TRIPLETS OF TRI-HOMOLOGICAL TRIANGLES

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In this article will prove some theorems in relation to the triplets of homological triangles two by two. These theorems will be used later to build triplets of triangles two by two tri-homological.

I Theorems on the triplets of homological triangles

Theorem 1

Two triangles are homological two by two and have a common homological center (their homological centers coincide) then their homological axes are concurrent.

Proof

Let's consider the homological triangles $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$ whose common homological center is O (see figure 1.)

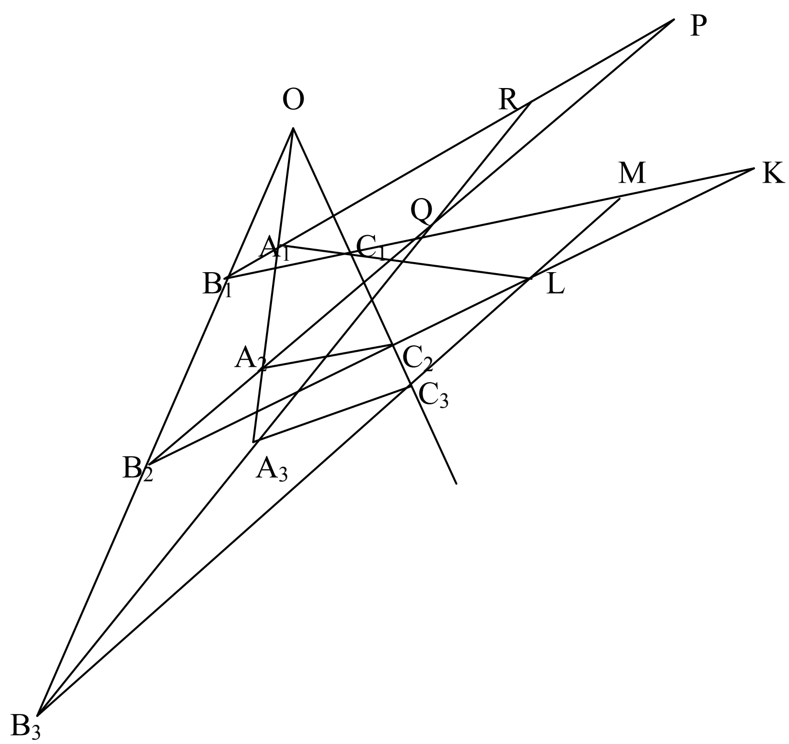

Fig. 1

We consider the triangle formed by the intersections of the lines: A_1B_1, A_2B_2, A_3B_3 and we note it PQR and the triangle formed by the intersection of the lines B_1C_1, B_2C_2, B_3C_3 and we'll note it KLM .

We observe that $PR \cap KM = \{B_1\}, RQ \cap ML = \{B_2\}$, PQ∩ KL=$\{B_3\}$ and because B_1, B_2, B_3 are collinear it results, according to the Desargues reciprocal theorem that the triangles PQR and KLM are homological, therefore PK, RM, QL are concurrent lines.

The line PK is the homological axes of triangles $A_1B_1C_1$ and $A_2B_2C_2$, the line RM is the homological axis for triangles $A_1B_1C_1$ and $A_3B_3C_3$, and the line QL is the homological axis for triangles $A_2B_2C_2$ and $A_3B_3C_3$, which proves the theorem.

Remark 1

Another proof of this theorem can be done using the spatial vision; if we imagine figure 1 as being the correspondent of a spatial figure, we notice that the planes $(A_1B_1C_1)$ and $(A_2B_2C_2)$ have in common the linePK, similarly the planes $(A_1B_1C_1)$ and $(A_3B_3C_3)$ have in common the line QL. If $\{O'\} = PK$ ∩LQ then O' will be in the plane $(A_2B_2C_2)$ and in the plane $(A_3B_3C_3)$, but these planes intersect by the line RM, therefore O' belongs to this line as well. The lines PK, RM, QL are the homological axes of the given triangles and therefore these are concurrent in O'.

Theorem 2

If three triangles are homological two by two and have the same homological axis (their homological axes coincide) then their homological axes are collinear.

