The Diophantine Equations $a^2 \pm mb^2 = c^n$, 

$a^3 \pm mb^3 = d^2$ and $y_1^4 \pm my_2^4 = R^2$

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Abstract

The Diophantine equations $a^2 \pm mb^2 = c^n$, and $a^3 \pm mb^3 = d^2$ have infinitely many nonzero integer solutions, Using the methods of infinite descent and infinite ascent we prove $y_1^4 \pm my_2^4 = R^2$.

The Diophantine equation

$$a^2 + b^2 = c^3, \quad (1)$$

has infinitely many nonzero integer solutions. But it is difficult to prove this [1,2]. In this paper we prove some theorems.

**Theorem 1.** The Diophantine equation

$$a^2 + mb^2 = c^n \quad (2)$$

has infinitely many nonzero integer solutions.

We define supercomplex number [3]

$$z = \begin{pmatrix} x & -my \\ y & x \end{pmatrix} = x + yJ, \quad (3)$$

where

$$J = \begin{pmatrix} 0 & -m \\ 1 & 0 \end{pmatrix}, \quad J^2 = -m.$$ 

Then from equation (3)

$$z^n = (x + yJ)^n = a + bJ. \quad (4)$$

Let $n$ be an odd number
Let \( n \) be an even number

\[
a = \sum_{k=0}^{n/2} \binom{n}{2k} (-m)^k x^{n-2k} y^{2k}, \quad b = \sum_{k=0}^{n/2} \binom{n}{2k+1} (-m)^k x^{n-2k-1} y^{2k+1},
\]

Then from (4) the circulant matrix

\[
\begin{pmatrix}
x & -my \\
y & x
\end{pmatrix}^n = \begin{pmatrix} a & -mb \\ b & a \end{pmatrix},
\]

(5)

Then from (5) circulant determinant

\[
\begin{vmatrix} x & -my \\ y & x \end{vmatrix} = \begin{vmatrix} a & -mb \\ b & a \end{vmatrix},
\]

(6)

Then from equation (6)

\[ c^n = a^2 + mb^2, \]

(7)

where

\[ c = x^2 + my^2. \]

We prove that (2) has infinitely many nonzero integer solutions.

**Theorem 2.** The Diophantine equation

\[ a^2 - mb^2 = c^n \]

has infinitely nonzero integer solutions.

Define supercomplex number \([3]\)

\[
z = \begin{pmatrix} x & my \\ y & x \end{pmatrix} = x + yJ,
\]

(9)

where

\[ J = \begin{pmatrix} 0 & m \\ 1 & 0 \end{pmatrix}, \quad J^2 = m. \]

Then from equation (9)

\[ z^n = (x + yJ)^n = a + bJ. \]

(10)

Let \( n \) be an odd number

\[
a = \sum_{k=0}^{(n-1)/2} \binom{n}{2k} m^k x^{n-2k} y^{2k}, \quad b = \sum_{k=0}^{(n-1)/2} \binom{n}{2k+1} m^k x^{n-2k-1} y^{2k+1}.
\]

Let \( n \) be an even number

\[
a = \sum_{k=0}^{n/2} \binom{n}{2k} m^k x^{n-2k} y^{2k}, \quad b = \sum_{k=0}^{n/2-1} \binom{n}{2k+1} m^k x^{n-2k-1} y^{2k+1}.
\]
Then from (10) circulant matrix
\[
\begin{pmatrix} x & my \\ y & x \end{pmatrix}^n = \begin{pmatrix} a & mb \\ b & a \end{pmatrix}.
\] (11)

Then from (11) circulant determinant
\[
\begin{vmatrix} x & my \\ y & x \end{vmatrix}^n = \begin{vmatrix} a & mb \\ b & a \end{vmatrix}.
\] (12)

Then from equation (12)
\[
c^n = a^2 - mb^2,
\] (13)
where
\[c = x^2 - my^2.\]

We prove that (8) has infinitely many nonzero integer solutions.

**Theorem 3.** The Diophantine equation
\[
a^3 + mb^3 + m^2 c^3 - 3mabc = d^n
\] (14)
has infinitely many nonzero integer solutions.

