Exceptional Jordan Strings/Membranes and Octonionic Gravity/p-branes

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Abstract

Nonassociative Octonionic Ternary Gauge Field Theories are revisited paving the path to an analysis of the many physical applications of Exceptional Jordan Strings/Membranes and Octonionic Gravity. The old octonionic gravity constructions based on the *split* octonion algebra $\mathbf{O}_s$ (which strictly speaking is not a division algebra) is extended to the full fledged octonion division algebra $\mathbf{O}$. A real-valued analog of the Einstein-Hilbert Lagrangian $\mathcal{L} = \mathcal{R}$ involving sums of all the possible contractions of the Ricci tensors plus their octonionic-complex conjugates is presented. A discussion follows of how to extract the Standard Model group (the gauge fields) from the *internal* part of the octonionic gravitational connection. The role of Exceptional Jordan algebras, their automorphism and reduced structure groups which play the roles of the rotation and Lorentz groups is also re-examined. Finally, we construct (to our knowledge) generalized novel octonionic string and p-brane actions and raise the possibility that our generalized 3-brane action (based on a quartic product) in octonionic flat backgrounds of $7, 8$ octonionic dimensions may display an underlying E_7, E_8 symmetry, respectively. We conclude with some final remarks pertaining to the developments related to Jordan exceptional algebras, octonions, black-holes in string theory and quantum information theory.

Keywords: Octonions, ternary algebras, Lie 3-algebras, membranes, nonassociative gauge theories, nonassociative geometry, Exceptional Jordan Algebras, Exceptional Groups, Grand Unification.

1 Introduction

Exceptional, Jordan, Division, Clifford, noncommutative and nonassociative algebras are deeply related and are essential tools in many aspects in Physics, see

[1], [2], [3], [4], [7], [8], [9], for references, among many others.

A thorough discussion of the relevance of ternary and nonassociative structures in Physics has been provided in [5], [10], [11]. The earliest example of nonassociative structures in Physics can be found in Einstein's special theory of relativity. Only colinear velocities are commutative and associative, but in general, the addition of non-colinear velocities is non-associative and non-commutative.

Recently, tremendous activity has been launched by the seminal works of Bagger, Lambert and Gustavsson (BLG) [12], [13] who proposed a Chern-Simons type Lagrangian describing the world-volume theory of multiple $M2$-branes. The original BLG theory requires the algebraic structures of generalized Lie 3-algebras and also of nonassociative algebras. Later developments by [14] provided a $3D$ Chern-Simons matter theory with $\mathcal{N} = 6$ supersymmetry and with gauge groups $U(N) \times U(N)$, $SU(N) \times SU(N)$. The original construction of [14] did not require generalized Lie 3-algebras, but it was later realized that it could be understood as a special class of models based on Hermitian 3-algebras [15], [16]. For more recent developments we refer to [17] and references therein.

In this work we explore further physical applications of Exceptional Jordan Strings/Membranes and Octonionic Gravity [24], [27], [18], [19]. The outline of this work is organized as follows. In section **2** we present a review of Octonionic Ternary Gauge Field Theories [28] and add new material pertaining octonionic-valued $SU(N)$ Yang-Mills and 3-Lie-algebra gauge field theories.

In section **3** we shall generalize the octonionic gravity construction based on the *split* octonion algebra $\mathbf{O}_s$ (which strictly speaking is not a division algebra) studied by [18], [19] to the full fledged octonion division algebra $\mathbf{O}$. A real-valued analog of the Einstein-Hilbert Lagrangian $\mathcal{L} = \mathcal{R}$ involving sums of all the possible contractions of the Ricci tensors plus their octonionic-complex conjugates is presented. Section **3** ends with a discussion of how to extract the Standard Model group (the gauge fields) from the *internal* part of the octonionic gravitational connection. The role of Exceptional Jordan algebras, their automorphism and reduced structure groups which play the roles of the rotation and Lorentz groups is also examined.

Finally, in section **4**, we briefly discuss Exceptional Jordan Strings/Membranes and provide a series of generalized octonionic string and p-brane actions (that are novel to our knowledge) and raise the possibility that our generalized 3-brane action (based on a quartic product) in octonionic flat backgrounds of $7, 8$ octonionic dimensions may display an underlying E_7, E_8 symmetry, respectively. We conclude with some final remarks pertaining to the developments related to Jordan exceptional algebras, octonions, black-holes in string theory and quantum information theory.

2 Octonionic Ternary Gauge Field Theories

Recently [28] , a novel (to our knowledge) nonassociative and noncommutative octonionic ternary gauge field theory was explicitly constructed that it is based on a ternary-bracket structure involving the octonion algebra. The ternary bracket obeying the fundamental identity (generalized Jacobi identity) was developed earlier by Yamazaki [29]. The field strength $F_{\mu\nu} = \partial_{[\nu} A_{\mu]} - [A_\mu, A_\nu, \mathbf{g}]$ is defined in terms of the 3-bracket $[A_\mu, A_\nu, \mathbf{g}]$ involving an octonionic-valued field $A_\mu = (A_\mu)^a e_a$, and an octonionic-valued coupling $\mathbf{g} = \mathbf{g}^a e_a$. In this section we shall review briefly the Octonionic Ternary Gauge Field Theory description [28] and add some new material.

Given an octonion $\mathbf{X}$ it can be expanded in a basis (e_o, e_m) as

$$\mathbf{X} = x^o\ e_o\ +\ x^m\ e_m, \quad m, n, p = 1, 2, 3, ...7. \tag{2.1}$$

where e_o is the identity element. The Noncommutative and Nonassociative algebra of octonions is determined from the relations

$$e_o^2 = e_o,\ e_o e_i = e_i e_o = e_i,\ e_i e_j = -\delta_{ij} e_o\ + c_{ijk} e_k,\ i, j, k = 1, 2, 3, ...7. \tag{2.2}$$

where the fully antisymmetric structure constants c_{ijk} are taken to be 1 for the combinations $(124), (235), (346), (457), (561), (672), (713)$ [30]. The octonion conjugate is defined by $\bar{e}_o = e_o$, $\bar{e}_m = -e_m$

$$\bar{\mathbf{X}} = x^o\ e_o\ -\ x^m\ e_m. \tag{2.3}$$

and the norm is

$$N(\mathbf{X})\ =\ |\ <\ \mathbf{X}\ \mathbf{X}\ >\ |^{\frac{1}{2}}\ = |\ Real\ (\bar{\mathbf{X}}\ \mathbf{X})\ |^{\frac{1}{2}}\ = |\ (x_o\ x_o\ + x_k\ x_k)\ |^{\frac{1}{2}}. \tag{2.4}$$

The inverse

$$\mathbf{X}^{-1}\ =\ \frac{\bar{\mathbf{X}}}{<\ \mathbf{X}\ \mathbf{X}\ >}, \quad \mathbf{X}^{-1}\mathbf{X} = \mathbf{X}\mathbf{X}^{-1} = 1. \tag{2.5}$$

The non-vanishing associator is defined by

$$(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = (\mathbf{X}\mathbf{Y})\mathbf{Z} - \mathbf{X}(\mathbf{Y}\mathbf{Z}) \tag{2.6}$$

In particular, the associator

$$(e_i, e_j, e_k)\ =\ (e_i e_j) e_k - e_i (e_j e_k)\ =\ 2\ d_{ijkl}\ e_l$$

$$d_{ijkl}\ =\ \frac{1}{3!}\ \epsilon_{ijklmnp}\ c^{mnp},\ i, j, k.... = 1, 2, 3, ...7 \tag{2.7}$$

Yamazaki [29] define the three-bracket as

$$[\, u,\ v,\ x\,] \ \equiv\ D_{u,v}\, x \ =\ \frac{1}{2}\ (\ u(vx) - v(ux)\ +\ (xv)u\ -\ (xu)v\ +\ u(xv)\ -\ (ux)v\ )\,. \tag{2.8}$$

For the octonionic algebra, after a straightforward calculation when the indices span the imaginary elements $a, b, c, d = 1, 2, 3, ..., 7$, one has that

$$[\ e_a,\ e_b,\ e_c\ ]\ =\ f_{abcd}\ e_d\ =\ -\ [\ d_{abcd}\ -\ \delta_{ac}\ \delta_{bd}\ +\ \delta_{bc}\ \delta_{ad}\ ]\ e_d \tag{2.9a}$$

whereas

$$[\ e_a,\ e_b,\ e_0\ ]\ =\ [\ e_a,\ e_0,\ e_b\ ]\ =\ [\ e_0,\ e_a,\ e_b\ ]\ =\ 0 \tag{2.9b}$$

The ternary bracket (2.8) obeys the fundamental identity

$$[\ [x,\ u,\ v],\ y,\ z\ ]\ +\ [\ x,\ [y,\ u,\ v],\ z\ ]\ +\ [\ x,\ y,\ [z,\ u,\ v]\ ]\ =\ [\ [x,\ y,\ z],\ u,\ v\ ] \tag{2.10}$$

A bilinear positive symmetric product $< u, v >=< v, u >$ is required such that that the ternary bracket/derivation obeys what is called the metric compatibility condition

$$<\ [u,\ v,\ x],\ y\ >\ =\ -\ <\ [u,\ v,\ y],\ x\ >\ =\ -\ <\ x,\ [u,\ v,\ y]\ >\ \Rightarrow$$

$$D_{u,v}\ < x, y >\ =\ 0 \tag{2.11}$$

The symmetric product remains invariant under derivations. There is also the additional symmetry condition required by [29]

$$<\ [u,\ v,\ x],\ y\ >\ =\ <\ [x,\ y,\ u],\ y\ > \tag{2.12}$$

Thus, the ternary product provided by Yamazaki (2.8) *obeys* the key fundamental identity (2.10) and leads to the structure constants f_{abcd} that are *pairwise* antisymmetric but are *not* totally antisymmetric in all of their indices : $f_{abcd} = -f_{bacd} = -f_{abdc} = f_{cdab}$; however : $f_{abcd} \neq f_{cabd}$; and $f_{abcd} \neq -\ f_{dbca}$. The associator ternary operation for octonions $(x, y, z) = (xy)z - x(yz)$ *does not obey* the fundamental identity (2.10) as emphasized by [29]. For this reason we cannot use the associator to construct the 3-bracket.