Proof

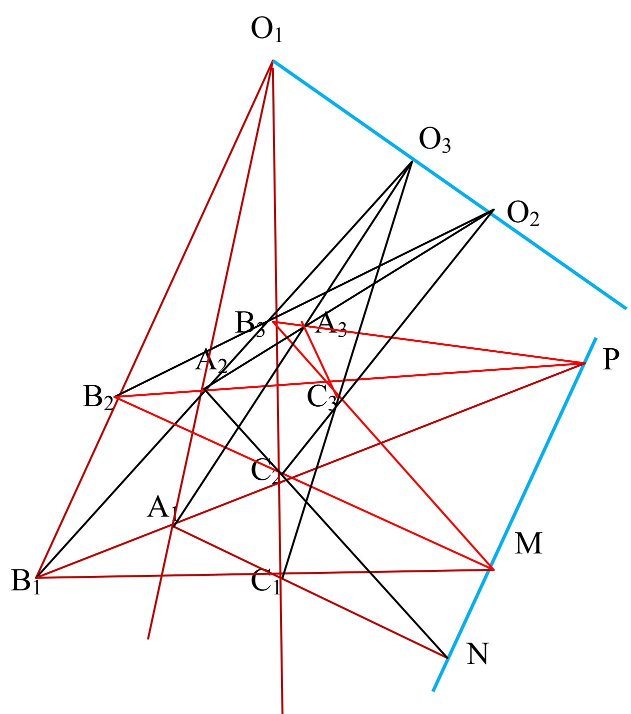

Fig. 2

Let’s consider the homological triangles two by two $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$. We note M, N, P their common homological axis (see figure 2). We note O_1 the homological center of the triangles $A_1B_1C_1$ and $A_2B_2C_2$, with O_2 the homological center of the triangles $A_2B_2C_2$ and $A_3B_3C_3$ and with O_3 the homological center of the triangles $A_3B_3C_3$ and $A_1B_1C_1$.

We consider the triangles $A_1A_2A_3$ and $B_1B_2B_3$, and we observe that these are homological because A_1B_1, A_2B_2, A_3B_3 intersect in the point P which is their homological center. The homological axis of these triangles is determined by the points

, $\{O_1\}=A_1A_2 \cap B_1B_2$, $\{O_2\} = A_2A_3 \cap B_2B_3$, $\{O_3\} = A_1A_3 \cap B_1B_3$

therefore the points O_1, O_2, O_3 are collinear and this concludes the proof of this theorem.

Theorem 3 (The reciprocal of theorem 2)

If three triangles are homological two by two and have their homological centers collinear, then these have the same homological axis.

Proof

We will use the triangles from figure 2. Let therefore O_1, O_2, O_3 the three homological collinear points. We consider the triangles $B_1B_2B_3$ and $C_1C_2C_3$, we observe that these admit as homological axis the line $O_1O_2O_3$.

Because

$$\{O_1\} = B_1B_2 \cap C_1C_2, \{O_2\} = B_2B_3 \cap C_2C_3, \{O_3\} = B_1B_3 \cap C_1C_3,$$

It results that these have as homological center the point $\{M\} = B_1C_1 \cap B_2C_2 \cap B_3C_3$.

Similarly for the triangles $A_1A_2A_3$ and $C_1C_2C_3$ have as homological axis $O_1O_2O_3$ and the homological center M. We also observe that the triangles $A_1A_2A_3$ and $B_1B_2B_3$ are homological and $O_1O_2O_3$ is their homological axis, and their homological center is the point P. Applying the theorem 2, it results that the points M, N, P are collinear, and the reciprocal theorem is then proved.

Theorem 4 (The Veronese theorem)

If the triangles $A_1B_1C_1$, $A_2B_2C_2$ are homological and

$\{A_3\} = B_1C_2 \cap B_2C_1, \{B_3\} = A_1C_2 \cap A_2C_1, \{C_3\} = A_1B_2 \cap A_2B_1$

then the triangle $A_3B_3C_3$ is homological with each of the triangles $A_1B_1C_1$ and $A_2B_2C_2$, and their homological centers are collinear.

Proof

Let O_1 be the homological center of triangles $A_1B_1C_1$ and $A_2B_2C_2$ (see figure 3) and A', B', C' their homological axis.