Define supercomplex number [3]
\[
w = \begin{pmatrix} x & mz & my \\ y & x & mz \\ z & y & x \end{pmatrix} = x + yJ + zJ^2,
\] (15)
where
\[
J = \begin{pmatrix} 0 & 0 & m \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad J^2 = \begin{pmatrix} 0 & m & 0 \\ 0 & 0 & m \\ 1 & 0 & 0 \end{pmatrix}, \quad J^3 = m.
\]

Then from (15)
\[
w^n = (x + yJ + zJ^2)^n = a + bJ + cJ^2
\] (16)

Then from equation (16) circulant matrix
\[
\begin{pmatrix} x & mz & my \\ y & x & mz \\ z & y & x \end{pmatrix}^n = \begin{pmatrix} a & mc & mb \\ b & a & mc \\ c & b & a \end{pmatrix}
\] (17)

Then from equation (17) circulant determinant
\[
\begin{vmatrix} x & mz & my \\ y & x & mz \\ z & y & x \end{vmatrix}^n = \begin{vmatrix} a & mc & mb \\ b & a & mc \\ c & b & a \end{vmatrix}
\] (18)

Then from equation (18)
\[
d^n = a^3 + mb^3 + m^2 c^3 - 3mabc
\] (19)
where
\[ d = x^3 + my^3 + m^2 z^3 - 3 mxyz \]  
(20)

We prove that (14) has infinitely many nonzero integer solutions.
Suppose \( n = 2 \) and \( c = 0 \). Then from (19)
\[ a^3 + mb^3 = d^2 \]  
(21)

when \( n = 2 \) from (16)
\[ a = x^2 + 2myz \neq 0, \quad b = 2xy + mz^2 \neq 0, \quad c = y^2 + 2xz = 0 \]  
(22)

Then from (22) \( y^2 = -2xz \).
Let \( z = -2, \quad x = P^2, \quad y = 2P \),
where \( P > 1 \) is an odd number.
Substituting (23) into (20) and (22)
\[ d = P^6 + 20mP^3 - 8m^2, \quad a = P^4 - 8mP, \quad b = 4P^3 + 4m \]  
(24)

Using equation (24) we prove that (21) has infinitely many nonzero integer solutions.

**Theorem 4.** The Diophantine equation
\[ a^3 - mb^3 + m^2 c^3 + 3mabc = d^a \]  
(25)

has infinitely many nonzero integer solutions.
Define supercomplex number [3]
\[ w = \begin{pmatrix} x & -mz & -my \\ y & x & -mz \\ z & y & x \end{pmatrix} = x + yJ + zJ^2, \]  
(26)

where
\[ J = \begin{pmatrix} 0 & 0 & -m \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad J^2 = \begin{pmatrix} 0 & -m & 0 \\ 0 & 0 & -m \\ 1 & 0 & 0 \end{pmatrix}, \quad J^3 = -m, \]

Then from (26)
\[ w^n = (x + yJ + zJ^2)^n = a + bJ + cJ^2 \]  
(27)

Then from (27) circulant matrix
\[ \begin{pmatrix} x & -mz & -my \\ y & x & -mz \\ z & y & x \end{pmatrix}^n = \begin{pmatrix} a & -mc & -mb \\ b & a & -mc \\ c & b & a \end{pmatrix}, \]  
(28)

Then from equation (28) circulant determinant
\[
\begin{vmatrix}
    x & -mz & -my \\
    y & x & -mz \\
    z & y & x
\end{vmatrix} = 
\begin{vmatrix}
    a & -mc & -mb \\
    b & a & -mc \\
    c & b & a
\end{vmatrix}.
\tag{29}
\]

Then from (29)
\[
d^n = a^3 - mb^3 + m^2c^3 + 3mabc
\tag{30}
\]

where
\[
d = x^3 - y^3 + m^2z^3 + 3mxyz.
\tag{31}
\]

We prove that (25) has infinitely many nonzero integer solutions.

