinatorial Euclidean spaces. Let $\widetilde{\mathbf{R}}(n_1, \cdots, n_m)$ be a combinatorially euclidean space and

$$\tau : \widetilde{\mathbf{R}}(n_1, \cdots, n_m) \to \widetilde{\mathbf{R}}(n_1, \cdots, n_m)$$

a $C^1$ differential mapping with

$$[\overline{y}] = [y^{\kappa\lambda}]_{m \times n_m} = [\tau^{\kappa\lambda}([x^{\mu\nu}])]_{m \times n_m}.$$

The *Jacobi matrix* of $f$ is defined by

$$\frac{\partial[\overline{y}]}{\partial[\overline{x}]} = [A_{(\kappa\lambda)(\mu\nu)}],$$

where $A_{(\kappa\lambda)(\mu\nu)} = \frac{\partial \tau^{\kappa\lambda}}{\partial x^{\mu\nu}}$.

Now let $\omega \in T_k^0(\widetilde{\mathbf{R}}(n_1, \cdots, n_m))$, a *pull-back* $\tau^*\omega \in T_k^0(\widetilde{\mathbf{R}}(n_1, \cdots, n_m))$ is defined by

$$\tau^*\omega(a_1, a_2, \cdots, a_k) = \omega(f(a_1), f(a_2), \cdots, f(a_k))$$

for $\forall a_1, a_2, \cdots, a_k \in \widetilde{R}$.

Denoted by $n = \sum_{i=1}^{m} n_i - \widehat{m}m$. If $0 \leq l \leq n$, recall([4]) that the basis of $\Lambda^l(\widetilde{\mathbf{R}}(n_1, \cdots, n_m))$ is

$$\{\mathbf{e}^{i_1} \wedge \mathbf{e}^{i_2} \wedge \cdots \wedge \mathbf{e}^{i_l} | 1 \leq i_1 < i_2 \cdots < i_l \leq n\}$$

for a basis $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n$ of $\widetilde{\mathbf{R}}(n_1, \cdots, n_m)$ and its dual basis $\mathbf{e}^1, \mathbf{e}^2, \cdots, \mathbf{e}^n$. Thereby the dimension of $\Lambda^l(\widetilde{\mathbf{R}}(n_1, \cdots, n_m))$ is

$$\left( \begin{array}{c} n \\ l \end{array} \right) = \frac{(\sum\limits_{i=1}^{m} n_i - \widehat{m}m)!}{l!(\sum\limits_{i=1}^{m} n_i - \widehat{m}m - l)!}.$$

Whence $\Lambda^n(\widetilde{\mathbf{R}}(n_1, \cdots, n_m))$ is one-dimensional. Now if $\omega_0$ is a basis of $\Lambda^n(\widetilde{R})$, we then know that its each element $\omega$ can be represented by $\omega = c\omega_0$ for a number $c \in \mathbf{R}$. Let $\tau : \widetilde{\mathbf{R}}(n_1, \cdots, n_m) \to \widetilde{\mathbf{R}}(n_1, \cdots, n_m)$ be a linear mapping. Then

$$\tau^* : \Lambda^n(\widetilde{\mathbf{R}}(n_1, \cdots, n_m)) \to \Lambda^n(\widetilde{\mathbf{R}}(n_1, \cdots, n_m))$$

is also a linear mapping with $\tau^*\omega = c\tau^*\omega_0 = b\omega$ for a unique constant $b = \det\tau$, called the determinant of $\tau$. It has been known that ([1])

$$\det\tau = \det(\frac{\partial[\overline{y}]}{\partial[\overline{x}]})$$

for a given basis $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n$ of $\widetilde{\mathbf{R}}(n_1, \cdots, n_m)$ and its dual basis $\mathbf{e}^1, \mathbf{e}^2, \cdots, \mathbf{e}^n$, where $n = \sum\limits_{i=1}^{m} n_i - \widehat{m}m$.