We observe that O_1 is a homological center also for the triangles $A_1B_1C_2$ and $A_2B_2C_1$. The homological axis of these triangles is C', A_3, B_3. Also O_1 is the homological center for the triangles

$B_1C_1A_2$and $B_2C_2A_1$, it results that their homological axis is A', B_3, C_3

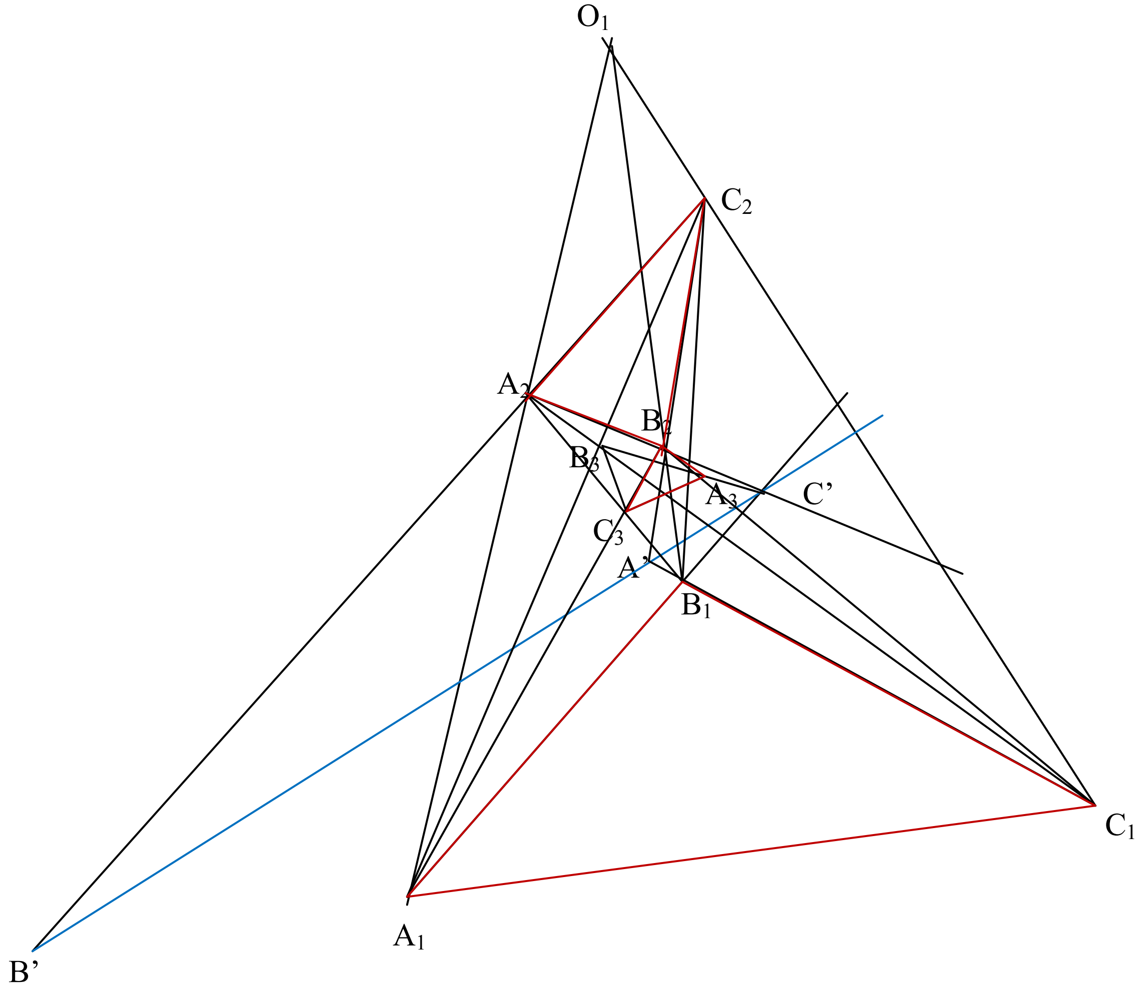

Fig. 3

Similarly, we obtain that the points B', A_3, C_3 are collinear, these being on a homological axis of triangle $C_1A_1B_2$ and$C_2A_2B_1$. The triplets of the collinear points (C', A_3, B_3), (B', A_3, C_3) and(A', B_3, C_3) show that the triangle $A_3B_3C_3$ is homological with triangle $A_1B_1C_1$ and with the triangle $A_2B_2C_2$.

The triangles $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$ are homological two by two and have the same homological axis A', B', C'. Using theorem 3, it results that their homological centers are collinear points.

II. Double-homological triangles

Definition 1

We say that the triangles $A_1B_1C_1$ and $A_2B_2C_2$ are double-homological or bi- homological if these are homological in two modes.