Suppose \( n = 2 \) and \( c = 0 \). Then from (30)
\[
a^3 - mb^3 = d^2
\tag{32}
\]

When \( n = 2 \) from (27)
\[
a = x^2 - 2myz \neq 0, \quad b = -mz^2 + 2xy \neq 0, \quad c = y^2 + 2xz = 0
\tag{33}
\]

Then from (33)
\[
y^2 = -2xz
\]

Let \( z = -2, \quad x = P^2, \quad y = 2P \),

where \( P > 1 \) is an odd number.

Substituting (34) into (31) and (33)
\[
d = P^6 - 20mP^3 - 8m^2, \quad a = P^4 + 8mP, \quad b = 4P^3 - 4m
\tag{35}
\]

Using (35) we prove that (32) has infinitely many nonzero integer solutions.

**Theorem 5.** Define supercomplex number
\[
w = \begin{pmatrix}
    x_1 & -mx_4 & -mx_3 & -mx_2 \\
    x_2 & x_1 & -mx_4 & -mx_3 \\
    x_3 & x_2 & x_1 & -mx_4 \\
    x_4 & x_3 & x_2 & x_1
\end{pmatrix} = x_1 + x_2J + x_3J^2 + x_4J^3,
\tag{36}
\]

where
\[
J = \begin{pmatrix}
    0 & 0 & 0 & -m \\
    1 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 1 & 0
\end{pmatrix}, \quad J^2 = \begin{pmatrix}
    0 & 0 & -m & 0 \\
    0 & 0 & 0 & -m \\
    1 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0
\end{pmatrix}, \quad J^3 = \begin{pmatrix}
    0 & -m & 0 & 0 \\
    0 & 0 & -m & 0 \\
    0 & 0 & 0 & -m \\
    1 & 0 & 0 & 0
\end{pmatrix}, \quad J^4 = -m
\]

Then from (36)
\[
w^n = (x_1 + x_2J + x_3J^2 + x_4J^3)^n = y_1 + y_2J + y_3J^2 + y_4J^3
\tag{37}
\]

Then from (37)
\[
R^n = |y_1|.
\tag{38}
\]
where
\[
R = x_1^4 + m(x_1^4 + 2x_1^2x_2^2 - 4x_1x_2x_3x_4 + 4x_2x_4^2) + m^2(x_1^4 + 2x_2^2x_4^2 + 4x_1x_3x_4^2 - 4x_1x_4^2) + m^3x_4^4,
\]
\[
= y_1^4 + m(y_1^4 + 2y_2^2y_3^2 - 4y_1^2y_2y_3y_4 + 4y_2^2y_4^2) + m^2(y_3^4 + 2y_4^2y_1^2 + 4y_3^2y_4^2 - 4y_2y_4y_3^2) + m^3y_4^4.
\]
(39)

We prove that (38) has infinitely many nonzero integer solutions.

Suppose \( n = 2, y_1 \neq 0, y_2 \neq 0, y_3 = 0 \) and \( y_4 = 0 \), from (38) and (39)
\[
R^2 = y_1^4 + my_2^4
\]
(40)

When \( n = 2 \) from (37)
\[
y_1 = x_1^2 - mx_3^2 - 2mx_2x_4 \neq 0,
\]
(41)
\[
y_2 = 2(x_1x_2 - mx_3x_4) \neq 0,
\]
(42)
\[
y_3 = x_2^2 - mx_4^2 + 2x_1x_3 = 0,
\]
(43)
\[
y_4 = 2(x_1x_4 + x_2x_3) = 0,
\]
(44)

Then from (44)
\[
x_3 = -\frac{x_1x_4}{x_2}.
\]
(45)

Substituting (45) into (43)
\[
x_4 = \frac{-x_1^2 \pm \sqrt{x_1^4 + mx_2^4}}{mx_2}.
\]
(46)

Then from (46)
\[
R_1^2 = x_1^4 + mx_2^4.
\]
(47)

If (47) has no nonzero integer solutions, \( R_1 < R \), using the method of infinite descent we prove
(40) has no nonzero integer solutions. If (47) has one nonzero integer solution, \( R_1 < R \), using the
method of infinite ascent we prove (40) has infinitely many nonzero integer solutions.

Suppose \( m = 1 \) from (47)
\[
R_1^2 = x_1^4 + x_2^4
\]
(48)

has no integer solutions.