**Definition** 3.3 *Let $\widetilde{\mathbf{R}}(n_1, n_2, \cdots, n_m)$ be a combinatorial Euclidean space,$n = \widehat{m} + \sum\limits_{i=1}^{m}(n_i - \widehat{m})$, $\widetilde{U} \subset \widetilde{\mathbf{R}}(n_1, n_2, \cdots, n_m)$ and $\omega \in \Lambda^n(U)$ have compact support with*

$$\omega(x) = \omega_{(\mu_{i_1}\nu_{i_1})\cdots(\mu_{i_n}\nu_{i_n})} dx^{\mu_{i_1}\nu_{i_1}} \wedge \cdots \wedge dx^{\mu_{i_n}\nu_{i_n}}$$

*relative to the standard basis $\mathbf{e}^{\mu\nu}, 1 \leq \mu \leq m, 1 \leq \nu \leq n_m$ of $\widetilde{\mathbf{R}}(n_1, n_2, \cdots, n_m)$ with $\mathbf{e}^{\mu\nu} = e^{\nu}$ for $1 \leq \mu \leq \widehat{m}$. An integral of $\omega$ on $\widetilde{U}$ is defined to be a mapping $\int_{\widetilde{U}} : f \to \int_{\widetilde{U}} f \in \mathbf{R}$ with*

$$\int_{\widetilde{U}} \omega = \int \omega(x) \prod_{\nu=1}^{\widehat{m}} dx^{\nu} \prod_{\mu \geq \widehat{m}+1, 1 \leq \nu \leq n_i} dx^{\mu\nu}, \qquad (3.2)$$

*where the right hand side of* (3.2) *is the Riemannian integral of $\omega$ on $\widetilde{U}$.*

For example, consider the combinatorial Euclidean space $\widetilde{\mathbf{R}}(3,5)$ with $\mathbf{R}^3 \cap \mathbf{R}^5 = \mathbf{R}$. Then the integration of an $\omega \in \Lambda^6(\widetilde{U})$ for an open subset $\widetilde{U} \in \widetilde{\mathbf{R}}(3,5)$ is

$$\int_{\widetilde{U}} \omega = \int_{\widetilde{U} \cap (\mathbf{R}^3 \cup \mathbf{R}^5)} \omega(x) dx^1 dx^{12} dx^{13} dx^{22} dx^{23} dx^{24} dx^{25}.$$

**Theorem** 3.2 *Let $U$ and $V$ be open subsets of $\widetilde{\mathbf{R}}(n_1,\cdots,n_m)$ and $\tau : U \to V$ is an orientation-preserving diffeomorphism. If $\omega \in \Lambda^n(V)$ has compact support for $n = \sum\limits_{i=1}^{m} n_i - \widehat{m}m$, then $\tau^*\omega \in \Lambda^n(U)$ has compact support and*

$$\int \tau^*\omega = \int \omega.$$

*Proof* Let $\omega(x) = \omega_{(\mu_{i_1}\nu_{i_1})\cdots(\mu_{i_n}\nu_{i_n})}dx^{\mu_{i_1}\nu_{i_1}} \wedge \cdots \wedge dx^{\mu_{i_n}\nu_{i_n}} \in \Lambda^n(V)$. Since $\tau$ is a diffeomorphism, the support of $\tau^*\omega$ is $\tau^{-1}(\mathrm{supp}\omega)$, which is compact by that of $\mathrm{supp}\omega$ compact.

By the usual change of variables formula, since $\tau^*\omega = (\omega \circ \tau)(\det\tau)\omega_0$ by definition, where $\omega_0 = dx^1 \wedge \cdots \wedge dx^{\widehat{m}} \wedge dx^{1(\widehat{m}+1)} \wedge dx^{1(\widehat{m}+2)} \wedge \cdots \wedge dx^{1n_1} \wedge \cdots \wedge dx^{mn_m}$, we then get that

$$\begin{aligned}\int \tau^*\omega &= \int (\omega \circ \tau)(\det\tau) \prod_{\nu=1}^{\widehat{m}} dx^{\nu} \prod_{\mu \geq \widehat{m}+1, 1 \leq \nu \leq n_\mu} dx^{\mu\nu} \\ &= \int \omega. \quad \natural\end{aligned}$$

**Definition** 3.4 *Let $\widetilde{M}$ be a smoothly combinatorial manifold. If there exists a family $\{(U_\alpha, [\varphi_\alpha]|\alpha \in \widetilde{I})\}$ of local charts such that*

(1) $\bigcup\limits_{\alpha \in \widetilde{I}} U_\alpha = \widetilde{M}$;

