On the relation between double summations and tetrahedral numbers

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Abstract In this paper we provide an inverse proof of the relation between a particular class of double sums and tetrahedral numbers. Thus, we present a compact formula to reduce the number of calculations necessary to solve such a kind of problems. The initial identity is confirmed "a posteriori" using the formula mentioned above.

§1. Introduction

There exist different methods to solve summations [1-2] depending on the particular problem we are working on. In this brief paper, we explain that it's possible to use a compact formula to minimize the calculations when we are involved in a particular class of double sums.
First of all we show a preliminary result, which we will use later, to prove that the double summations we have to calculate can be written as the difference of two tetrahedral numbers.

If we use elementary algebra [1] to calculate the value of

$$\sum_{i=m}^{n}(n-i)+\sum_{i=m+1}^{n}(n-i)$$

we have

$$2*\sum_{i=m}^{n}(n-i)-(n-m)=2*\sum_{i=m}^{n}n-2*\sum_{i=m}^{n}i-n+m=$$
$$=2n(n-m+1)-2*\frac{n^2+n-m^2+m}{2}-n+m=$$
$$=2n^2-2nm+2n-n^2-n+m^2-m-n+m=$$
$$=\boldsymbol{n^2-2nm+m^2}=$$
$$=(\boldsymbol{n-m})^2.$$

So, we have seen, without omitting any step, that (for every n<m, where both n and m are positive integers - $\forall n \in \aleph\backslash\{0\}, m \in \aleph_0$ -)

$$\sum_{\mathbf{i=m}}^{\mathbf{n}}(\mathbf{n-i})+\sum_{\mathbf{i=m+1}}^{\mathbf{n}}(\mathbf{n-i})=(\mathbf{n-m})^2. \tag{1}$$

This result let us easily find the value of

$$\sum_{j=0}^{n-m}\sum_{i=m+j}^{n}(n-i)=\sum_{i=m}^{n}(n-i)+\sum_{i=m+1}^{n}(n-i)+\sum_{i=m+2}^{n}(n-i)+\cdots+\sum_{i=n-1}^{n}(n-i)+(n-n) \qquad (2)$$

showing that, in general,

$$\sum_{j=0}^{t}\sum_{i=m+j}^{n}(n-i)$$ with $t \leq n-m$

is equal to the difference between the $(n-m)^{th}$ tetrahedral number [3] and the $(n-m-t-1)^{th}$. In fact, the (2) is, by definition, equal to the $(n-m)^{th}$ tetrahedral number (we consider only positive terms of the sequence A000292 of OEIS [4]).

§2. Proof and main result

The (2) is given by the sum of n-m+1 terms, of which the last one is null (n-n=0). We use the (1) to make the relation explicit, rewriting the (2) as $(n-m)^2+(n-(m+2))^2+(n-(m+4))^2+(n-(m+6))^2+\ldots+6^2+4^2+2^2+0$ if n-m is even and $(n-m)^2+(n-(m+2))^2+(n-(m+4))^2+(n-(m+6))^2+\ldots+5^2+3^2+1^2$ if n-m is an odd number.
Thus, the value of the (2) is given by the sum of the squares of the even numbers $\leq$ n-m, when n-m is even, and by the sum of the squares of the odd numbers $\leq$ n-m, for n-m odd. Unifying the two previous exhaustive occurrences, we get the $(n-m)^{th}$ positive term of the sequence A000292. In particular, let T_i:=0,1,4,10,20,35,56,84,… for i=0,1,2,3,4,5,6,7,… , T_i is even iff $i \pmod 4 \equiv 1$. So, we have proved that the (2) – plus the trivial case t=n-m – is equal to the $(n-m)^{th}$ tetrahedral number (let we assume $T_1=1$ as the first tetrahedral number).

If the upper limit of the index j is t<n-m, we simply have to subtract the value of the $(n-m-t-1)^{th}$ positive tetrahedral number to the previous result: $T_{n-m-t+1}:=\binom{n-m-t+1}{3}$.

Thus:

$$\sum_{j=0}^{t}\sum_{i=m+j}^{n}(n-i)=\binom{n-m}{3}-\binom{n-m-t+1}{3}=$$

$$=\frac{(n-m)(n-m+1)(n-m+2)-(n-m-(t+1))(n-m-(t+1)+1)(n-m-(t+1)+2)}{6}=$$

$$=\frac{\mathbf{(n-m)(n-m+1)(n-m+2)-(n-m-t-1)(n-m-t)(n-m-t+1)}}{\mathbf{6}}$$

Doing some calculations, we are able to rewrite it as:

$$\sum_{j=0}^{t}\sum_{i=m+j}^{n}(n-i)=\frac{3n^2t+3n^2-6nmt-6nm-3nt^2+3n+3m^2t+3m^2+3mt^2-3m+t^3-t}{6}=$$

$$=\frac{(\sqrt{t+1}\,n-\sqrt{t+1}\,m)^2+\ m(t^2-1)-n(t^2-1)+\frac{t(t^2-1)}{3}}{2}=$$

$$=\frac{(\mathbf{t}+\mathbf{1})(\mathbf{n}-\mathbf{m})^2+\left(\mathbf{t}^2-\mathbf{1}\right)\left(\mathbf{m}-\mathbf{n}+\frac{\mathbf{t}}{\mathbf{3}}\right)}{\mathbf{2}}=$$

$$=\frac{\mathbf{t}+\mathbf{1}}{\mathbf{2}}\left[(\mathbf{n}-\mathbf{m})^2+(\mathbf{t}-\mathbf{1})\left(\mathbf{m}-\mathbf{n}+\frac{\mathbf{t}}{\mathbf{3}}\right)\right] \qquad (3)$$

Referring to the initial problem, we can now verify the property (1) substituting t=1 inside the (3):

$$\frac{(1+1)\left[(n-m)^2+(1-1)\left(m-n+\frac{1}{3}\right)\right]}{2}=\frac{2(n-m)^2}{2}=(n-m)^2$$

□

Thus, we have proved that the (1) is a particular instance of the (3).

References

[1] http://en.wikipedia.org/wiki/Summation

[2] http://mathworld.wolfram.com/DoubleSeries.html

[3] http://en.wikipedia.org/wiki/Tetrahedral_number

[4] N. J. A. Sloane, The Online Encyclopedia of Integer Sequences, http://oeis.org/A000292, 2011.