Suppose \( m = 2 \) from (47)
\[
R_1^2 = x_1^4 + 2x_2^4
\]
(49)
has no nonzero integer solutions.

Suppose $m = 8$ [4] from (47)

$$R_1^2 = x_1^4 + 8x_2^4$$ \hspace{1cm} (50)

We have a solution $3^2 = 1^4 + 8(1^4)$. Let $x_1 = 1, x_2 = 1$. Then from (45) and (46) $x_3 = 1/2$,

$x_4 = -1/2$. Then from (41) and (42) $y_1 = 7, y_2 = 6, 7^4 + 8 \times 6^4 = 113^2$

Let $x_1 = 7, x_2 = 6, x_3 = -14/9, x_4 = 4/3$. Then from (41) and (42) $y_1 = -\frac{7967}{81}$,

$y_2 = \frac{9492}{81}$. $7967^4 + 8 \times 9492^4 = 262621633^2$.

Suppose $m = 73$ [5]. From (47)

$$R_1^2 = x_1^2 + 73x_2^2.$$ \hspace{1cm} (51)

We have $6^4 + 73(1^4) = 37^2$, $1223^4 + 73 \times 444^4 = 2252593^2$.

Suppose $m = 89$ [5]. From (47),

$$R_1^2 = x_1^4 + 89x_2^4.$$ \hspace{1cm} (52)

We have $2^4 + 89 \times 3^4 = 85^2$, $7193^4 + 89 \times 1020^4 = 52662001^2$.

If $m = R_1^2 - 1$ and $m = R_1^2 - x_1^4$, then (40) has infinitely many nonzero integer solutions.

**Theorem 6.** The Diophantine equation

$$y_1^4 - my_2^4 = R^2,$$ \hspace{1cm} (53)

where

$$y_1 = x_1^2 + mx_2^2 + 2mx_4x_3 \neq 0,$$ \hspace{1cm} (54)

$$y_2 = 2(x_1x_2 + mx_3x_4) \neq 0,$$ \hspace{1cm} (55)

$$y_3 = x_2^2 + mx_4^2 + 2x_1x_3 = 0,$$ \hspace{1cm} (56)

$$y_4 = 2(x_1x_4 + x_2x_3) = 0.$$ \hspace{1cm} (57)

Then from (57)

$$x_3 = -\frac{x_1x_4}{x_2}.$$ \hspace{1cm} (58)

Substituting (58) into (56)
Then from (59)
\[ x_4 = \frac{x_1^4 \pm \sqrt{x_1^4 - mx_2^4}}{mx_2}. \]  

(59)

If (60) has no nonzero integer solutions. \( R_1 < R \), using the method of infinite descent we prove

(53) has no nonzero integer solutions. If (60) has one nonzero integer solution, \( R_1 < R \), using the method of infinite ascent we prove (53) has infinitely many nonzero integer solutions. Suppose \( m = 1 \) from (60)

\[ x_1^4 - x_2^4 = R_1^2. \]  

(60)

has no nonzero integer solutions. Suppose \( m = 2 \), from (60)

\[ x_1^4 - 2x_2^4 = R_1^2. \]  

(61)

has no nonzero integer solutions.

Suppose \( m = 7 \) from (60)

\[ x_1^4 - 7x_2^4 = R_1^2. \]  

(62)

We have one solution

\[ 2^4 - 7(1^4) = 3^2. \]

Let \( x_1 = 2, x_2 = 1 \). Then from (58) and (59) \( x_3 = -2, x_4 = 1 \). Then from (54) and (55)

\[ y_1 = 2 \times 23, \quad y_2 = -2 \times 12. \quad 23^4 - 7 \times 12^4 = 367^2. \]

We prove (63) has infinitely many nonzero integer solutions.

If \( m = x_1^4 - R_1^2 \), then (53) has infinitely many nonzero integer solutions. Our method [3] is used in studies of the Diophantine equations

\[ y_1^n \pm my_2^n = R^e, \quad n = 2, 3, 4, \ldots; e = 2, 3, 4, \ldots; m = 1, 2, 3, \ldots \]  

(63)

References


[3] Chun-Xuan Jiang, Foundations of Santilli’s isonumber theory with applications to new


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