We defined in [28] the ternary field strength in terms of the *ternary* bracket as

$$F_{\mu\nu}\ =\ \partial_\mu B_\nu\ -\ \partial_\nu B_\mu\ +\ [\ B_\mu,\ B_\nu,\ \mathbf{g}\ ] \tag{2.13}$$

where $\mathbf{g} = g^a e_a$ is an octonionic-valued "coupling" function which is not inert under octonionic gauge transformations. Only the scalar part of $\mathbf{g}$ remains invariant. It was shown in [28] after some algebra that under the local gauge transformations

$$\delta(B_\mu^m\ e_m)\ =\ \Lambda^{ab}(x)\ [e_a, e_b, B_\mu^c\ e_c] \tag{2.14}$$

and

$$\delta(g^m\ e_m)\ =\ \Lambda^{ab}(x)\ [e_a, e_b, g^c\ e_c] \tag{2.15}$$

one can ensure that the ternary field strength $F_{\mu\nu}$ defined in terms of the 3-brackets (2.13) transforms properly (homogeneously) under the ternary gauge transformations if, and only if, the bivector gauge parameters $\Lambda^{ab}(x)$ obey the "self-duality" equations $\frac{1}{2}d_{abcm}\Lambda^{ab}(x) = \Lambda_{cm}(x)$. If this is so then $F_{\mu\nu}$ transforms *homogeneously* under the infinitesimal ternary gauge transformations as

$$\delta(F^m_{\mu\nu}\ e_m)\ =\ \Lambda^{ab}\,[\,e_a,\ e_b,\ F^c_{\mu\nu}\ e_c\,]\ =\ \Lambda^{ab}\ F^c_{\mu\nu}\ f_{abc}{}^m\ e_m\ \Rightarrow\ \delta F^m_{\mu\nu}\ =\ \Lambda^{ab}\ F^c_{\mu\nu}\ f_{abc}{}^m \tag{2.16}$$

The result (2.16) is a direct consequence of the fundamental identity (2.10) because the 3-bracket (2.8) is defined as a derivation

$$[\ [e_a,\ e_b,\ B_\mu],\ B_\nu,\ \mathbf{g}\ ]\ +\ [\ B_\mu,\ [e_a,\ e_b,\ B_\nu],\ \mathbf{g}\ ]\ +\ [\ B_\mu,\ B_\nu,\ [e_a,\ e_b,\ \mathbf{g}]\ ]\ =$$

$$[\ e_a,\ e_b,\ [B_\mu,\ B_\nu,\ \mathbf{g}]\ ] \tag{2.17}$$

The parameter $\Lambda^o(x)$ involved in the transformation $\delta B^o_\mu\ =\ \partial_\mu\Lambda^o(x)$, corresponding to the real (identity) element e_0 of the octonion algebra, leads to $\delta F^0_{\mu\nu}\ =\ 0$ where the field strength component is Abelian-Maxwell-like $F^0_{\mu\nu}\ =\ \partial_\mu B^0_\nu - \partial_\nu B^0_\mu$.

One can verify that the expression for $U = exp\ (-\alpha\Lambda^{ab}[e_a, e_b]);\ U^{-1} = \bar{U} = exp\ (\alpha\Lambda^{ab}[e_a, e_b])$, where one excludes the identity element e_0 from the above definition of U because it yields the trivial transformation $U = 1$, and $\alpha = \frac{1}{4}$ is a real numerical constant, yields the finite gauge transformations

$$F'\ =\ e^{-\alpha\Lambda^{ab}[\ e_a,\ e_b\ ]}\ (F^c\ t_c)\ e^{\alpha\Lambda^{ab}[\ t_a,\ t_b\ ]}. \tag{2.18}$$

which agree with the *ternary* ones (2.17) when the real parameters Λ^{ab} are infinitesimals

$$\delta F\ =\ F'\ -F\ =\ \Lambda^{ab}\ F^c\ [\ e_a,\ e_b,\ e_c\ ]\ =\ -\ \alpha\ \Lambda^{ab}\ F^c\ [\ [\ e_a,\ e_b\ ],\ e_c]\ \Rightarrow$$

$$\Lambda^{ab}\ F^c\ f_{abcm}\ e_m\ =\ -\,\alpha\ \Lambda^{ab}\ F^c\ (2c_{abd})\ (2c_{dcm})\ e_m\ \Rightarrow\ -4\,\alpha\ c_{abd}\ c_{dcm}\ =\ f_{abcm}. \tag{2.19}$$

Therefore, by choosing $\alpha = \frac{1}{4}$ one arrives at the condition among the structure constants given by $c_{abd}\ c_{dcm}\ =\ -\ f_{abcm}$ and which is indeed *obeyed* for the octonion algebra as shown in [30]; i.e. the Yamazaki 3-bracket (2.8) satisfies the identity for octonions when $a, b, c, m = 1, 2, 3, ..., 7$

$$[\ e_a,\ e_b,\ e_c\ ]\ =\ f_{abcm}\ e_m =\ -\ [\ d_{abcm}\ -\ \delta_{ac}\ \delta_{bm}\ +\ \delta_{bc}\ \delta_{am}\ ]\ e_m\ =$$

$$-\ \frac{1}{4}\ [\ [\ e_a,\ e_b\ ],\ e_c\ ]\ =\ -\ c_{abd}\ c_{dcm}\ e_m\ \Rightarrow$$

$$c_{abd}\ c_{dcm}\ =\ d_{abcm}\ -\ \delta_{ac}\ \delta_{bm}\ +\ \delta_{bc}\ \delta_{am} \tag{2.20}$$

d_{abcm} are the associator structure constants given by the duals to the octonion structure constants as shown in eq-(2.7). A series of identities involving the structure constants of octonions can be found in [30]. Therefore, by choosing $\alpha = \frac{1}{4}$, the equality in eq-(2.20) is indeed satisfied for the octonion algebra and such that for infinitesimal real valued parameters Λ^{ab} eq-(2.18) yields to lowest order $\delta F = F' - F = \Lambda^{ab}[e_a, e_b, F]$ recovering the homogeneous ternary infinitesimal gauge transformations for the field strengths as expected.

Given the octonionic valued field strength $F_{\mu\nu} = F^a_{\mu\nu}\ e_a$, with *real valued* components $F^0_{\mu\nu}, F^i_{\mu\nu};\ \ i = 1, 2, 3, ..., 7$, a gauge invariant action under ternary infinitesimal gauge transformations in D-dim is

$$S \ = \ - \ \frac{1}{4\kappa^2} \int d^D x \ < \ F_{\mu\nu} \ F^{\mu\nu} \ > \tag{2.21}$$

κ is a numerical parameter introduced to make the action dimensionless and it can be set to unity for convenience. The $<\quad>$ operation is defined as $< XY >= Real(\bar{X}Y) =< YX >= Real(\bar{Y}X)$. Under infinitesimal ternary gauge transformations of the action one has

$$\begin{aligned}
\delta\ S \ = \ -\frac{1}{4} \int d^D x \ < \ F_{\mu\nu}\ (\delta F^{\mu\nu}) \ + \ (\delta F_{\mu\nu})\ F^{\mu\nu} \ > \ = \\
-\frac{1}{4} \int d^D x \ < \ F^c_{\mu\nu}\ e_c\ \Lambda^{ab}\ [e_a,\ e_b,\ F^{\mu\nu\ n}\ e_n] \ > \ + \\
-\frac{1}{4} \int d^D x \ < \ \Lambda^{ab}\ [e_a,\ e_b,\ F^c_{\mu\nu}\ e_c]\ F^{\mu\nu\ n}\ e_n \ > \ = \\
-\frac{1}{4} \int d^D x\, \Lambda^{ab}\ F^c_{\mu\nu}\ F^{\mu\nu\ n}\ (\ < \ e_c\ f_{abnk}\ e_k \ > \ + \ < \ f_{abck}\ e_k\ e_n \ > \ ) \ = \ 0.
\end{aligned} \tag{2.22}$$

since

$$< \ e_c\ f_{abnk}\ e_k \ > + < \ f_{abck}\ e_k\ e_n \ > \ = \ f_{abnk}\ \delta_{ck} + f_{abck}\ \delta_{kn} \ = \ f_{abnc} + f_{abcn} \ =$$

$$-\ [\ d_{abnc} \ - \ \delta_{an}\ \delta_{bc} \ + \ \delta_{bn}\ \delta_{ac}\ ] \ - \ [\ d_{abcn} \ - \ \delta_{ac}\ \delta_{bn} \ + \ \delta_{bc}\ \delta_{an}\ ] \ = \ 0 \tag{2.23}$$

because $d_{abnc} + d_{abcn} = 0$; $d_{nabc} + d_{cabn} = 0$, due to the total antisymmetry of the associator structure constant d_{nabc} under the exchange of any pair of indices. Invariance $\delta S = 0$, only occurs if, and only if, $\delta F = \Lambda^{ab}[e_a, e_b, F^c e_c] \ \neq \ \Lambda^{ab}[F^c e_c, e_a, e_b]$. The ordering inside the 3-bracket is crucial. One can check that if one sets $\delta F = \Lambda^{ab}[F^c e_c, e_a, e_b]$, the variation δS leads to a term in the integral which is *not* zero

$$f_{nabc} + f_{cabn} \ = \ -\ [\ d_{nabc} - \delta_{nb}\ \delta_{ac} + \delta_{ab}\ \delta_{nc}\ ] - [\ d_{cabn} - \delta_{cb}\ \delta_{an} + \delta_{ab}\ \delta_{cn}\ ] \ \neq \ 0 \tag{2.24}$$

However, under $\delta F = \Lambda^{ab}[e_a, e_b, F^c e_c]$, the variation δS is indeed zero as shown. This is a consequence of the fact that $[e_a, e_b, e_c] \neq [e_c, e_a, e_b]$ when the 3-bracket is given by eq-(2.8).

To show that the action is invariant under finite ternary gauge transformations requires to follow a few steps. Firstly, one defines

$$< x\, y > \;\equiv\; Real\,[\,\bar x\, y\,] \;=\; \frac{1}{2}\,(\,\bar x\, y + \bar y\, x\,) \;\Rightarrow\; < x\, y > \;=\; < y\, x > \quad (2.25)$$

Despite nonassociativity, the *very special conditions*

$$x(\bar x u) \;=\; (x\bar x)u; \quad x(u\bar x) \;=\; (xu)\bar x; \quad x(xu) \;=\; (xx)u; \quad x(ux) \;=\; (xu)x \quad (2.26)$$

are obeyed for octonions resulting from the Moufang identities. Despite that $(xy)z \neq x(yz)$ one has that their real parts obey

$$Real\;[\;(x\;y)\;z\;] \;=\; Real\;[x\;(y\;z)\;] \quad (2.27)$$

Due to the nonassociativity of the algebra, in general one has that $(UF)U^{-1} \neq U(FU^{-1})$. However, if and only if $U^{-1} = \bar U \Rightarrow \bar U U = U\bar U = 1$, as a result of the the *very special conditions* (2.26) one has that $F' = (UF)U^{-1} = U(FU^{-1}) = UFU^{-1} = UF\bar U$ is *unambiguously* defined.

Dropping the spacetime indices for convenience in the expressions for $F^{\mu\nu}, F_{\mu\nu}$, and by repeated use of eqs-(2.25-27), when $U^{-1} = \bar U$, the action density is also invariant under finite gauge transformations of the form

$$< F'\,F' > \;=\; Re\,[\bar F'\,F'] \;=\; Re\,[(U\bar F U^{-1})\,(UFU^{-1})] \;=\; Re\,[(U\bar F)\,(\,U^{-1}\,(UF\,U^{-1})\,)] \;=$$

$$Re\;[(U\;\bar F)\;(U^{-1}\;U)\;(FU^{-1})] \;=\; Re\;[(U\bar F)\;(FU^{-1})] \;=\; Re\;[(FU^{-1})\;(U\bar F)] \;=$$

$$Re\,[F\,(\,U^{-1}\,(U\,\bar F)\,)] \;=\; Re\,[F\,(U^{-1}U)\,\bar F] \;=\; Re\,[F\,\bar F] \;=\; Re\,[\bar F\,F] \;=\; <\,F\,F\,> . \quad (2.28)$$

Since the action (2.21) is invariant under finite and infinitesimal ternary gauge transformations, this means that $S[A^a_\mu; g^a] = S[(A^a_\mu)'; (g^a)' = C^a]$, where $\mathbf{C} = C^a e_a$ is a constant octonionic-valued coupling which can be obtained from gauging the octonionic-valued coupling function $\mathbf{g}(x)$ to a constant $\mathbf{C}$. This can be attained by performing a finite gauge transformation with $\bar U = U^{-1}$ such that $\mathbf{C} = U(x)\mathbf{g}(x)U^{-1}(x) \Rightarrow \mathbf{g}(x) = U^{-1}(x)\mathbf{C}U(x)$ and whose components are $g^a =< e^a(U^{-1}(x)\mathbf{C}U(x)) >$. Because the real parts $g^o = C^o$ remain invariant one may identify $g^o = C^o$ with a physical coupling constant. The physical interpretation of the remaining 7 vector charges/couplings $C^i, i = 1, 2, 3, ...7$ deserves further investigation.

