Logistic equation of Human population growth
(generalization to the case of reactive environment).

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A key points of new approach for modeling of the population dynamics in reactive environment are presented here: 1) generalization of the Logistic equation to the case of reactive environment for modeling of population dynamics (or for the fulfilling of the ecological niches); 2) new type of asymptotic solution for such equation (*which is tested on human population growth*); 3) reduction to the Abel ODE in general case.
Due to a very special character of Abel ODE, it's general solution is proved to have a jumping or the break-down of the components for such a solution.
It means an existence of continuous general solution only at some definite, restricted range of time-parameter, or a possibility of sudden gradient catastrophe in regard to the components of solution (*population growth*), at the definite moment of time-parameter.

According to the results [1-3], logistic equation describes how population evolves over time. Such an equation actually determines a linear dependence of self-similar rate of evolution process (*or dynamics of population*) in regard to the proper residual capacity of non-filled part of niche. The last is assumed to be proportional to the "difference of the potentials", defining a proper rate of population dynamics, as below [4]:

$$\left(\frac{d\,N\,/\,d\,t}{N}\right) \;=\; b\cdot(K-N)\ ,$$

- here t – is the time-parameter, N – is the population total (or a proper level of niche saturation), $N = N\,(t)$; $b = b\,(t)$ – is the Malthusian parameter (the rate of maximum population growth); K – is the carrying capacity (i.e., the maximum sustainable population at the total saturation of a proper niche), $K = K\,(t)$.

Let us assume that a key point in modeling of such a population dynamics processes - is to take into consideration the moment of reactive environment. In this case, the function of resistance of environment should be presented in a form below:

$$R_{active} + R_{reactive}\ ,$$

- where R_{active} – is the active, constant resistance of environment, or a proper resistance of environment due to a saturation of niche by extraneous elements (*or in modeling a process of an exhaustion of main resources, which means the exhaustion of the niche in regard to it's own elements*); $R_{reactive}$ - is the reactive, non-constant resistance of environment, i.e. the proper resistance of environment as a reaction in regard to increasing of the extraneous elements, incorporated into this population $N\,(t)$ (*or into the non-filled part of a proper niche*), where $R_{reactive} = R_{reactive}\,(N)$.

Active resistance R_{active} above does not depend on the amounts of elements (i.e., it has a stable, constant value). For example, in the case of human population dynamics, such an active resistance R_{active} means the world-wide accidents which unfortunately take place every year in the world ~ like aviation-accidents or technical catastrophes, catastrophes of natural or other character (as a result, we have a decreasing of human population by millions every year).

We should also take into consideration that the level of saturation of a proper niche is known to be determined by the level of demographic pressure [1-2]:

$$P(t) \;=\; N(t) \,/\, K(t) \,,$$

- where according to the principle “*counteraction should be like action*” we assume:

$$R_{\,reactive}\,(N) \;\sim\; N(t) \,,$$

- so, the entire environment resistance should be defined as below:

$$R(t) \;\sim\; R_{0}(t) \;+\; (\,N(t) \,/\, K(t)\,) \;.$$

Summarizing all the assumptions above, we could state a principle which is to define the dynamics of such a processes of evolution: *a self-similar rate of the evolution process is to be directly proportional to the residual capacity of the non-filled part of a proper niche, but also it should simultaneously be in inverse proportion to the function of resistance of the environment space.*

Taking into consideration the universal principle above, we should represent the logistic equation of evolution in a form below:

$$\left(\frac{d\,N \,/\, d\,t}{N}\right) \;=\; b\cdot\left(\frac{1 \;-\; (N\,/\,K)}{R \;+\; (N\,/\,K)}\right) ,$$

- here R – is the function of active, constant resistance of environment $R_{0}\,(t)$, but for simplicity we will denote it as $R\,(t)$. Besides, let us represent the last equation as below:

$$(K\cdot R + N)\cdot\frac{d\,N}{d\,t} \;=\; b\cdot(K\cdot N - N^{\,2}) \qquad (1.1)$$

- which is proved to be a proper *Abel* ordinary differential equation of the 2-nd kind [4].