Theorem 5

Let's consider the triangles $A_1B_1C_1$ and $A_2B_2C_2$ such that

$$B_1C_1 \cap B_2C_2 = \{P_1\}, B_1C_1 \cap A_2C_2 = \{Q_1\}, B_1C_1 \cap A_2B_2 = \{R_1\}$$

$$A_1C_1 \cap A_2C_2 = \{P_2\}, A_1C_1 \cap A_2B_2 = \{Q_2\}, A_1C_1 \cap B_2C_2 = \{R_2\}$$

$$A_1B_1 \cap A_2B_2 = \{P_3\}, A_1B_1 \cap B_2C_2 = \{Q_3\}, A_1B_1 \cap C_2A_2 = \{R_3\}$$

Then:

$$\frac{P_1B_1 \cdot P_2C_1 \cdot P_3A_1}{P_1C_1 \cdot P_2A_1 \cdot P_3B_1} \cdot \frac{Q_1B_1 \cdot Q_2C_1 \cdot Q_3A_1}{Q_1C_1 \cdot Q_2A_1 \cdot Q_3B_1} \cdot \frac{R_1B_1 \cdot R_2C_1 \cdot R_3A_1}{R_1C_1 \cdot R_2A_1 \cdot R_3B_1} = 1 \qquad (1)$$

Proof

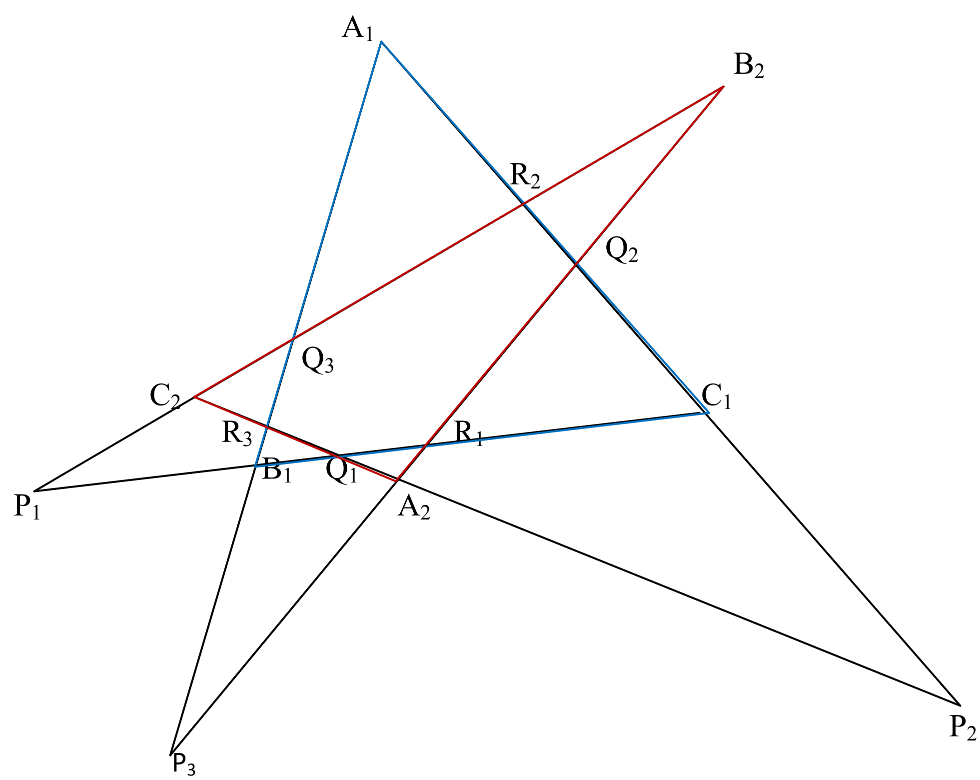

Fig. 4

We'll apply the Menelaus' theorem in the triangle $A_1B_1C_1$ for the transversals $P_1Q_3R_2$, $P_2Q_1R_3$, $P_3Q_2R_1$, (see figure 4).

We obtain

$$\frac{P_1B_1 \cdot R_2C_1 \cdot Q_3A_1}{P_1C_1 \cdot R_2A_1 \cdot Q_3B_1} = 1$$

$$\frac{P_2C_1 \cdot Q_1B_1 \cdot R_3A_1}{P_2A_1 \cdot Q_1C_1 \cdot R_3B_1} = 1$$

$$\frac{P_3A_1 \cdot R_1B_1 \cdot Q_2C_1}{P_3B_1 \cdot R_1C_1 \cdot Q_2A_1} = 1$$

Multiplying these relations side by side and re-arranging the factors, we obtain relation (1).