As is well known, the ordinary 2-bracket does *not* obey the Jacobi identity

$$[\,e_i,\;[\,e_j,\;e_k\,]\,] \;+\; [\,e_j,\;[\,e_k,\;e_i\,]\,] \;+\; [\,e_k,\;[\,e_i,\;e_j\,]\,] \;=\; 3\,d_{ijkl}\,e_l \;\neq\; 0 \quad (2.29)$$

If one has the ordinary Yang-Mills expression for the field strength

$$F_{\mu\nu} \;=\; \partial_\mu A_\nu \;-\; \partial_\nu A_\mu \;+\; [\;A_\mu,\;A_\nu\;] \quad (2.30)$$

because the 2-bracket does *not* obey the Jacobi identity, one has an extra (spurious) term in the expression for

$$[\ D_\mu,\ D_\nu\ ]\ \Phi\ =\ [\ F_{\mu\nu},\ \Phi\ ]\ +\ (\ A_\mu,\ A_\nu,\ \Phi\ ) \tag{2.31}$$

given by the crucial contribution of the non-vanishing associator $(A_\mu, A_\nu, \Phi) = (A_\mu A_\nu)\Phi - A_\mu(A_\nu \Phi) \neq 0$. For this reason, due to the non-vanishing condition (2.29), the ordinary Yang-Mills field strength does *not* transform homogeneously under ordinary gauge transformations involving the parameters $\Lambda = \Lambda^a e_a$

$$\delta A_\mu =\ \partial_\mu \Lambda\ +\ [A_\mu, \Lambda] \tag{2.32}$$

and it yields an extra contribution of the form

$$\delta F_{\mu\nu}\ =\ [F_{\mu\nu}, \Lambda]\ +\ (\ \Lambda,\ A_\mu,\ A_\nu) \tag{2.33}$$

As a result of the additional contribution (Λ, A_μ, A_ν) in eq-(2.33), the ordinary Yang-Mills action $S = \int < F_{\mu\nu} F^{\mu\nu} >$ will *no* longer be gauge invariant. Under infinitesimal variations eqs-(2.33), the variation of the action is *no* longer zero but receives spurious contributions of the form $\delta S = -4 F^l_{\mu\nu} \Lambda^i A^{\mu j} A^{\nu k} d_{ijkl} \neq 0$ due to the non-associativity of the octonion algebra.

Antisymmetric tensor field theories based on octonionic valued fields $A_{\mu\nu} = A^a_{\mu\nu} e_a$, $A_{\mu\nu} = -A_{\nu\mu}$ can also be constructed. The rank-three octonionic-valued antisymmetric tensor field strength is defined in terms of the 3-bracket as

$$F_{\mu\nu\rho}\ =\ \partial_\mu\ A_{\nu\rho}\ +\ \partial_\nu\ A_{\rho\mu}\ +\ \partial_\rho\ A_{\mu\nu}\ +$$
$$[\ A_\mu,\ A_{\nu\rho},\ \mathbf{g}\ ]\ +\ [\ A_\nu,\ A_{\rho\mu},\ \mathbf{g}\ ]\ +\ [\ A_\rho,\ A_{\mu\nu},\ \mathbf{g}\ ] \tag{2.34}$$

There is another rank-three tensor given by

$$H_{\mu\nu\rho}\ =\ [\ A_\mu,\ A_\nu,\ A_\rho\ ]. \tag{2.35}$$

which is only anti-symmetric in the first pair of indices $H_{\mu\nu\rho} = -H_{\nu\mu\rho}$. Since the 3-bracket obeys the Fundamental identity, under ternary gauge transformations

$$\delta A_\mu\ =\ \Lambda^{ab}\ [\ e_a,\ e_b,\ A_\mu\ ],\quad \delta A_{\mu\nu}\ =\ \Lambda^{ab}\ [\ e_a,\ e_b,\ A_{\mu\nu}\ ] \tag{2.36}$$

one has that

$$\delta F_{\mu\nu\rho}\ =\ \Lambda^{ab}\ [\ e_a,\ e_b,\ F_{\mu\nu\rho}\ ],\quad \delta H_{\mu\nu\rho}\ =\ \Lambda^{ab}\ [\ e_a,\ e_b,\ H_{\mu\nu\rho}\ ] \tag{2.37}$$

if, and only if, the bivector gauge parameters obey the "self-duality" conditions $\Lambda_{ab} = \frac{1}{2} d_{abcd} \Lambda^{cd}$ as shown in [28]. An invariant action involving the octonionic valued fields $F_{\mu\nu\rho}$ and $H_{\mu\nu\rho}$ in D-dim is of the form

$$S\ =\ \frac{1}{2\kappa^2} \int\ d^D x\ <\ \frac{1}{3!} F_{\mu\nu\rho}\ F^{\mu\nu\rho}\ + \frac{1}{2!}\ H_{\mu\nu\rho}\ H^{\mu\nu\rho}\ > \tag{2.38}$$

where κ is a parameter with the suitable dimensions to render the action dimensionless.

To finalize this section we discuss further constructions, like having an octonionic-valued and $SU(N)$-valued gauge field $\mathbf{A}_\mu = A_\mu^{am}(e_a \otimes T_m)$ involving the $SU(N)$ algebra generators $T_m, m = 1, 2, 3, ..., N^2 - 1$ and the octonion algebra generators $e_a, a = 0, 1, 2, 3, ..., 7$; i.e. one has octonionic-valued components for the $SU(N)$ gauge fields. The commutator is

$$[\ \mathbf{A}_\mu,\ \mathbf{A}_\nu\ ]\ =\ [\ A_\mu^{am}\ (e_a \otimes T_m),\ A_\nu^{bn}\ (e_b \otimes T_n)\ ]\ =$$

$$\frac{1}{2}\ A_\mu^{am}\ A_\nu^{bn}\ \{e_a,\ e_b\} \otimes [T_m,\ T_n]\ +\frac{1}{2}\ A_\mu^{am}\ A_\nu^{bn}\ [e_a,\ e_b] \otimes \{T_m,\ T_n\} \tag{2.38}$$

where

$$\{e_a,\ e_b\}\ =\ -2\ \delta_{ab}\ e_o, \quad [e_a,\ e_b]\ =\ 2\ c_{abc}\ e_c \tag{2.39}$$

and

$$\{T_m,\ T_n\}\ =\ \frac{1}{N}\ \delta_{mn}\ +\ d_{mnp}\ T_p, \quad [T_m,\ T_n]\ =\ f_{mnp}\ T_p \tag{2.40}$$

One may note that the r.h.s of (2.38) involves both commutators and anti-commutators. Due to the fact that the octonion algebra does not obey the Jacobi identities this will spoil the gauge invariance of typical Yang-Mills actions as described before. Let us have instead a ternary Lie algebra (3-Lie algebra) obeying the ternary commutation relations

$$[\ T_m,\ T_n,\ T_p\ ]\ =\ f_{mnpq}\ T_q \tag{2.41}$$

and such that the ternary-bracket structure-constants f_{mnpq} obey the fundamental identity. A 3-Lie-algebra and octonionic-valued field is defined by $\mathbf{A}_\mu \equiv A_\mu^{ma}\ (T_m \otimes e_a)$. However, the triple commutator

$$[\ \mathbf{A}_\mu,\ \mathbf{A}_\nu,\ \mathbf{A}_\rho\ ]\ =\ [\ A_\mu^{mi}\ (T_m \otimes e_i),\ A_\nu^{nj}\ (T_n \otimes e_j),\ A_\rho^{pk}\ (T_p \otimes e_k)\ ] \tag{2.42}$$

would furnish a very *complicated* expression for the r.h.s of eq-(2.42). To simplify matters one could define the ternary bracket as

$$[\ \mathbf{A}_\mu,\ \mathbf{A}_\nu,\ \mathbf{A}_\rho\ ]\ =\ A_\mu^{mi}\ A_\nu^{nj}\ A_\rho^{pk}\ [T_m,\ T_n,\ T_p] \otimes [e_i, e_j, e_k]\ =$$

$$A_\mu^{mi}\ A_\nu^{nj}\ A_\rho^{pk}\ f_{mnpq}\ f_{ijkl}\ (T_q \otimes e_l) \tag{2.43}$$

so that one has closure in the r.h.s of (2.43). It is warranted to explore further these generalized ternary gauge field theories involving 3-Lie algebras and octonions.

3 Octonionic Gravity

In this section we shall generalize the octonionic gravity construction based on the *split* octonion algebra $\mathbf{O}_s$ (which strictly speaking is not a division algebra) [18], [19] to the full fledged octonion division algebra $\mathbf{O}$. $\mathbf{G}_{\mu\nu}$ is an octonionic-valued metric $(\mathbf{G}_{\mu\nu})^o e_o + (\mathbf{G}_{\mu\nu})^i e_i$ obeying the Hermiticity condition $\mathbf{G}^\dagger_{\mu\nu} = \bar{\mathbf{G}}_{\nu\mu} = \mathbf{G}_{\mu\nu}$, and from which one can infer that the real part of the metric is symmetric $(\mathbf{G}_{(\mu\nu)})^o$, and the 7 imaginary components are anti-symmetric $(\mathbf{G}_{[\mu\nu]})^i$ in their μ, ν indices. The bar denotes octonionic "complex" conjugation : $\bar{e}_o = e_o; \bar{e}_i = -e_i; i = 1, 2, 3, ..., 7$. The diagonal components of $\mathbf{G}_{\mu\nu}$ are comprised of real-valued entries; and the off-diagonal ones are comprised of octonionic-valued entries.