Due to a very special character of *Abel* equations, it's general solution is known to have a jumping or the break-down of the function $N(t)$ at some moment t_o. It means the existence of continuous solution only at some definite, restricted range of parameter t, or possibility of sudden gradient catastrophe [5] at some moment t_o.

An appropriate change of variables in (1.1): $N(t) + K \cdot R = 1 / y(t)$, let us we obtain the *Abel* ODE of the 1-st kind [4] as below:

$$y' = b \cdot \left(K^2 \cdot R \cdot (1+R) \cdot y^3 - \left(K \cdot (1+2R) + \frac{(K \cdot R)'}{b} \right) \cdot y^2 + y \right) \qquad (1.2)$$

If we assume: $K(t) = const = K$, $b(t) = const = b$, $R(t) = const = R$, equation (1.2) could be reduced as below:

$$y' = b \cdot \left(K^2 \cdot R \cdot (1+R) \cdot y^3 - K \cdot (1+2R) \cdot y^2 + y \right) .$$

- which has a proper analytical solution [4].

Besides, let us assume $R = 1 / K$; it means that we consider a case when the level of resistance of the environment should be directly proportional to the carrying capacity. Such an assumption should properly simplify the right part of equation (1.2) as below:

$$y' = b \cdot \left(\frac{1}{R} \cdot (1+R) \cdot y^3 - \frac{1}{R} \cdot (1+2R) \cdot y^2 + y \right) ,$$

- or:

$$\left(\frac{R}{1+R} \right) \cdot \int \frac{d\,y}{(y-1) \cdot y \cdot \left(y - \dfrac{R}{R+1} \right)} = b \cdot \int d\,t \ ,$$

- the analytical integration of equation above yields:

$$\frac{y\cdot(y-1)^{R}}{\left(y-\frac{R}{1+R}\right)^{(1+R)}} = e^{b\cdot\Delta t} ,$$

- or

$$\left(\frac{N(t)}{R\cdot N(t)-1}\right)^{(1+R)}\cdot\left(-\frac{1}{N(t)}\right) = \frac{e^{b\cdot\Delta t}}{(1+R)^{(1+R)}} ,$$

Let us represent equation above in other form (*R* = *1* / *K*):

$$\left(\frac{1}{K}-\frac{1}{N(t)}\right)^{\left(1+\frac{1}{K}\right)}\cdot\left(-N(t)\right) = \left(1+\frac{1}{K}\right)^{\left(1+\frac{1}{K}\right)}\cdot e^{-b\cdot\Delta t} ,$$

- so, we have obtained the analytical expression for $N(t)$.

Besides, if we take into consideration that carrying capacity K is a large enough for the case of modeling of human population (*K ~ 18 billions of persons*), the last expression could be easily simplified under the appropriate condition ($1 + (1/K)) \to 1$ as below (Fig. 1):

$$N(t) = \left(1-e^{-b\cdot\Delta t}\right)\cdot K .$$

Thus, we have obtained the simple asymptotic solution for the final prognosis of Human population dynamics. Finally, let us especially note the analytical representation of solution of equation (1.2) for the case of separated variables, $R = 1/K = const$, $b(t) \neq const$, $(1/K) \to 1$:

$$N(t) = \left(1-e^{-\int b(t)dt}\right)\cdot K$$

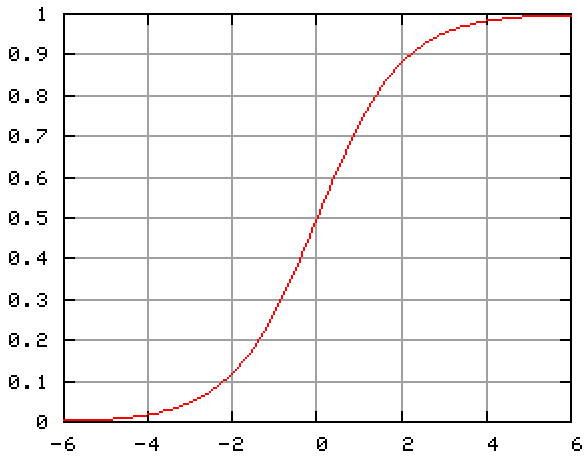

Fig.1. Logistic curve.

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