Theorem 6

The triangles $A_1B_1C_1$ and $A_2B_2C_2$ are homological (the lines A_1A_2, B_1B_2, C_1C_2 are concurrent) if and only if:

$$\frac{Q_1B_1 \cdot Q_2C_1 \cdot Q_3A_1}{Q_1C_1 \cdot Q_2A_1 \cdot Q_3B_1} = \frac{R_1C_1 \cdot R_2A_1 \cdot R_3B_1}{R_1B_1 \cdot R_2C_1 \cdot R_3A_1} \tag{2}$$

Proof

Indeed, if A_1A_2, B_1B_2, C_1C_2 are concurrent then the points P_1, P_2, P_3 are collinear and the Menelaus' theorem for the transversal $P_1P_2P_3$ in the triangle $A_1B_1C_1$ gives:

$$\frac{P_1B_1 \cdot P_2C_1 \cdot P_3A_1}{P_1C_1 \cdot P_2A_1 \cdot P_3B_1} = 1 \tag{3}$$

This relation substituted in (1) leads to (2)

Reciprocal

If the relation (2) takes place then substituting it in the relation (1) we obtain (3) which shows that P_1, P_2, P_3 is the homology axis of the triangles $A_1B_1C_1$ and $A_2B_2C_2$.

Remark 2

If in relation (1) two fractions are equal to 1, then the third fraction will be equal to 1, and this leads to the following:

Theorem 7

If the triangles $A_1B_1C_1$ and $A_2B_2C_2$ are homological in two modes (are double-homological) then these are homological in three modes (are tri-homological).

Remark 3

The precedent theorem can be formulated in a different mod that will allow us to construct tri-homological triangles with a given triangle and of some tri-homological triangles.

Here is the theorem that will do this:

Theorem 8

(i) Let ABC a given triangle and P, Q two points in its plane such that BP intersects CQ in A_1, CP intersects AQ in B_1 and AP intersects BQ in C_1.

Then AA_1, BB_1, CC_1 intersect in a point R.

(ii) If $\cap\, CP = \{A_2\}, CQ \cap AP = \{B_2\}$, BP$\cap\, AQ = \{C_2\}$then the triangles ABC, $A_1B_1C_1$, $A_2B_2C_2$ are two by two homological and their homological centers are collinear.

Proof

(i). From the way how we constructed the triangle $A_1B_1C_1$, we observe that ABC and $A_1B_1C_1$ are double homological, their homology centers being two given points P,Q (see figure 5). Using theorem 7 it results that the triangles ABC, $A_1B_1C_1$ are tri-homological, therefore AA_1, BB_1, CC_1 are concurrent in point noted R.