Furthermore, instead of having octonionic-valued metric functions of the form [18] $(\mathbf{G}_{\mu\nu})^o(x^\rho)\ e_o + (\mathbf{G}_{\mu\nu})^i(x^\rho)\ e_i$, where x^ρ are ordinary real-valued coordinates of a real manifold $\mathcal{M}$ whose real-dimension is $dim_R(\mathcal{M}) = D$, we have octonionic-valued metric functions of the form

$$\mathbf{G}_{\mu\nu} = (\mathbf{G}_{\mu\nu})^o(\mathbf{Z}^\rho, \bar{\mathbf{Z}}^\rho)\ e_o + (\mathbf{G}_{\mu\nu})^i(\mathbf{Z}^\rho, \bar{\mathbf{Z}}^\rho)\ e_i \qquad (3.1a)$$

$$\mathbf{G}_{\bar{\mu}\nu} = (\mathbf{G}_{\bar{\mu}\nu})^o(\mathbf{Z}^\rho, \bar{\mathbf{Z}}^\rho)\ e_o + (\mathbf{G}_{\bar{\mu}\nu})^i(\mathbf{Z}^\rho, \bar{\mathbf{Z}}^\rho)\ e_i \qquad (3.1b)$$

$$\mathbf{G}_{\mu\bar{\nu}} = (\mathbf{G}_{\mu\bar{\nu}})^o(\mathbf{Z}^\rho, \bar{\mathbf{Z}}^\rho)\ e_o + (\mathbf{G}_{\mu\bar{\nu}})^i(\mathbf{Z}^\rho, \bar{\mathbf{Z}}^\rho)\ e_i \qquad (3.1c)$$

$$\mathbf{G}_{\bar{\mu}\bar{\nu}} = (\mathbf{G}_{\bar{\mu}\bar{\nu}})^o(\mathbf{Z}^\rho, \bar{\mathbf{Z}}^\rho)\ e_o + (\mathbf{G}_{\bar{\mu}\bar{\nu}})^i(\mathbf{Z}^\rho, \bar{\mathbf{Z}}^\rho)\ e_i \qquad (3.1d)$$

where $\mathbf{Z}^\rho, \bar{\mathbf{Z}}^\rho$ are octonionic-valued coordinates. The (real and 7 imaginary) components of the octonionic-valued metric

$$[(\mathbf{G}_{\mu\nu})^o(\mathbf{Z}^\rho, \bar{\mathbf{Z}}^\rho),\ (\mathbf{G}_{\mu\nu})^i(\mathbf{Z}^\rho, \bar{\mathbf{Z}}^\rho)];\ [(\mathbf{G}_{\bar{\mu}\bar{\nu}})^o(\mathbf{Z}^\rho, \bar{\mathbf{Z}}^\rho),\ (\mathbf{G}_{\bar{\mu}\bar{\nu}})^i(\mathbf{Z}^\rho, \bar{\mathbf{Z}}^\rho)]; ...... \qquad (3.2)$$

are real-valued functions. In the bi-octonions case $C \times O$, one can have *complex*-valued functions for the components of the metric in eq-(3.2) with $i = 1, 2, 3, ..., 7$. For the time being we concentrate in the octonions case.

The determinants of non-Hermitian matrices over the division algebras $\mathbf{H}, \mathbf{O}$ are *not* well defined. However the determinant of a 2×2 and 3×3 Hermitian matrix over $\mathbf{H}, \mathbf{O}$ is well defined and *real*-valued [23]. In the 3×3 Hermitian matrix X case one requires to use the Freudenthal's determinant definition given by the trace of the cubic form $detX = \frac{1}{3} Tr(X *_J (X \times_F X))$ in terms of the the Jordan nonassociative (but commutative) $*_J$ product and the Freudental $\times_F$ product. This is one of the reasons why it is important to impose the Hermiticity condition on the octonionic-valued metric $\mathbf{G}_{\mu\nu}$. The indices μ, ν range over the number of plausible octonionic dimensions $D = 1, 2, 3$ where a determinant can be defined and correspond to $8, 16, 24$ real dimensions, respectively.

If one has now an octonionic-valued metric $\mathbf{G}_{\mu\nu} \neq \mathbf{G}_{\nu\mu}$ instead of the real-valued metric $\eta_{\mu\nu} = \eta_{\nu\mu}$, due to the nonassociativity and noncommutativity, the real-valued metric interval ds^2 is defined to be

$$ds^2 \;=\; \frac{1}{2}\;[\;(d\mathbf{Z}^\mu\;\mathbf{G}_{\mu\nu})\;d\mathbf{Z}^\nu\;+\;d\mathbf{Z}^\mu\;(\mathbf{G}_{\mu\nu}\;d\mathbf{Z}^\nu)\;]\;+$$

$$\frac{1}{2}\;[\;(d\bar{\mathbf{Z}}^\nu\;\mathbf{G}_{\bar{\mu}\bar{\nu}})\;d\bar{\mathbf{Z}}^\mu\;+\;d\bar{\mathbf{Z}}^\nu\;(\mathbf{G}_{\bar{\mu}\bar{\nu}}\;d\bar{\mathbf{Z}}^\mu)\;]\;+$$

$$\frac{1}{2}\;[\;(d\mathbf{Z}^\mu\;\mathbf{G}_{\mu\bar{\nu}})\;d\bar{\mathbf{Z}}^\nu\;+\;d\mathbf{Z}^\mu\;(\mathbf{G}_{\mu\bar{\nu}}\;d\bar{\mathbf{Z}}^\nu)\;]\;+$$

$$\frac{1}{2}\;[\;(d\mathbf{Z}^\nu\;\mathbf{G}_{\bar{\mu}\nu})\;d\bar{\mathbf{Z}}^\mu\;+\;d\mathbf{Z}^\nu\;(\mathbf{G}_{\bar{\mu}\nu}\;d\bar{\mathbf{Z}}^\mu)\;] \qquad (3.3)$$

where the octonionic (and octonionic-complex conjugate) coordinates are

$$\mathbf{Z}^\mu \;=\; (Z^\mu)^o\;e_o\;+\;(Z^\mu)^i\;e_i, \quad \mu = 1,2,3,\ldots\ldots,D \qquad (3.4a)$$

$$\bar{\mathbf{Z}}^\mu \;=\; (Z^\mu)^o\;e_o\;-\;(Z^\mu)^i\;e_i, \quad \mu = 1,2,3,\ldots\ldots,D \qquad (3.4b)$$

The components $(Z^\mu)^o, (Z^\mu)^i; i = 1,2,3,\ldots.,7$ are real-valued entries. In the bi-octonions case $C \times O$ they could be complex-valued. The interval ds^2 in eq-(3.3) is comprised of sums of terms involving pairs of octonionic-complex conjugates and for this reason it is real-valued. The first two terms, and the last two terms of (3.3), are respective pairs of octonionic-complex conjugates. For example, in ordinary complex manifolds one has a real-valued interval

$$ds^2 \;=\; g_{\mu\nu}\;dz^\mu\;dz^\nu\;+\;g_{\bar{\mu}\bar{\nu}}\;d\bar{z}^\mu\;d\bar{z}^\nu\;+\;g_{\bar{\mu}\nu}\;d\bar{z}^\mu\;dz^\nu\;+\;g_{\mu\bar{\nu}}\;dz^\mu\;d\bar{z}^\nu \qquad (3.5)$$

due to the conditions under complex conjugation $(g_{\mu\nu})^* = g_{\bar{\mu}\bar{\nu}}$, $(g_{\bar{\mu}\nu})^* = g_{\mu\bar{\nu}}$, $(g_{\mu\bar{\nu}})^* = g_{\bar{\mu}\nu}$, the interval (3.5) is comprised of sums of terms involving pairs of complex conjugates and for this reason it is real-valued.

One may define the analog of a phase rotation or unitary transformation in terms of

$$U \;=\; e^{\;\Lambda^i(\mathbf{Z}^\mu,\bar{\mathbf{Z}}^\mu)\;e_i}, \quad i = 1,2,3,\ldots\ldots\ldots,7 \qquad (3.6)$$

where $\Lambda^i(\mathbf{Z}^\mu, \bar{\mathbf{Z}}^\mu)$, for $i = 1,2,3,\ldots,7$, are 7 *real*-valued functions of the octonionic spacetime coordinates $\mathbf{Z}^\mu, \bar{\mathbf{Z}}^\mu; \mu = 1,2,3,\ldots\ldots,D$; i.e. the functions $\Lambda^i(\mathbf{Z}^\mu, \bar{\mathbf{Z}}^\mu)$ under octonionic conjugation $\bar{e}_i = -e_i$, $\bar{e}_o = e_o$, obey the conditions $\bar{\Lambda}^i(\bar{\mathbf{Z}}^\mu, \mathbf{Z}^\mu) = \Lambda^i(\mathbf{Z}^\mu, \bar{\mathbf{Z}}^\mu)$ due to the reality condition imposed on the Λ^i. As a result, one has that U satisfies the condition $U^{-1} = \bar{U}$ which is the analog of a unitary transformation (unitary matrix). A rigorous definition of an octonionic exponential function, an octonionic Taylor expansion, the Fourier transform and the Paley-Wiener theorem was provided by [20].

In section **2** ...... we learned from the Moufang identities that when $U^{-1} = \bar{U}$, the new coordinate

$$Z'_\mu \;=\; (U\;Z_\mu)\;\bar{U}\;=\;U\;(Z_\mu\;\bar{U})\;=\;U\;Z_\mu\;\bar{U}\;=\;U\;Z_\mu\;U^{-1} \qquad (3.7)$$

are *unambiguously* defined. Dropping the spacetime indices, and the bold face notation for convenience in the octonionic-valued coordinates $\mathbf{Z}^\mu, \mathbf{Z}_\mu$, one can again show, after a repeated use of eqs-(2.25-2.27) involving the Moufang identities, that the interval

$$ds^2 \;=\; \eta_{\mu\nu} \;< d\mathbf{Z}^\mu \; d\mathbf{Z}^\nu > \;\;=\;\; < d\mathbf{Z}^\mu \; d\mathbf{Z}_\mu > \tag{3.8}$$

is invariant under the U-transformations (3.7) when $U^{-1} = \bar{U}$,

$$(ds')^2 \;=\; <\; dZ' \; dZ' \;\; > \;=\; Re \; [d\bar{Z}' \; dZ'] \;=\; Re \; [(Ud\bar{Z}U^{-1}) \; (UdZU^{-1})] \;=$$
$$Re \; [(Ud\bar{Z}) \; (\; U^{-1} \; (UdZ \; U^{-1}) \;)] \;=\; Re \; [(U \; d\bar{Z}) \; (U^{-1} \; U) \; (dZU^{-1})] \;=$$
$$Re \; [(Ud\bar{Z}) \; (dZU^{-1})] \;=\; Re \; [(dZU^{-1}) \; (Ud\bar{Z})] \;=$$
$$Re \, [dZ \; (\; U^{-1} \; (U \; d\bar{Z}) \;)] \;=\; Re \, [dZ \; (U^{-1}U) \; d\bar{Z}] \;=\; Re \, [dZ \; d\bar{Z}] \;=\; Re \, [d\bar{Z} \; dZ] \;=$$
$$<\; dZ \; dZ \;\; > \;\;=\; ds^2 \tag{3.9}$$

Therefore the interval $ds^2 = (ds')^2$ remains invariant under the U-transformations (3.7).

The octonionic-valued connection can be decomposed as

$$\mathbf{Y}^\sigma_{\mu\rho} \;=\; (\Gamma^\sigma_{\mu\rho})^o \; e_o \;+\; (\Theta^\sigma_{\mu\rho})^i \; e_i \tag{3.10}$$

There are other components $\mathbf{Y}^{\bar{\sigma}}_{\bar{\mu}\bar{\rho}}$, $\mathbf{Y}^{\bar{\sigma}}_{\bar{\mu}\rho}$, $\mathbf{Y}^{\sigma}_{\bar{\mu}\rho}$, ... that must be included as well. For simplicity we shall not write them down. In complex Hermitian manifolds one has $g_{\mu\nu} = g_{\bar{\mu}\bar{\nu}} = 0$, and $g_{\mu\bar{\nu}}, g_{\bar{\mu}\nu}$ are not zero [21]. The only non-vanishing connection components are $\Gamma^\sigma_{\mu\rho}; \Gamma^{\bar{\sigma}}_{\bar{\mu}\bar{\rho}}$; the only non-vanishing curvature components are $R^\sigma_{\mu\bar{\nu}\rho}; R^{\bar{\sigma}}_{\bar{\mu}\nu\bar{\rho}}$. Lowering indices with the non-vanishing metric components $g_{\tau\bar{\sigma}}, g_{\bar{\tau}\sigma}$ yields the non-vanishing curvature components $R_{\bar{\tau}\mu\bar{\nu}\rho}; R^{\bar{\sigma}}_{\tau\bar{\mu}\nu\bar{\rho}}$. The Ricci tensor is $R_{\mu\bar{\nu}}$. In the octonionic case matters are more complicated due to nonassociativity/noncommutativity.