(2) *for $\forall \alpha, \beta \in \widetilde{I}$, either $U_\alpha \bigcap U_\beta = \emptyset$ or $U_\alpha \bigcap U_\beta \neq \emptyset$ but for $\forall p \in U_\alpha \bigcap U_\beta$, the Jacobi matrix*

$$det(\frac{\partial[\varphi_\beta]}{\partial[\varphi_\alpha]}) > 0,$$

*then $\widetilde{M}$ is called an oriently combinatorial manifold and $(U_\alpha, [\varphi_\alpha])$ an oriented chart for $\forall \alpha \in \widetilde{I}$.*

For a smoothly combinatorial manifold $\widetilde{M}(n_1,\cdots,n_m)$, it must be finite by definition. Whence, there exists an atlas $\mathscr{C} = \{(\widetilde{U}_\alpha, [\varphi_\alpha])|\alpha \in \widetilde{I}\}$ on $\widetilde{M}(n_1,\cdots,n_m)$ consisting of positively oriented charts such that for $\forall \alpha \in \widetilde{I}$, $\widehat{s}(p) + \sum\limits_{i=1}^{s(p)} (n_i - \widehat{s}(p))$ is an constant $n_{\widetilde{U}_\alpha}$ for $\forall p \in \widetilde{U}_\alpha$. Denote such atlas on $\widetilde{M}(n_1,\cdots,n_m)$ by $\mathscr{C}_{\widetilde{M}}$ and an integer family $\mathscr{H}_{\widetilde{M}} = \{n_{\widetilde{U}_\alpha}|\alpha \in \widetilde{I}\}$.

Now for any integer $n \in \mathscr{H}_{\widetilde{M}}$, we can define an integral of $n$-forms on a smoothly combinatorial manifold $\widetilde{M}(n_1,\cdots,n_m)$.

**Definition** 3.5 *Let $\widetilde{M}$ be a smoothly combinatorial manifold with orientation $\mathscr{O}$ and $(\widetilde{U};[\varphi])$ a positively oriented chart with a constant $n_{\widetilde{U}}$. Suppose $\omega \in \Lambda^{n_{\widetilde{U}}}(\widetilde{M}), \widetilde{U} \subset \widetilde{M}$ has compact support $\widetilde{C} \subset \widetilde{U}$. Then define*

$$\int_{\widetilde{C}} \omega = \int \varphi_*(\omega|_{\widetilde{U}}). \qquad (3.3)$$

*Now if $\mathscr{C}_{\widetilde{M}}$ is an atlas of positively oriented charts with an integer set $\mathscr{H}_{\widetilde{M}}$, let $\widetilde{P} = \{(\widetilde{U}_\alpha, \varphi_\alpha, g_\alpha)|\alpha \in \widetilde{I}\}$ be a partition of unity subordinate to $\mathscr{C}_{\widetilde{M}}$. For $\forall\omega \in \Lambda^n(\widetilde{M})$, $n \in \mathscr{H}_{\widetilde{M}}$, an integral of $\omega$ on $\widetilde{P}$ is defined by*

$$\int_{\widetilde{P}} \omega = \sum_{\alpha\in\widetilde{I}} \int g_\alpha\omega. \qquad (3.4)$$

The next result shows that the integral of $n$-forms, $n \in \mathscr{H}_{\widetilde{M}}$ is well-defined.

**Theorem** 3.3 *Let $\widetilde{M}(n_1, \cdots, n_m)$ be a smoothly combinatorial manifold. For $n \in \mathscr{H}_{\widetilde{M}}$, the integral of $n$-forms on $\widetilde{M}(n_1, \cdots, n_m)$ is well-defined, namely, the sum on the right hand side of (3.4) contains only a finite number of nonzero terms, not dependent on the choice of $\mathscr{C}_{\widetilde{M}}$ and if $P$ and $Q$ are two partitions of unity subordinate to $\mathscr{C}_{\widetilde{M}}$, then*

$$\int_{\widetilde{P}} \omega = \int_{\widetilde{Q}} \omega.$$

*Proof* By definition for any point $p \in \widetilde{M}(n_1, \cdots, n_m)$, there is a neighborhood $\widetilde{U}_p$ such that only a finite number of $g_\alpha$ are nonzero on $\widetilde{U}_p$. Now by the compactness of supp$\omega$, only a finite number of such neighborhood cover supp$\omega$. Therefore, only a finite number of $g_\alpha$ are nonzero on the union of these $\widetilde{U}_p$, namely, the sum on the right hand side of (3.4) contains only a finite number of nonzero terms.

