Brocard’s Problem. Variants of Brocard’s Problem

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Abstract

In this article we considered an open problem. One of the problems in the list of open problems of General Number Theory, existing in [1], [2] is the Brocard’s Problem, asking to find integer values of \( n \), for which \( n! + 1 = m^2 \). 'Introduction' section is dedicated to the statement of the main problem. We presented some historical overview and known facts about this problem in the 'Historical overview and known facts' section, based on information presented in the web [1], [2]. In the section 'Variants of the Problem' several variants of the Problem are presented by author based on more general \( n! + A = k^2 \) [4] equation and asked to find solutions for them.

Short Introduction to the problem

One of the open problems in General Number Theory as well as in Mathematics is the Brocard’s Problem. Brocard’s Problem asks to find integer values of \( n \), for which \( n! + 1 = m^2 \), where \( n! \) is the factorial. It was posed by Henri Brocard in a pair of articles in 1876 and 1885, and independently in 1913 by Ramanujan.

More generally the problem has the following form

**Do integers, \( n, m \), exist such that \( n! + 1 = m^2 \), other than \( n = 4, 5, 7 \)?**

Historical overview and known facts

Pairs of the numbers \( (n, m) \) that solve Brocard’s problem are called Brown numbers. The only known solutions are \( n = 4, 5 \) and \( 7 \). There are only three known pairs of Brown numbers: \( (4,5), (5,11), \) and \( (7,71) \).

1. In 1906, Gérardin claimed that, if \( n > 71 \), then \( m \) must have at least 20 digits.
2. In 1935, Gupta stated that calculations of \( n! \) up to \( n = 63 \) gave no future solutions.
3. In 1986, Wells claimed, that there are no other solutions with \( n < 10^7 \).
4. Paul Erdős conjectured that no other solutions exist.
5. In 1993, Overholt showed that there are only finitely many solutions provided that the abc conjecture [3] is true.
6. In 1994, Guy claimed, that it is virtually certain that there are no more solutions...
7. In 2000, Berndt and Galway [6] performed calculations for \( n \) up to \( 10^9 \) and found no further solutions.

Wilson has also computed the least \( k \) such that \( n! + k^2 \) is square starting at \( n = 4 \), giving 1, 1, 3, 1, 9, 27, 15, 18, 288, 288, 420, 464, 1856... (Sloane’s A038202).

**Variants of the Problem**

It is natural to consider the more general diophantine equation \( n! + A = k^2 \) (1). An information about solutions and properties of general equation can be found in [4].

From this point, We would like to present some new versions of equation for the Brocard’s Problem.

It seems, from general point of view, that equations presented below, are also has right to be considered as an equation for Brocard’s problem

1. \((n + \alpha)! + A = k^2\). This equation is \(\alpha\)-Brocard’s Problem, where \(\alpha > 0\) or \(\alpha < 0\) an integer, parameter. When \(\alpha = 0\) we have Brocard’s problem. Find solutions for this problem.
2. \(n! + A = (k + \beta)^2\). This equation is \(\beta\)-Brocard’s Problem, where \(\beta > 0\) or \(\beta < 0\) an integer, parameter. When \(\beta = 0\) we have Brocard’s problem. Find solutions for this problem.
3. \((n + \alpha)! + A = (k + \beta)^2\). This equation can be considered as \(\alpha, \beta\)-Brocard’s Problem. \(\alpha, \beta\) are integers. Find solutions for this problem.
4. Find solutions for \((n + \alpha)! = (k + \beta)^2\) equation, i.e. \(A = 0\).
5. Find solutions for \((n! + A)! + (k + \beta) = (k + \beta)^{(n!+A)}\) equation. It is clear, that when we have \((n! + A) = 2\), our equation comes to \(2! + (k + \beta) = (k + \beta)^2\), which in its turn can be considered as Brocard’s Problem, when \(A = k\) in (1).

**References**