If one *restricts* the internal part of the octonionic connection in the following form $\Theta^\sigma_{\mu\rho} = \delta^\sigma_\rho \Theta_\mu$ then eq- (3.10) becomes

$$\mathbf{Y}^\sigma_{\mu\rho} \;=\; (\Gamma^\sigma_{\mu\rho})^o \; e_o \;+\; \delta^\sigma_\rho \; (\Theta_\mu)^i \; e_i \tag{3.11}$$

The scalar (real) part is given by the spacetime connection

$$\Gamma^\sigma_{\mu\rho} \;=\; \Gamma^\sigma_{(\mu\rho)} \;+\; \Gamma^\sigma_{[\mu\rho]} \tag{3.12}$$

comprised of a symmetric $\Gamma^\sigma_{(\mu\rho)}$ and antisymmetric (torsion) piece $T^\sigma_{\mu\rho} = \Gamma^\sigma_{[\mu\rho]}$. The internal (purely imaginary) part of the connection is given by $\delta^\sigma_\rho \; \Theta^i_\mu \; e_i$, with $i = 1, 2, 3, ..., 7$, so that the commutator becomes $[\Theta_\mu, \Theta_\nu] = 2 \; \Theta^i_\mu \; \Theta^j_\nu \; c_{ijk} \; e_k$. The octonionic-valued curvature, when one restricts the internal part of the connection to be $\Theta^\sigma_{\mu\rho} = \delta^\sigma_\rho \Theta_\mu$, is given by

$$\mathbf{R}^\sigma_{\mu\nu\rho} \;=\; R^\sigma_{\mu\nu\rho} \; e_o \;+\; (\; \mathbf{P}^\sigma_{\mu\nu\rho} \;)^k \; e_k \;\Rightarrow$$
$$R^\sigma_{\mu\nu\rho} \;=\; \partial_\mu \; \Gamma^\sigma_{\nu\rho} \;-\; \partial_\nu \; \Gamma^\sigma_{\mu\rho} \;+\; \Gamma^\sigma_{\mu\tau} \; \Gamma^\tau_{\nu\rho} \;-\; \Gamma^\sigma_{\nu\tau} \; \Gamma^\tau_{\mu\rho} \tag{3.13}$$

is the standard real part of the curvature. The internal (purely imaginary) part of the curvature tensor can be written in terms of

$$\mathbf{P}_{\mu\nu} \ = \ \partial_\mu \ \Theta_\nu \ - \ \partial_\nu \ \Theta_\mu \ + \ [\ \Theta_\mu, \ \Theta_\nu \ ] \ =$$

$$(\ \partial_\mu \ \Theta^k_\nu \ - \ \partial_\nu \ \Theta^k_\mu \ ) \ e_k \ + \ 2 \ \Theta^i_\mu \ \Theta^j_\nu \ c_{ijk} \ e_k. \tag{3.14}$$

such that

$$\mathbf{P}^\sigma_{\mu\nu\rho} \ = \ (\ \mathbf{P}^\sigma_{\mu\nu\rho} \ )^k \ e_k \ = \ \delta^\sigma_\rho \ (\mathbf{P}_{\mu\nu})^k \ e_k \ =$$

$$\delta^\sigma_\rho \ \left(\ \partial_\mu \ \Theta_\nu \ - \ \partial_\nu \ \Theta_\mu \ + \ [\ \Theta_\mu, \ \Theta_\nu \ ] \ \right)^k \ e_k, \quad e_k \ = \ e_1, e_2, e_3, ..., e_7. \tag{3.15}$$

By derivatives in eqs-(3.13-3.15) it is understood that $\partial_\mu = (\partial/\partial \mathbf{Z}^\mu), \partial_{\bar\mu} = (\partial/\partial \bar{\mathbf{Z}}^\mu)$. There are other components of the curvature involving derivatives of the remaining connection components $\mathbf{Y}^{\bar\sigma}_{\bar\mu\bar\rho}$, $\mathbf{Y}^\sigma_{\bar\mu\rho}$, ... that must be included as well and yielding $\mathbf{R}^{\bar\sigma}_{\bar\mu\bar\nu\bar\rho}, \mathbf{R}^\sigma_{\bar\mu\nu\rho}$, ...

When one does *not* restrict the internal part of the connection to be $\Theta^\sigma_{\mu\rho} = \delta^\sigma_\rho \Theta_\mu = \delta^\sigma_\rho \Theta^i_\mu e_i$, but instead if it is given by $\Theta^\sigma_{\mu\rho} = (\Theta^\sigma_{\mu\rho})^i e_i$, then the expression for the curvature is more complicated. In this case, due to $c_{ijk} = -c_{jik}$ one has modified contributions to the curvature of the form

$$(\Theta^\sigma_{\mu\tau})^i \ e_i \ (\Theta^\tau_{\nu\rho})^j \ e_j \ - \ (\Theta^\sigma_{\nu\tau})^j \ e_j \ (\Theta^\tau_{\mu\rho})^i \ e_i \ =$$

$$-\delta_{ij} \ \left(\ (\Theta^\sigma_{\mu\tau})^i \ (\Theta^\tau_{\nu\rho})^j \ - \ (\Theta^\sigma_{\nu\tau})^j \ (\Theta^\tau_{\mu\rho})^i \ \right) \ e_o \ +$$

$$\left(\ (\Theta^\sigma_{\mu\tau})^i \ (\Theta^\tau_{\nu\rho})^j \ + \ (\Theta^\sigma_{\nu\tau})^j \ (\Theta^\tau_{\mu\rho})^i \ \right) c_{ijk} \ e_k \tag{3.16}$$

Namely, the *real* part of the curvature will receive an *additional* contribution to the prior real part in eq-(3.13) which is given by the second line of eq-(3.16). And the third line of eq-(3.16) differs now from the commutator term $\delta^\sigma_\rho[\Theta_\mu, \Theta_\nu]$ in eq-(3.15).

Covariant derivatives are defined by

$$\nabla_\mu \ \mathbf{V}^\rho \ = \ \partial_\mu \ \mathbf{V}^\rho \ + \ \Gamma^\rho_{\sigma\mu} \ \mathbf{V}^\sigma \ + \ [\ \Theta_\mu, \ \mathbf{V}^\rho \ ] \tag{3.17a}$$

$$\nabla_{\bar\mu} \ \mathbf{V}^{\bar\rho} \ = \ \partial_{\bar\mu} \ \mathbf{V}^{\bar\rho} \ + \ \Gamma^{\bar\rho}_{\bar\sigma\bar\mu} \ \mathbf{V}^{\bar\sigma} \ + \ [\ \Theta_{\bar\mu}, \ \mathbf{V}^{\bar\rho} \ ] \tag{3.17b}$$

etc .... For instance, the commutator will receive additional contributions

$$[\ \nabla_\mu, \ \nabla_\nu \ ] \ \mathbf{V}^\tau \ = \ \mathbf{R}^\tau_{\mu\nu\sigma} \ \mathbf{V}^\sigma \ + \ \Gamma^\sigma_{[\mu\nu]} \ (\nabla_\sigma \ \mathbf{V}^\tau) \ +$$

$$[\ \mathbf{P}_{\mu\nu}, \ \mathbf{V}^\tau \ ] \ + \ 6 \ (\ \Theta_\mu, \ \Theta_\nu, \ \mathbf{V}^\tau \ ) \tag{3.18}$$

One may notice the contribution of the non-vanishing associator $(\Theta_\mu, \Theta_\nu, \mathbf{V}^\tau)$ to the r.h.s of eq-(3.18) . There is also the contribution of the term $[\mathbf{P}_{\mu\nu}, \mathbf{V}^\tau]$ due to the noncommutativity of the octonions, as well as the contribution $\Gamma^\sigma_{[\mu\nu]}(\nabla_\sigma \mathbf{V}^\tau)$. The antisymmetric part of the connection $\Gamma^\sigma_{[\mu\nu]}$ contributes to a non-vanishing torsion.

In ordinary Riemannian geometry the Bianchi identities $\nabla_{[\rho} R^\tau_{\mu\nu]\sigma} = 0$ are obeyed as well as $R^\tau_{[\mu\nu\sigma]} = 0$ [22], after an antisymmetrization of three indices is performed. However, due to the non-associativity of octonions, this is *no* longer the case for the octonionic-valued curvature tensor $\mathbf{R}^{\tau}_{\mu\nu\sigma}$. In particular the non-vanishing associator of three covariant derivatives acting on an octonionic-valued vector is of the form

$$(\nabla_\mu, \ \nabla_\nu, \ \nabla_\rho) \ \mathbf{V}^\tau \ = \ a_1 \ \nabla_{[\rho} \mathbf{R}^{\tau}_{\mu\nu]\sigma} \ \mathbf{V}^\sigma \ + a_2 \ \mathbf{R}^{\sigma}_{[\mu\nu\rho]} \ \nabla_\sigma \ \mathbf{V}^\tau \ \neq \ 0 \qquad (3.19)$$

the numerical coefficients a_1, a_2 are real-valued. The antisymmetrization with respect to the indices $[\mu\nu\rho]$ is a result of the total antisymmetry of the associator structure constant d_{ijkl}, for example

$$(\Theta_\mu, \ \Theta_\nu, \ \Theta_\rho) \ = \ 2 \ \Theta^i_\mu \ \Theta^j_\nu \ \Theta^k_\rho \ d_{ijkl} \ e_l \qquad (3.20)$$

There are many possible contractions of $\mathbf{R}^{\tau}_{\mu\nu\sigma}, \mathbf{R}^{\bar\tau}_{\bar\mu\bar\nu\bar\sigma}, ...$ due to the non-commutativity/nonassociativity of octonions and to the position of the indices in the metric because it is not symmetric. It obeys the Hermiticity condition $\mathbf{G}^{\dagger}_{\mu\nu} = \bar{\mathbf{G}}_{\nu\mu} = \mathbf{G}_{\mu\nu}$. Therefore, there are many possible contractions from the family of possible octonionic-valued Ricci tensors to obtain octonionic-valued curvature scalars. For example, given $\mathbf{R}_{\mu\sigma} = \delta^\nu_\tau \mathbf{R}^\tau_{\mu\nu\sigma}$ one has

$$\mathbf{R}_{\mu\sigma} \ \mathbf{G}^{\sigma\mu} \ \neq \ \mathbf{R}_{\mu\sigma} \ \mathbf{G}^{\mu\sigma} \ \neq \ \mathbf{G}^{\sigma\mu} \ \mathbf{R}_{\mu\sigma} \ \neq \ \mathbf{G}^{\mu\sigma} \ \mathbf{R}_{\mu\sigma} \qquad (3.21)$$

As a result of the identities $< XY > \ = \ < YX > \ = Re[\bar{X} Y] \ = \ Re[\bar{Y} X]$ one has

$$< \ \mathbf{R}_{\mu\sigma} \ \mathbf{G}^{\sigma\mu} \ > \ = \ < \ \mathbf{G}^{\sigma\mu} \ \mathbf{R}_{\mu\sigma} \ > \qquad (3.22)$$

and which is not equal to

$$< \ \mathbf{R}_{\mu\sigma} \ \mathbf{G}^{\mu\sigma} \ > \ = \ < \ \mathbf{G}^{\mu\sigma} \ \mathbf{R}_{\mu\sigma} \ > \qquad (3.23)$$

Because there is torsion, and nonmetricity in general $\nabla_\rho \mathbf{G}_{\mu\nu} \neq 0; \nabla_\rho \mathbf{G}_{\bar\mu\bar\nu} \neq 0; ....$ the Ricci tensor can be decomposed into a symmetric $\mathbf{R}_{(\mu\sigma)}$ and antisymmetric piece $\mathbf{R}_{[\mu\sigma]}$, for example. If one sets the torsion and nonmetricity to zero, it will yield a relationship among the metric and the connection as in ordinary Riemann geometry.