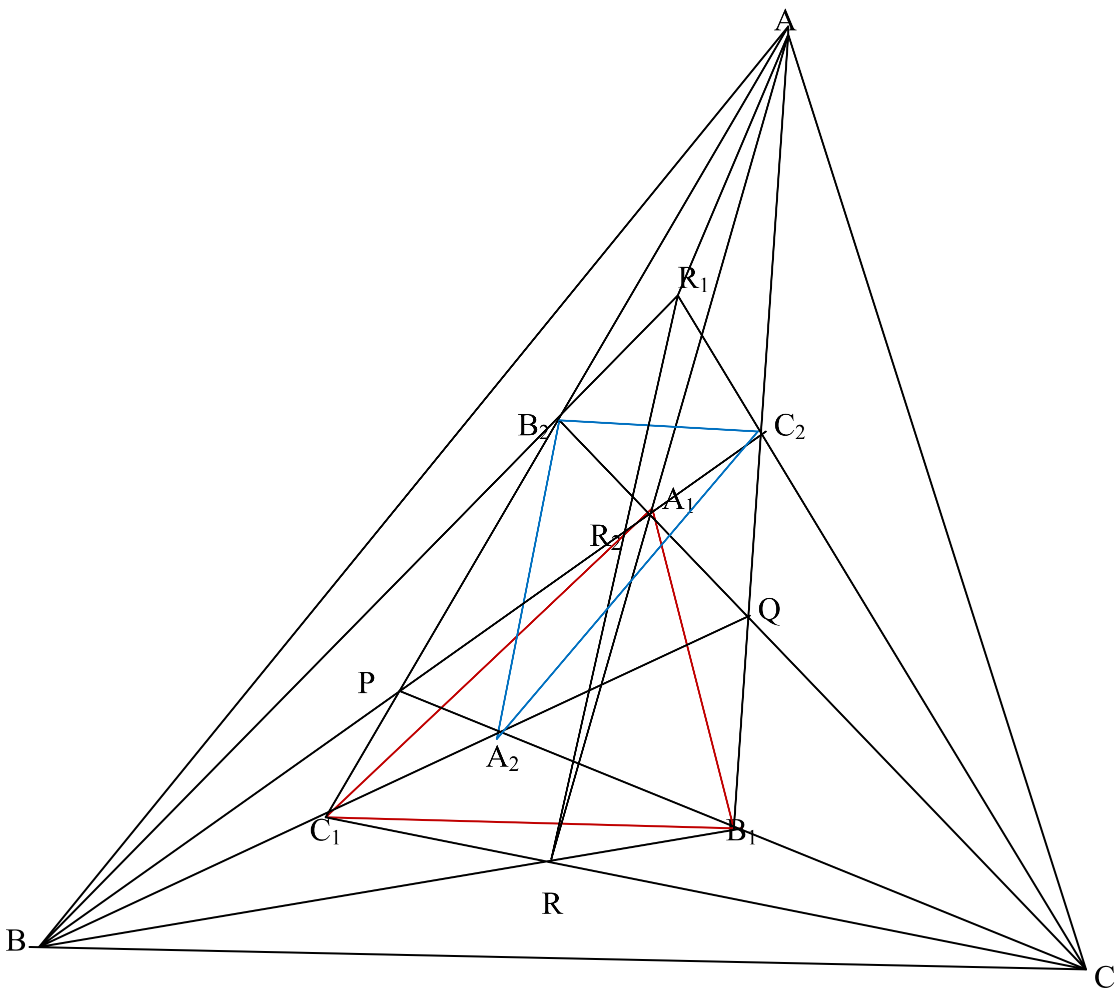

Fig. 5

(ii) The conclusion results by applying the Veronese theorem for the homological triangles ABC, $A_1B_1C_1$ that have as homological center the point R.

Remark 4

We observe that the triangles ABC and $A_2B_2C_2$ are bi-homological, their homological centers being the given points P,Q. It results that these are tri-homological and therefore AA_2, BB_2, CC_2 are concurrent in the third homological center of these triangle, which we'll note R_1.

Similarly we observe that the triangles $A_1B_1C_1$, $A_2B_2C_2$ are double homological with the homological centers P, Q; it results that these are tri-homological, therefore A_1A_2, B_2B_2, C_2C_2 are concurrent, their concurrence point being notated with R_2. In accordance to the Veronese's theorem, applied to any pair of triangles from the triplet $(ABC,\ A_1B_1C_1,\ A_2B_2C_2)$ we find that the points $R,\ R_1,\ R_2$ are collinear.

Remark 5

Considering the points P, R and making the same constructions as in theorem 8 we obtain the triangle $A_3B_3C_3$ which along with the triangles ABC, $A_1B_1C_1$ will form another triplet of triangles tri-homological two by two.

Remark 6

The theorem 8 provides us a process of getting a triplet of tri-homological triangles two by two beginning with a given triangle and from two given points in its plane. Therefore if we consider the triangle ABC and as given points the two points of Brocard $\Omega\Omega$ and Ω', the triangle $A_1B_1C_1$ constructed as in theorem 8 will be the first Brocard's triangle and we'll find that this is a theorem of J. Neuberg: the triangle ABC and the first Brocard triangle are tri-homological. The third homological center of these triangles is noted $\Omega''$and it is called the Borcard's third point and Ω'' is the isometric conjugate of the simedian center of the triangle ABC

Open problems

1) If T_1, T_2, T_3 are triangles in a plane, such that (T_1, T_2) are tri-homological, (T_2, T_3) are tri-homological, then are the (T_1, T_3) tri-homological?

2) If T_1, T_2, T_3 are triangles in a plane such that (T_1, T_2) are tri-homological, (T_2, T_3) are tri-homological, (T_1, T_3) are tri-homological and these pairs of triangles have in common two homological centers, then are the three remaining non-common homological centers collinear?

References

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