Notice that the integral of $n$-forms on a smoothly combinatorial manifold $\widetilde{M}(n_1, \cdots, n_m)$ is well-defined for a local chart $\widetilde{U}$ with a constant $n_{\widetilde{U}} = \widehat{s}(p) + \sum\limits_{i=1}^{s(p)}(n_i - \widehat{s}(p))$ for $\forall p \in \widetilde{U} \subset \widetilde{M}(n_1, \cdots, n_m)$ by (3.3) and Definition 3.3. Whence each term on the right hand side of (3.4) is well-defined. Thereby $\int_{\widetilde{P}} \omega$ is well-defined.

Now let $\widetilde{P} = \{(\widetilde{U}_\alpha, \varphi_\alpha, g_\alpha)|\alpha \in \widetilde{I}\}$ and $\widetilde{Q} = \{(\widetilde{V}_\beta, \varphi_\beta, h_\beta)|\beta \in \widetilde{J}\}$ be partitions of unity subordinate to atlas $\mathscr{C}_{\widetilde{M}}$ and $\mathscr{C}^*_{\widetilde{M}}$ with respective integer sets $\mathscr{H}_{\widetilde{M}}$ and $\mathscr{H}^*_{\widetilde{M}}$. Then these functions $\{g_\alpha h_\beta\}$ satisfy $g_\alpha h_\beta(p) = 0$ except only for a finite number of index pairs $(\alpha, \beta)$ and

$$\sum_\alpha \sum_\beta g_\alpha h_\beta(p) = 1, \quad \text{for } \forall p \in \widetilde{M}(n_1, \cdots, n_m).$$

Since $\sum\limits_{\beta} = 1$, we then get that

$$\begin{aligned}\int_{\widetilde{P}} &= \sum_{\alpha}\int g_{\alpha}\omega\\ &= \sum_{\beta}\sum_{\alpha}\int h_{\beta}g_{\alpha}\omega\\ &= \sum_{\alpha}\sum_{\beta}\int g_{\alpha}h_{\beta}\omega\\ &= \int_{\widetilde{Q}}\omega. \quad \natural\end{aligned}$$

Now let $n_1, n_2, \cdots, n_m$ be a positive integer sequence. For any point $p \in \widetilde{M}$, if there is a local chart $(\widetilde{U}_p, [\varphi_p])$ such that $[\varphi_p] : U_p \rightarrow B^{n_1}\bigcup B^{n_2}\bigcup\cdots\bigcup B^{n_m}$ with $B^{n_1}\bigcap B^{n_2}\bigcap\cdots\bigcap B^{n_m} \neq \emptyset$, then $\widetilde{M}$ is called a *homogenously combinatorial manifold*. Particularly, if $m = 1$, a homogenously combinatorial manifold is nothing but a manifold. We then get consequences for the integral of $(\widehat{m}+\sum\limits_{i=1}^{m}(n_i-\widehat{m}))$-forms on $n$-manifolds.

**Corollary** 3.2 *The integral of $(\widehat{m} + \sum\limits_{i=1}^{m}(n_i - \widehat{m}))$-forms on a homogenously combinatorial manifold $\widetilde{M}(n_1, n_2, \cdots, n_m)$ is well-defined, particularly, the integral of $n$-forms on an $n$-manifold is well-defined.*

Similar to Theorem 3.2 for the *change of variables formula of integral* in combinatorial Euclidean space, we get that of formula in smoothly combinatorial manifolds.