A real-valued analog of the Einstein-Hilbert Lagrangian $\mathcal{L} = \mathcal{R}$ should involve sums of all the possible contractions of the Ricci tensors plus their octonionic-complex conjugates

$$\mathcal{L} \ = \ c_1 \ (\mathbf{R}_{\mu\bar\sigma} \ \mathbf{G}^{\bar\sigma\mu} + \mathbf{G}^{\sigma\bar\mu} \ \mathbf{R}_{\bar\mu\sigma}) \ + \ c_2 \ (\mathbf{R}_{\bar\mu\sigma} \ \mathbf{G}^{\sigma\bar\mu} + \mathbf{G}^{\bar\sigma\mu} \ \mathbf{R}_{\mu\bar\sigma}) \ +$$
$$c_3 \ (\mathbf{R}_{\mu\bar\sigma} \ \mathbf{G}^{\mu\bar\sigma} + \mathbf{G}^{\bar\mu\sigma} \ \mathbf{R}_{\bar\mu\sigma}) \ + \ c_4 \ (\mathbf{R}_{\bar\mu\sigma} \ \mathbf{G}^{\bar\mu\sigma} + \mathbf{G}^{\mu\bar\sigma} \ \mathbf{R}_{\mu\bar\sigma}) \ +$$
$$d_1 \ (\mathbf{R}_{\mu\sigma} \ \mathbf{G}^{\sigma\mu} + \mathbf{G}^{\bar\sigma\bar\mu} \ \mathbf{R}_{\bar\mu\bar\sigma}) \ + \ d_2 \ (\mathbf{R}_{\mu\sigma} \ \mathbf{G}^{\mu\sigma} + \mathbf{G}^{\bar\mu\bar\sigma} \ \mathbf{R}_{\bar\mu\bar\sigma}) \ +$$
$$d_3 \ (\mathbf{G}^{\sigma\mu} \ \mathbf{R}_{\mu\sigma} + \mathbf{R}_{\bar\mu\bar\sigma} \ \mathbf{G}^{\bar\sigma\bar\mu}) \ + \ d_4 \ (\mathbf{G}^{\mu\sigma} \ \mathbf{R}_{\mu\sigma} + \mathbf{R}_{\bar\mu\bar\sigma} \ \mathbf{G}^{\bar\mu\bar\sigma}) \qquad (3.24)$$

where $c_1, c_2, c_3, c_4, d_1, d_2, d_3, d_4$ are real numerical coefficients. From (3.24) one may notice that in this octonionic case one can construct 8 different real-valued curvature scalars from the contractions of the Ricci curvature tensors due to the nonassociativity and noncommutativity. It is likely that due to symmetries not all of these 8 quantities are independent from each other.

By analogy with what occurs in complex Hermitian manifolds endowed with a Hermitian metric, if the non-vanishing components of the octonionic metric are $\mathbf{G}_{\mu\bar{\nu}}, \mathbf{G}_{\bar{\mu}\nu}$ one can define the determinant in $D = 2, 3$ octonionic dimensions as

$$det\ (\mathbf{G})\ =\ \sqrt{det\ (\mathbf{G}_{\mu\bar{\nu}})\ det\ (\mathbf{G}_{\bar{\mu}\nu})} \tag{3.25}$$

and the analog of the Einstein-Hilbert action is

$$\frac{1}{16\pi G_N}\ \int\ [d\Omega]\ \sqrt{|det(\mathbf{G})|}\ \mathcal{L} \tag{3.26}$$

where the (real-valued) $\mathcal{L}$ in this particular case is obtained after setting the last two lines in eq-(3.24) to *zero*. $[d\Omega]$ is a suitable measure in an octonionic D-dim spacetime. The values $D = 2, 3$ have a correspondence to $16, 24$ real-dimensions, respectively. For example, in $D = 3$ octonionic dimensions, the measure is

$$(dx^0 \wedge dx^1 \wedge\ ....\ \wedge dx^7)\ \wedge\ (dy^0 \wedge dy^1 \wedge\ ....\ \wedge dy^7)\ \wedge\ (dz^0 \wedge dz^1 \wedge\ ....\ \wedge dz^7). \tag{3.27}$$

The components of the metric $\mathcal{G} = \mathbf{G}_{\mu\nu}$ are

$$\mathcal{G}\ \equiv\ \begin{pmatrix} g_{11} & g^o_{(12)}e_o\ +\ g^i_{[12]}e_i & g^o_{(13)}e_o\ +\ g^i_{[13]}e_i \\ g^o_{(21)}e_o\ +\ g^i_{[21]}e_i & g_{22} & g^o_{(23)}e_o\ +\ g^i_{[23]}e_i \\ g^o_{(31)}e_o\ +\ g^i_{[31]}e_i & g^o_{(32)}e_o\ +\ g^i_{[32]}e_i & g_{33} \end{pmatrix}\ =$$

$$\begin{pmatrix} g_{11} & g^o_{(12)}e_o\ +\ g^i_{[12]}e_i & g^o_{(13)}e_o\ +\ g^i_{[13]}e_i \\ g^o_{(12)}e_o\ -\ g^i_{[12]}e_i & g_{22} & g^o_{(23)}e_o\ +\ g^i_{[23]}e_i \\ g^o_{(13)}e_o\ -\ g^i_{[13]}e_i & g^o_{(23)}e_o\ -\ g^i_{[23]}e_i & g_{33} \end{pmatrix} \tag{3.28}$$

It is clear from the second matrix in (3.28) that it is Hermitian from the octonionic point of view. The 3×3 Hermitian matrix $\mathcal{G}$ has the same structure as a Jordan matrix belonging to the exceptional Jordan-Albert algebra $J_3[O]$. The real-valued Freudenthal determinant is given by the trace of the cubic form

$$det\ \mathcal{G}\ =\ \frac{1}{3}\ Tr(\ \mathcal{G} *_J (\ \mathcal{G} \times_F \mathcal{G}\ )\ ) \tag{3.29}$$

The nonassociative but commutative Jordan product of two Jordan matrices X, Y is

$$X\ *_J\ Y\ =\ \frac{1}{2}\ (\ X\ Y\ +\ Y\ X\ ) \tag{3.30}$$

The (symmetric) Freudenthal product is

$$X \times_F Y \ = \ X *_J Y - \frac{1}{2} \left[Y \, Tr(X) + X \, Tr(Y) \right] + \frac{1}{2} \left[Tr(X) \, Tr(Y) - Tr(X *_J Y) \right] \mathbf{1} \tag{3.31}$$

the last term involves the unit matrix $\mathbf{1}$. Hence, the Freudental determinant allows to construct the Einstein-Hilbert action (3.26) in this case. In $D = 2$ octonionic-dimensions one has the Exceptional Jordan algebra $J_2[O]$ involving 2×2 Hermitian matrices with a well defined determinant.

Similar 3×3 Hermitian matrix representations as eq-(3.28) exist for the other components of the octonionic metric as described by eqs-(3.1) and obeying the relations $\mathbf{G}_{\bar{\mu}\bar{\nu}} = \bar{\mathbf{G}}_{\mu\nu} = \mathbf{G}_{\nu\mu}$; $\mathbf{G}_{\bar{\mu}\nu} = \bar{\mathbf{G}}_{\mu\bar{\nu}} = \mathbf{G}_{\bar{\nu}\mu}$; $\mathbf{G}_{\mu\bar{\nu}} = \bar{\mathbf{G}}_{\bar{\mu}\nu} = \mathbf{G}_{\nu\bar{\mu}}$. Their Freudenthal determinant will be computed in the same way as in eq-(3.29). In particular, this will allow us to evaluate the expression for the measure in eq-(3.25) to be used in the action (3.26) for the very special case that the non-vanishing components of the metric are $\mathbf{G}_{\mu\bar{\nu}}, \mathbf{G}_{\bar{\mu}\nu}$.

One could add torsion and curvature squared terms to the action (3.26), and other terms if one wishes, like a cosmological constant and derivatives of curvature terms. At the moment we shall focus on the simplest of actions linear in the curvature. Clearly, due to the nonassociativity and noncommutativity of octonions there are clear generalizations and modifications of ordinary gravity.

Before we finalize this section one ought to mention what is the analog of the rotation and Lorentz group when octonions and Exceptional Jordan algebras are the ingredients of a physical model. There are several ways in which Lie groups can be defined by Jordan algebras. The analog of the rotation group is the automorphism group of a Jordan algebra consisting of linear transformations preserving its multiplication table. These transformations can be expressed in terms of the associator $(A, B, C) = (AB)C - A(BC)$ as [25]

$$M' \ = \ M \ + (\ A, \ M, \ B \ ) \ + \ \frac{1}{2!} (\ A, \ (A, \ M, \ B), \ B \ ) \ + \ ..... \tag{3.32}$$

where A, B are traceless elements of the Jordan algebra. The transformation (3.32) is just the "exponentiation" of the infinitesimal transformation $\delta M = (A, M, B)$. The analog of the Lorentz group is the reduced structure group with infinitesimal transformations of the form $\delta M \ = (A, M, B) + C *_J M$ where C is also a traceless element of the Jordan algebra. The finite transformations under the reduced structure group obtained by "exponentiation" are defined as [25]

$$\delta_{finite} M \ \ = \ \delta M \ + \ \frac{1}{2!} \delta(\delta M) \ + \frac{1}{3!} (\delta(\delta(\delta M))) \ + \ .... \ \Rightarrow$$

$$M' - M \ = \ (\ A, \ M, \ B \ ) + C *_J M + \frac{1}{2!} \ [\ (\ A, \ [\ (\ A, \ M, \ B \ ) \ + \ C *_J M \ ], \ B \ ] \ +$$

$$\frac{1}{2!} \ C *_J \ \ [\ (\ A, \ M, \ B \ ) \ + \ C *_J M \ ] \ + \ .... \tag{3.33}$$

The automorphism (rotation) and reduced structure group (Lorentz) of the $J_3[O]$ algebra are $F_4, E_{6(-26)}$ respectively. Inspired by the magic Tits square, Gunaydin et al [33] extended the automorphism (rotation) and reduced structure (Lorentz) groups to the so-called conformal and quasi-conformal case leading to non-compact forms of E_7, E_8, respectively. The authors [23] provided a different approach to the Lorentz group. In particular in $D = 10$, the analog of Octonionic Mobius (nested) transformations and 3-component Cayley spinors were constructed.

The key question is : can one extract the Standard Model group (the gauge fields) from the *internal* part $\delta^{\rho}_{\sigma} (\Theta_{\mu})^i e_i$ of the octonionic gravitational connection ? Instead of having pure octonionic gravity based solely on octonions, one could have the metric and connection fields taking values in a *composition* algebra of the form $C \times \mathbf{H} \times \mathbf{O}$ a la Dixon [7]. The complex numbers are related to a $U(1)$ symmetry associated with the circle S^1. The quaternions are related to a $SU(2)$ symmetry associated with the 3-sphere S^3. The octonions are related to a $SU(3)$ symmetry associated with the 7-sphere S^7. The $SU(3)$ is a subgroup of the 14-dim automorphism group of the octonions G_2 which leaves invariant the idempotents $\frac{1}{2}(e_o \pm i e_7)$. One may include the algebra R of the real numbers which is associated to S^0 and corresponds to the two end-points ± 1 of a line segment. It is more reminiscent of a discrete symmetry like C, P, T.

Therefore, a gravitational theory based on *composition* algebra of the form $C \times \mathbf{H} \times \mathbf{O}$ a la Dixon [7] would encode the Standard Model group $SU(3) \times SU(2) \times U(1)$. Naturally, since the octonionic gravity program described in this section involves $D = 2, 3$ octonionic dimensions, which correspond to $16, 24$ real dimensions, it is amenable to a Kaluza-Klein compactification mechanism to generate the Yang-Mills (GUT, Standard Model) group in lower dimensions from the isometry group of the internal space. A Kaluza-Klein theory without extra dimensions involving a metric in curved Clifford space was analyzed by [34].

4 Octonionic Branes, Exceptional Jordan Strings and Membranes

Given an ordinary string's world-sheet parametrized by the real-valued coordinates σ^1, σ^2 and embedded in an octonionic-valued target spacetime background $\mathbf{Z}^{\mu}, \ \mu = 1, 2, 3, ...D$, in the form $\mathbf{Z}^{\mu}(\sigma^1, \sigma^2) = (Z^{\mu})^o(\sigma^1, \sigma^2)\ e_o + (Z^{\mu})^i(\sigma^1, \sigma^2)\ e_i$, allows to formulate the octonionic analog of the Eguchi-Schild string action (area-squared) as

$$S \ = \ - \frac{T}{4} \int d^2\sigma \ < \ [\ \{ \ \mathbf{Z}_{\mu}, \ \mathbf{Z}_{\nu} \ \}_{PB} \ ] \ [\ \{ \ \mathbf{Z}^{\mu}, \ \mathbf{Z}^{\nu} \ \}_{PB} \ ] \ > \ =$$

$$\frac{T}{4} \int d^2\sigma \ Real \ \left(\ [\ \{ \ \bar{\mathbf{Z}}_{\nu}, \ \bar{\mathbf{Z}}_{\mu} \ \}_{PB} \ ] \ [\ \{ \ \mathbf{Z}^{\mu}, \ \mathbf{Z}^{\nu} \ \}_{PB} \ ] \ \right) \qquad (4.1)$$

where T is the string's tension and the Poisson bracket (PB) is defined as usual

$$\{ \mathbf{Z}^\mu, \ \mathbf{Z}^\nu \}_{PB} \ = \ (\partial_a \mathbf{Z}^\mu) \ (\partial_b \mathbf{Z}^\nu) \ - \ (\partial_b \mathbf{Z}^\mu) \ (\partial_a \mathbf{Z}^\nu); \ ..... \tag{4.2}$$

One should note the ordering of the indices in eq-(4.2) due to the noncommutativity. Another (real-valued) action that can be constructed is

$$S \ = \ -\frac{T}{8} \int d^2\sigma \ (\ [\ \{ \ \mathbf{Z}_\mu, \ \mathbf{Z}_\nu \ \}_{PB} \ ] \ [\ \{ \ \mathbf{Z}^\mu, \ \mathbf{Z}^\nu \ \}_{PB} \ ] \ + \ o.c.c \ ). \tag{4.3}$$

by adding the octonionic-complex conjugate (o.c.c).

Despite the nonassociativity, the *real* parts of the cubic product obey the equality

$$Re \ [\ (e_i \ e_j) \ e_k \ ] \ = \ Re \ [\ e_i \ (e_j \ e_k) \ ] \tag{4.4}$$

however the real parts of the quartic products $differ$

$$Re \ [\ (e_i \ e_j) \ (e_k \ e_l) \ ] \ \neq \ Re \ [\ e_i \ (e_j \ e_k) \ e_l \ ] \tag{4.5}$$

For this reason one must specify the ordering of the quartic products in the specific form as shown in eqs-(4.1, 4.3) .

The analog of the Polyakov-Howe-Tucker action for an octonionic p-brane action is

$$S = -\frac{T_p}{2} \int d^{p+1}\sigma \ \sqrt{|h|} \ \frac{h^{ab}}{4} \ [\ (\partial_a \mathbf{Z}^\mu \ \mathbf{G}_{\mu\nu}) \ \partial_b \mathbf{Z}^\nu \ + \ \partial_a \mathbf{Z}^\mu \ (\mathbf{G}_{\mu\nu} \ \partial_b \mathbf{Z}^\nu) + ... \ ] \ +$$

$$\frac{(p-1)}{2} \ T_p \int d^{p+1}\sigma \ \sqrt{|h|} \tag{4.6}$$

where the ellipsis .... denote the other contributions from the $\mathbf{G}_{\bar\mu\bar\nu}, \mathbf{G}_{\bar\mu\nu}, ...$ components of the metric and the $\bar{\mathbf{Z}}^\mu$ coordinates. The terms inside the brackets in eq-(4.6) are comprised of sums of terms involving pairs of octonionic-complex conjugates in order to render the action real-valued, exactly as it was done in eq-(3.3) to obtain a real-valued interval ds^2. T_p is the p-brane's tension of mass dimension $(mass)^{p+1}$; h^{ab} is the auxiliary $p+1$-dim world-volume metric corresponding to the p-brane. When $p = 1$, one recovers the string action. After the algebraic elimination of the auxiliary $p+1$-dim world-volume metric h^{ab} via its equations of motion, one will recover the Nambu-Goto-Dirac action for the p-brane. The (real-valued) induced world-volume metric $\tilde{h}_{ab}$ is

$$\tilde{h}_{ab} \ = \ \frac{1}{4} \ [\ (\partial_a \mathbf{Z}^\mu \ \mathbf{G}_{\mu\nu}) \ \partial_b \mathbf{Z}^\nu \ + \ \partial_a \mathbf{Z}^\mu \ (\mathbf{G}_{\mu\nu} \ \partial_b \mathbf{Z}^\nu) \ + \ \partial_b \bar{\mathbf{Z}}^\nu \ (\mathbf{G}_{\bar\mu\bar\nu} \ \partial_a \bar{\mathbf{Z}}^\mu) \ + \ .... \ ] \tag{4.7}$$

after inserting this on-shell value for $\tilde{h}_{ab}$ back into the action (4.5) $h_{ab} = \tilde{h}_{ab}$, it yields the Nambu-Goto-Dirac p-brane action

$$S \ = \ - \ T_p \int d^{p+1}\sigma \ \sqrt{ \ det \ |\tilde{h}_{ab}| } \ . \tag{4.8}$$

Naturally, when the background metric is real-valued $\mathbf{G}_{\mu\nu} = g_{\mu\nu}$ there is no ambiguity in the ordering in eqs-(4.5).

Other constructions of string and membrane actions based on the nonassociative octonion algebra are possible. For example, a nonassociative formulation of Exceptional Jordan bosonic strings in $D = 26$ was presented by [24] where the string embedding coordinates belong to the 3×3 matrix elements of the (traceless) Jordan algebra $\mathcal{J}_3(O)$ and carry internal charges belonging to representations of the exceptional F_4 algebra. The automorphism group of $\mathcal{J}_3(O)$ is F_4. The traceless condition is required to obtain a 26-dim algebra since the Exceptional Jordan-Albert algebra $J_3[O]$ is 27-dimensional [26].

A construction of the nonassociative Chern-Simons membrane action from the large N limit of an Exceptional Jordan Matrix Model, corresponding to the direct product of the Exceptional Jordan algebra $\mathcal{J}_3(C \times O)$ with the Lie algebra $SU(N)$, was advanced by [27]. Such Chern-Simons membrane action had a rigid global E_6 invariance. The E_6 Exceptional Jordan Matrix Model was developed by [31] and the F_4 Jordan Matrix Model was studied by [32]. In [27] we proposed also that the generalized spacetime coordinates $\mathbf{X}$ may belong to a real Freudenthal algebra defined in Zorn matrix notation as

$$\begin{pmatrix} a & J_3[O] \\ J_3[O] & b \end{pmatrix} \tag{4.9}$$

the two real variables a, b lie along the diagonal The dimension of the algebra $Fr[O]$ is $2 + 2 \times 27 = 2 + 54 = 2 \times 28 = 56$. In [27] we argued why the latter 56 dimensions may permit the complexified formulation of the bosonic 28-dimensional version of F theory. The connection between the 28-dim bosonic formulation F theory and quaternionic Jordan algebras of degree four $J_4[H]$ has also been raised by Smith [26]. The automorphism group of the Freudenthal algebra $Fr[O]$ is E_6 which is a Grand Unification group.

The generalized spacetime coordinates $\mathbf{X}$ may belong also to the $complexified$ Freudenthal algebra :

$$\begin{pmatrix} a_1 + i\ a_2 & J_3[C \times O] \\ J_3[C \times O] & b_1 + i\ b_2 \end{pmatrix} \tag{4.10}$$

with two complex variables $a_1 + ia_2; b_1 + ib_2$ along the diagonal. The dimension of the algebra $Fr[C \times O]$ is $2 \times (2 + 54) = 4 \times 28 = 112$ which may permit a quaternionic formulation of the bosonic 28-dimensional version of F theory. The automorphism group of the complexified Freudenthal algebra $Fr[C \times O]$ is E_7.