**Theorem** 3.4 *Let $\widetilde{M}$ and $\widetilde{N}$ be oriently combinatorial manifolds and $\tau : \widetilde{M} \rightarrow \widetilde{N}$ an orientation-preserving diffeomorphism. If $\omega \in \Lambda(\widetilde{N})$ has compact support, then $\tau^*\omega$ has compact support and*

$$\int \omega = \int \tau^*\omega.$$

*Proof* Notice that $\mathrm{supp}\tau^*\omega = \tau^{-1}(\mathrm{supp}\omega)$. Thereby $\tau^*\omega$ has compact support since $\omega$ has so. Now let $\{(U_i, \varphi_i)|i \in \widetilde{I}\}$ be an atlas of positively oriented charts of $\widetilde{M}$ and $\widetilde{P} = \{g_i|i \in \widetilde{I}\}$ a subordinate partition of unity with constants $n_{U_i}$. Then $\{(\tau(U_i), \varphi_i \circ \tau^{-1})|i \in \widetilde{I}\}$ is an atlas of positively oriented charts of $\widetilde{N}$ and $\widetilde{Q} = \{g_i \circ \tau^{-1}\}$ is a partition of unity subordinate to the covering $\{\tau(U_i)|i \in \widetilde{I}\}$ with constants $n_{\tau(U_i)}$. Whence, we get that

$$\int \tau^*\omega \;=\; \sum_{i}\int g_i\tau^*\omega = \sum_{i}\int \varphi_{i*}(g_i\tau^*\omega)$$

$$= \sum_i \int \varphi_{i*}(\tau^{-1})_*(g_i \circ \tau^{-1})\omega$$

$$= \sum_i \int (\varphi_i \circ \tau^{-1})_*(g_i \circ \tau^{-1})\omega$$

$$= \int \omega. \qquad \natural$$

§4. **A generalization of Stokes' theorem**

**Definition** 4.1 *Let $\widetilde{M}$ be a smoothly combinatorial manifold. A subset $\widetilde{D}$ of $\widetilde{M}$ is with boundary if its points can be classified into two classes following.*

**Class** 1**(interior point Int$\widetilde{D}$)** *For $\forall p \in \mathrm{Int}D$, there is a neighborhood $\widetilde{V}_p$ of $p$ enable $\widetilde{V}_p \subset \widetilde{D}$.*

**Case** 2**(boundary $\partial\widetilde{D}$)** *For $\forall p \in \partial\widetilde{D}$, there is integers $\mu, \nu$ for a local chart $(U_p; [\varphi_p])$ of $p$ such that $x^{\mu\nu}(p) = 0$ but*

$$\widetilde{U}_p \cap \widetilde{D} = \{q | q \in U_p, x^{\kappa\lambda} \geq 0 \ for \ \forall \{\kappa, \lambda\} \neq \{\mu, \nu\}\}.$$

Then we generalize the famous *Stokes theorem* on manifolds in the next.

**Theorem** 4.1 *Let $\widetilde{M}$ be a smoothly combinatorial manifold with an integer set $\mathscr{H}_{\widetilde{M}}$ and $\widetilde{D}$ a boundary subset of $\widetilde{M}$. For $n \in \mathscr{H}_{\widetilde{M}}$ if $\omega \in \Lambda^n(\widetilde{M})$ has compact support, then*

$$\int_{\widetilde{D}} d\omega = \int_{\partial\widetilde{D}} \omega$$

*with the convention $\int_{\partial\widetilde{D}} \omega = 0$ while $\partial\widetilde{D} = \emptyset$.*

*Proof* By Definition 3.5, the integration on a smoothly combinatorial manifold was constructed with partitions of unity subordinate to an atlas. Let $\mathscr{C}_{\widetilde{M}}$ be an atlas of positively oriented charts with an integer set $\mathscr{H}_{\widetilde{M}}$ and $\widetilde{P} = \{(\widetilde{U}_\alpha, \varphi_\alpha, g_\alpha) | \alpha \in \widetilde{I}\}$ a partition of unity subordinate to $\mathscr{C}_{\widetilde{M}}$. Since supp$\omega$ is compact, we know that

$$\int_{\widetilde{D}} d\omega = \sum_{\alpha \in \widetilde{I}} \int_{\widetilde{D}} d(g_{\alpha\omega}),$$

$$\int_{\partial\widetilde{D}} \omega = \sum_{\alpha \in \widetilde{I}} \int_{\partial\widetilde{D}} g_{\alpha\omega}.$$

and there are only finite nonzero terms on the right hand side of the above two formulae. Thereby, we only need to prove

$$\int_{\widetilde{D}} d(g_\alpha \omega) = \int_{\partial \widetilde{D}} g_\alpha \omega$$

for $\forall \alpha \in \widetilde{I}$.