Gunaydin et al [33] have constructed the conformal (like $SO(10, 2)$, "two times") and quasi-conformal (like $SO(10 + 2, 4) = SO(12, 4)$, "four times") nonlinear realizations of the Exceptional Lie groups $E_{7(7)}, E_{8(8)}$ based on the 3-grading and 5-grading decompositions of the noncompact groups $E_{7(7)}$ and $E_{8(8)}$, respectively. The **56** dim representation of $E_{7(7)}$ admits the 3-grading decomposition under the $E_{6(6)} \times \mathcal{D}(dilations)$ subgroup

$$\mathbf{56} \ = \ \mathbf{1} \oplus (\bar{\mathbf{27}} \oplus \mathbf{27}) \oplus \mathbf{1}. \tag{4.11}$$

The 5-grading decomposition of $E_{8(8)}$ w.r.t the subgroup $E_{7(7)} \times \mathcal{D}$ is

$$\mathbf{248} \ = \ \mathbf{1} \oplus \mathbf{56} \oplus (\mathbf{133} \oplus \mathbf{1}) \oplus \mathbf{56} \oplus \mathbf{1} \qquad (4.12)$$

There are *no* quadratic $E_{7(7)}$ invariants in the **56** representation, nevertheless a real *quartic* $E_{7(7)}$ invariant I_4 can be constructed by means of the Freudenthal *ternary* product $[X, Y, Z] \rightarrow W$ and a skew-symmetric bilinear form $< X, Y >$ as [33]

$$I_4 = \ \frac{1}{48} < \ [X, X, X], \ X > \ = \ X^{ij} X_{jk} X^{kl} X_{li} - \frac{1}{4} X^{ij} X_{ij} X^{kl} X_{kl} \ +$$
$$\frac{1}{96} \epsilon^{ijklmnpq} X_{ij} X_{kl} X_{mn} X_{pq} + \frac{1}{96} \epsilon_{ijklmnpq} X^{ij} X^{kl} X^{mn} X^{pq}. \qquad (4.13)$$

the symplectic invariant of two **56** representations, like the area element in phase space $\int dp \wedge dq$, is given by

$$< X, Y > \ = \ X^{ij} \ Y_{ij} \ - \ X_{ij} \ Y^{ij} \qquad (4.14)$$

where the fundamental **56** dimensional representation of $E_{7(7)}$ is spanned by the anti-symmetric real tensors (bi-vectors) X^{ij}, X_{ij} built from $SL(8, R)$ indices $1 \leq i, j \leq 8$ so that $56 = 28 + 28$. An $SL(8, R)$ bi-vector has 28 independent components. The triple product $[X, Y, Z]$ is defined as [33]

$$[X, Y, Z]^{ij} \ = \ - 8 \ X^{ik} \ Y_{kl} \ Z^{lj} \ - 8 \ Y^{ik} \ X_{kl} \ Z^{lj} \ - 8 \ Y^{ik} \ Z_{kl} \ X^{lj} \ - \ i \leftrightarrow j \ -$$
$$2 \ Y^{ij} \ X^{kl} \ Z_{kl} \ - 2 \ X^{ij} \ Y^{kl} \ Z_{kl} \ - 2 \ Z^{ij} \ Y^{kl} \ X_{kl} \ +$$
$$\frac{1}{2} \ \epsilon^{ijklmnpq} \ X_{kl} \ Y_{mn} \ Z_{pq} \qquad (4.15a)$$

$$[X, Y, Z]_{ij} \ = \ 8 \ X_{ik} \ Y^{kl} \ Z_{lj} \ + 8 \ Y_{ik} \ X^{kl} \ Z_{lj} \ + 8 \ Y_{ik} \ Z^{kl} \ X_{lj} \ - \ i \leftrightarrow j \ +$$
$$2 \ Y_{ij} \ Z^{kl} \ X_{kl} \ + 2 \ X_{ij} \ Z^{kl} \ Y_{kl} \ + 2 \ Z_{ij} \ X^{kl} \ Y_{kl} \ -$$
$$\frac{1}{2} \ \epsilon_{ijklmnpq} \ X^{kl} \ Y^{mn} \ Z^{pq} \qquad (4.15b)$$

Gunaydin et al [33] have shown that one may exhibit a nonlinear realization of the algebra $E_{8(8)}$ on a $1 + 56 = 57$-dimensional real vector space where the generalized spacetime coordinates belong to the algebra $Fr[O] \oplus R$ and such that $\mathbf{X} = (X^{ij}, X_{ij}, x)$. The $\mathbf{X}$ forms the $\mathbf{56} \oplus \mathbf{1}$ representation of $E_{7(7)}$. The *quartic* invariant under the action of the $E_{8(8)}$ group is given by

$$\mathcal{N}_4 \ = \ I_4(X^{ij}, X_{ij}) \ - \ x^2. \qquad (4.16)$$

The finite displacement in the 57-dim generalized spacetime is defined as

$$\delta(X,Y) = (X^{ij} - Y^{ij},\ X_{ij} - Y_{ij},\ x - y\ +\ < X, Y >\ ) = (Z^{ij}, Z_{ij}, z) \quad (4.17)$$

the "light-cone" in 57-dim which is invariant under $E_{8(8)}$ is defined by the null condition

$$\mathcal{N}_4\ [\delta(X,Y)] = I_4(Z^{ij}, Z_{ij})\ - z^2 = 0. \quad (4.18)$$

The z generalized coordinate in (4.18) was interpreted as "entropy" by [33]. To sum up, Gunaydin et al [33] concluded that a nonlinear realization of $E_{8(8)}$ on a space of 57 real dimensions is quasi-conformal in the sense that it leaves invariant the "light" cone in (4.18). A further complexification leads to a "light cone" in C^{57} dimensions which is invariant under the complex group $E_8(C)$.

Okubo, de Wit-Nicolai and Gursey-Tze constructed (independently) an octonionic triple-product among three octonionic x, y, z variables [4]

$$[\ x,\ y,\ z\ ]_{Okubo}\ =\ \frac{1}{2}\ (\ (x,y,z)\ +\ < x|e_o >\ [y,z]\ +\ < y|e_o >\ [z,x]\ )\ +$$

$$\frac{1}{2}\ (\ < z|e_o >\ [x,y]\ -\ < z|[x,y]\ >\ e_o\ ). \quad (4.19)$$

where e_o is the Octonion unit element; $< x|y > = < y|x > = Re[\bar{x}y] = Re[\bar{y}x]$ is a symmetric bilinear non-degenerate form, and $(x,y,z) = (xy)z - x(yz)$ is the non-vanishing associator for the octonionic variables. The triple product (4.19) is totally *antisymmetric* in x, y, z whereas the triple product (4.15a, 4.15b) is not. A *quartic* invariant among four octonionic variables can be defined from the triple product (4.19) and the bilinear form as $< w|[x,y,z] >$. It is totally *antisymmetric* in the four variables x, y, z, w. This latter total antisymmetry property will permit us to construct a Nambu-Goto-Dirac 3-brane action corresponding to a $3+1$-dim world volume (our world ?) which is embedded into an octonionic spacetime background with a real-valued flat metric $\eta_{\mu\nu}$, and whose octonionic coordinates are $\mathbf{Z}^\mu, \mu = 1, 2, ..., D$. Hence, we propose the following 3-brane action on a octonionic flat background

$$S\ =\ -T_3 \int d^4\sigma \sqrt{\ [\ <\ \partial_{\sigma^1}\mathbf{Z}_{\mu_1}\ |\ [\ \partial_{\sigma^2}\mathbf{Z}_{\mu_2},\ \partial_{\sigma^3}\mathbf{Z}_{\mu_3},\ \partial_{\sigma^4}\mathbf{Z}_{\mu_4}\ ]\ >\ ]^2} \quad (4.20)$$

T_3 is the 3-brane tension. The square $[....]^2$ is

$$<\ \partial_{\sigma^1}\mathbf{Z}_{\mu_1}\ |\ [\ \partial_{\sigma^2}\mathbf{Z}_{\mu_2},\ \partial_{\sigma^3}\mathbf{Z}_{\mu_3},\ \partial_{\sigma^4}\mathbf{Z}_{\mu_4}\ ]\ >\ <\ \partial^{\sigma^1}\mathbf{Z}^{\mu_1}\ |\ [\ \partial^{\sigma^2}\mathbf{Z}^{\mu_2},\ \partial^{\sigma^3}\mathbf{Z}^{\mu_3},\ \partial^{\sigma^4}\mathbf{Z}^{\mu_4}\ ]\ > \quad (4.21)$$

where $\mu_1, \mu_2, \mu_3, \mu_4 = 1, 2, 3, ..., D \geq 4$. The quantity inside the square root (4.20) plays the same role in ordinary p-branes actions as the determinant of the induced world-volume metric resulting from the embedding of the $p+1$-dim world-volume into a flat target spacetime background

$$det\ h_{ab} \ = \ det\ (\ \partial_a X^\mu\ \partial_b X^\nu\ \eta_{\mu\nu}) \ =$$
$$\{\ X_{\mu_1}, X_{\mu_2}, ..., X_{\mu_{p+1}}\ \}_{NPB}\ \{\ X^{\mu_1}, X^{\mu_2}, ..., X^{\mu_{p+1}}\ \}_{NPB} \qquad (4.22)$$

where the brackets NPB are given by the standard Nambu-Poisson brackets. If the target background is not flat, one must contract the spacetime indices of (4.22) in the following form

$$\{\ X^{\mu_1}, X^{\mu_2}, ..., X^{\mu_{p+1}}\ \}\ \{\ X^{\nu_1}, X^{\nu_2}, ..., X^{\nu_{p+1}}\ \}\ g_{\mu_1\nu_1}(X)\ g_{\mu_2\nu_2}(X)\ ......\ g_{\mu_{p+1}\nu_{p+1}}(X) \qquad (4.23)$$

An octonionic background endowed with an octonionic-valued metric $\mathbf{G}_{\mu\nu}$ will complicate the expression inside the square root of eq-(4.20) resulting from the nonassociativity and noncommutativity. For this reason we opted to choose a real-valued flat metric $\eta_{\mu\nu}$ which permits us to contract spacetime indices in a simple fashion leading to a suitable sum of brackets squared (4.21).

An interesting question would be if the 3-brane action (4.20) involving a quartic product displays any symmetry under E_7 when the number of background octonionic dimensions is 7 and corresponding to $8 \times 7 = 56$ real-dimensions. The latter 56 real dimensions were required to construct the $E_{7(7)}$ quartic I_4 invariant $< [X, X, X], X >$ described by eq-(4.13). For a 3-brane action to display a symmetry under E_8 one would require (at least) a background whose octonionic dimensions is 8 and corresponding to $8 \times 8 = 64$ real-dimensions; i.e. the rank $7, 8$ algebras E_7, E_8 would require a background of $7, 8$ octonionic dimensions, respectively where the 3-brane lives.

To summarize the main results of this section, we have provided generalized octonionic string and brane actions given by eqs-(4.1, 4.3, 4.6, 4.20) that are novel to our knowledge and raised the possibility that the 3-brane action (4.20) (based on the quartic product) in octonionic flat backgrounds of $7, 8$ octonionic dimensions may display a E_7, E_8 symmetry. We conclude with some final remarks pertaining to the developments related to Jordan exceptional algebras, octonions, black-holes in string theory and quantum information theory.

The E_7 Cartan quartic invariant was used by [36] to construct the entanglement measure associated with the tripartite entanglement of seven quantum-bits represented by the group $SL(2, C)^3$ and realized in terms of $2 \times 2 \times 2$ cubic matrices. It was shown by [37] that this tripartite entanglement of seven quantum-bits is entirely decoded into the discrete geometry of the octonion Cayley-Fano plane. The analogy between quantum information theory and supersymmetric black holes in $4D$ string theory compactifications was extended further by [37]. The role of Jordan algebras associated with the homogeneous symmetric spaces present in the study of extended supergravities, BPS black holes, quantum attractor flows and automorphic forms can be found in [35]. An extensive review of the established relationships between black hole entropy in string theory and the quantum entanglement of qubits and qutrits in quantum information theory can be found in [36].

The classification of symmetric spaces associated with the scalars of N extended Supergravity theories, emerging from compactifications of $11D$ supergravity to lower dimensions, and the construction of the U-duality groups as spectrum-generating symmetries for four-dimensional BPS black-holes [35], [38] also involved exceptional symmetries associated with the exceptional magic Jordan algebras $J_3[R, C, H, O]$. The discovery of the anomaly free 10-dim heterotic string for the algebra $E_8 \times E_8$ was another hallmark of the importance of Exceptional Lie groups in Physics. Exceptional Jordan strings [24] carry internal charges that might bear a relation to the charges associated with the BPS states in the black hole solutions of $N = 8$ Supergravity. Finally, we may ask if the four-dim measure (4.20) associated with the 3 brane action, and defined by an invariant quartic product over the octonions, bears any connection to an entanglement Entropy function given by the square root of a hyper-determinant [36]. All this deserves further investigation.

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