Not loss of generality we can assume that $\omega$ is an $n$-forms on a local chart $(U, \varphi)$ with compact support. Now write

$$\omega(x) = \omega_{(\mu_{i_1}\nu_{i_1})\cdots(\mu_{i_n}\nu_{i_n})} dx^{\mu_{i_1}\nu_{i_1}} \wedge \cdots \wedge dx^{\mu_{i_n}\nu_{i_n}}$$

$$\omega = \sum_{h=1}^{n} (-1)^{h-1} \omega_{\mu_{i_h}\nu_{i_h}} dx^{\mu_{i_1}\nu_{i_1}} \wedge \cdots \wedge \widehat{dx^{\mu_{i_h}\nu_{i_h}}} \wedge \cdots \wedge dx^{\mu_{i_n}\nu_{i_n}},$$

where $\widehat{dx^{\mu_{i_h}\nu_{i_h}}}$ means that $dx^{\mu_{i_h}\nu_{i_h}}$ is deleted, where

$$i_h \in \{1, \cdots, \widehat{n}_U, (1(\widehat{n}_U + 1)), \cdots, (1n_1), (2(\widehat{n}_U + 1)), \cdots, (2n_2), \cdots, (mn_m)\}.$$

Then

$$d\omega = \sum_{i=1}^{n} \frac{\partial \omega_i}{\partial x^i} dx^{\mu_{i_1}\nu_{i_1}} \wedge \cdots \wedge dx^{\mu_{i_h}\nu_{i_h}} \wedge \cdots \wedge dx^{\mu_{i_n}\nu_{i_n}}. \qquad (4.1)$$

Consider the appearance of chart $U$. There are two cases must be considered.

**Case** 1 $\quad U \bigcap \partial U = \emptyset$

In this case, $\int_{\partial U} \omega = 0$ and $U$ is in $\widetilde{M} \setminus \widetilde{D}$ or in $\mathbf{Int}\widetilde{D}$. The former is naturally implies that $\int_{\widetilde{D}} d(g_\alpha \omega) = 0$. For the later, we find that

$$\int_{\widetilde{D}} d\omega = \sum_{i=1}^{n} \int_U \frac{\partial \omega_i}{\partial x^i} dx^{\mu_{i_1}\nu_{i_1}} \cdots dx^{\mu_{i_n}\nu_{i_n}}. \qquad (4.2)$$

Notice that $\int_{-\infty}^{+\infty} \frac{\partial \omega_i}{\partial x^i} dx^i = 0$ since $\omega_i$ has compact support. Thus $\int_U d\omega = 0$ as desired.

**Case** 2 $\quad \partial U \neq \emptyset$

In this case we can do the same trick for each term except the last. Without loss of generality, assume that

$$U \bigcap \widetilde{D} = \{q | q \in U, x^n(q) \geq 0\}$$

and

$$U \bigcap \partial \widetilde{D} = \{q | q \in U, x^n(q) = 0\}.$$

Then we get that

$$\begin{aligned}\int_{\partial\widetilde{D}}\omega &= \int_{U\cap\partial\widetilde{D}}\omega \\ &= \sum_{h=1}^{n}(-1)^{h-1}\int_{U\cap\partial\widetilde{D}}\omega_{\mu_{i_h}\nu_{i_h}}dx^{\mu_{i_1}\nu_{i_1}}\wedge\cdots\wedge\widehat{dx^{\mu_{i_h}\nu_{i_h}}}\wedge\cdots\wedge dx^{\mu_{i_n}\nu_{i_n}} \\ &= (-1)^{n-1}\int_{U\cap\partial\widetilde{D}}\omega_{\mu_n\nu_n}dx^{\mu_{i_1}\nu_{i_1}}\wedge\cdots\wedge dx^{\mu_{i_{n-1}}\nu_{i_{n-1}}}\end{aligned}$$

since $dx^n(q)=0$ for $q\in U\cap\partial\widetilde{D}$. Notice that $\mathbf{R}^{n-1}=\partial\mathbf{R}^n_+$ but the usual orientation on $\mathbf{R}^{n-1}$ is not the boundary orientation, whose outward unit normal is $-\mathbf{e}_n=(0,\cdots,o,-1)$. Hence

$$\int_{\partial\widetilde{D}}\omega=-\int_{\partial\mathbf{R}^n_+}\omega_{\mu_n\nu_n}(x^{\mu_{i_1}\nu_{i_1}},\cdots,x^{\mu_{i_{n-1}}\nu_{i_{n-1}}},0)dx^{\mu_{i_1}\nu_{i_1}}\cdots dx^{\mu_{i_{n-1}}\nu_{i_{n-1}}}.$$

On the other hand, by the fundamental theorem of calculus,

$$\begin{aligned}&\int_{\mathbf{R}^{n-1}}(\int_0^\infty\frac{\partial\omega_{\mu_n\nu_n}}{\partial x^{\mu_n\nu_n}})dx^{\mu_{i_1}\nu_{i_1}}\cdots dx^{\mu_{i_{n-1}}\nu_{i_{n-1}}} \\ &=-\int_{\mathbf{R}^{n-1}}\omega_{\mu_n\nu_n}(x^{\mu_{i_1}\nu_{i_1}}\cdots x^{\mu_{i_n}\nu_{i_n}},0)dx^{\mu_{i_1}\nu_{i_1}}\cdots dx^{\mu_{i_{n-1}}\nu_{i_{n-1}}}.\end{aligned}$$

Since $\omega_{\mu_{i_n}\nu_{i_n}}$ has compact support, thus

$$\int_U\omega=-\int_{\mathbf{R}^{n-1}}\omega_{\mu_n\nu_n}(x^{\mu_{i_1}\nu_{i_1}}\cdots x^{\mu_{i_n}\nu_{i_n}},0)dx^{\mu_{i_1}\nu_{i_1}}\cdots dx^{\mu_{i_{n-1}}\nu_{i_{n-1}}}.$$

Therefore, we get that

$$\int_{\widetilde{D}}d\omega=\int_{\partial\widetilde{D}}\omega$$

This completes the proof. ♮

Corollaries following are immediately obtained by Theorem 4.1

**Corollary** 4.1 *Let $\widetilde{M}$ be a smoothly and homogenously combinatorial manifold with an integer set $\mathscr{H}_{\widetilde{M}}$ and $\widetilde{D}$ a boundary subset of $\widetilde{M}$. For $n\in\mathscr{H}_{\widetilde{M}}$ if $\omega\in\Lambda^n(\widetilde{M})$ has compact support, then*

$$\int_{\widetilde{D}}d\omega=\int_{\partial\widetilde{D}}\omega,$$

*particularly, if $\widetilde{M}$ is nothing but a manifold, the Stokes theorem holds.*

**Corollary** 4.2 *Let $\widetilde{M}$ be a smoothly combinatorial manifold with an integer set $\mathscr{H}_{\widetilde{M}}$. For $n \in \mathscr{H}_{\widetilde{M}}$, if $\omega \in \Lambda^n(\widetilde{M})$ has a compact support, then*

$$\int_{\widetilde{M}} \omega = 0.$$

Similar to the case of manifolds, we find a generalization for *Gauss theorem* in the next.

**Theorem** 4.2 *Let $\widetilde{M}$ be a smoothly combinatorial manifold with an integer set $\mathscr{H}_{\widetilde{M}}$, $\widetilde{D}$ a boundary subset of $\widetilde{M}$ and $\mathbf{X}$ a vector field on $\widetilde{M}$ with compact support. Then*

$$\int_{\widetilde{D}} (\mathrm{div}\mathbf{X})\mathbf{v} = \int_{\partial\widetilde{D}} \mathbf{i}_{\mathbf{X}}\mathbf{v},$$

*where $\mathbf{v}$ is a volume form on $\widetilde{M}$, i.e., nonzero elements in $\Lambda^n(\widetilde{M})$ for $n \in \mathscr{H}_{\widetilde{M}}$.*

*Proof* This result is also a consequence of Theorem 4.1. Notice that

$$(\mathrm{div}\mathbf{X})\mathbf{v} = d\mathbf{i}_{\mathbf{X}}\mathbf{v} + \mathbf{i}_{\mathbf{X}}d\mathbf{v} = d\mathbf{i}_{\mathbf{X}}.$$

According to Theorem 4.1, we then get that

$$\int_{\widetilde{D}} (\mathrm{div}\mathbf{X})\mathbf{v} = \int_{\partial\widetilde{D}} \mathbf{i}_{\mathbf{X}}\mathbf{v}. \quad \natural$$

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