Sharp concentration of the rainbow connection of random graphs

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Abstract

An edge-colored graph $G$ is rainbow edge-connected if any two vertices are connected by a path whose edges have distinct colors. The rainbow connection of a connected graph $G$, denoted by $rc(G)$, is the smallest number of colors that are needed in order to make $G$ rainbow connected. Similarly, a vertex-colored graph $G$ is rainbow vertex-connected if any two vertices are connected by a path whose internal vertices have distinct colors. The rainbow vertex-connection of a connected graph $G$, denoted by $rvc(G)$, is the smallest number of colors that are needed in order to make $G$ rainbow vertex-connected. We prove that both $rc(G)$ and $rvc(G)$ have sharp concentration in classical random graph model $G(n, p)$.

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1. Introduction

We follow the terminology and notation of [4] in this letter. A natural and interesting connectivity measure of a graph was recently introduced in [6] and has attracted many attention of researchers. An edge-colored graph $G$ is called rainbow edge-connected if any two vertices are connected by a path whose edges have distinct colors. Hence, if a graph is rainbow edge-connected, then it must also be connected. Also notice that any connected graph has a trivial edge coloring that makes it rainbow edge-connected. The rainbow connection of a connected graph $G$, denoted $rc(G)$, is the smallest number of colors that are needed in order to make $G$ rainbow edge-connected.
If \( G \) has \( n \) vertices then \( rc(G) \leq n - 1 \), since one can color the edges of a given spanning tree of \( G \) with distinct colors, and color the remaining edges with one of the already used colors. Obviously, \( rc(G) = 1 \) if and only if \( G \) is a complete graph, and that \( rc(G) = n - 1 \) if and only if \( G \) is a tree. An easy observation gives \( rc(G) \geq diam(G) \), where \( diam(G) \) denotes the diameter of \( G \). The behavior of \( rc(G) \) with respect to the minimum degree \( \delta(G) \) has been addressed in the work [5, 10, 11], which indicate that \( rc(G) \) is upper bounded by the reciprocal of \( \delta(G) \) up to a multiplicative constant (which we will discuss later). Some related concepts such as rainbow path [9], rainbow tree [8] and rainbow \( k \)-connectivity [7] have also been investigated recently.

The authors in [10] introduce a vertex coloring edition. A vertex-colored graph \( G \) is called rainbow vertex-connected if any two vertices are connected by a path whose internal vertices have distinct colors. Denote the rainbow vertex-connection of a connected graph \( G \) by \( rvc(G) \), which is defined as the smallest number of colors that are needed in order to make \( G \) rainbow vertex-connected. It is clear that \( rvcG \leq n - 2 \), and \( rvcG = 0 \) if and only if \( G \) is complete. Similarly, we have \( rvcG \geq diam(G) - 1 \).

Note that \( rc(G) \) and \( rvc(G) \) are both monotonic property in the sense that if we add an edge to \( G \) we cannot increase its rainbow edge/vertex-connection. Therefore, it is desirable to study the random graph setting [3]. Motivating this idea, in this paper we consider the rainbow edge/vertex-connection in Erdős-Rényi random graph model \( G(n, p) \) with \( n \) vertices and edge probability \( p \in [0, 1] \). Based on some known bounds of diameter and degree of \( G(n, p) \), we establish the following concentration results:

**Theorem 1.** Suppose that \( \omega = \omega(n) \to -\infty \) and \( c = c(n) \to 0 \). Let \( d = d(n) \geq 2 \) be a natural number and \( 0 < p = p(n) < 1 \). If

\[
np = \ln n + \frac{20n \ln \ln n}{d + 1} - \omega, \quad (1)
\]

\[
p^d n^{d-1} = \ln \left( \frac{n^2}{c} \right) \quad (2)
\]

and

\[
\frac{pn}{(\ln n)^3} \to \infty \quad (3)
\]

hold, then \( rc(G(n, p)) = d \) almost surely as \( n \to \infty \).

**Theorem 2.** Suppose that \( \omega = \omega(n) \to -\infty \) and \( c = c(n) \to 0 \). Let \( d = d(n) \geq 2 \) be a
natural number and $0 < p = p(n) < 1$. If
\[
np = \ln n + \frac{11n\ln \ln n}{d} - \omega,
\]  
(4)
\[
p^d n^{d-1} = \ln \left(\frac{n^2}{c}\right)
\]  
(5)
and
\[
\frac{pn}{(\ln n)^d} \to \infty
\]  
(6)
hold, then $rvc(G(n,p)) = d - 1$ almost surely as $n \to \infty$.

2. Proof of Theorem 1 and 2

In this section, we will first prove Theorem 1 and then Theorem 2 can be derived similarly.

Let $\delta(G)$ be the minimum degree of a graph $G$. The following lemma gives upper bounds of rainbow edge/vertex-connection.

**Lemma 1.** ([10]) A connected graph $G$ with $n$ vertices has $rc(G) < 20n/\delta(G)$ and $rvc(G) < 11n/\delta(G)$.

**Proof of Theorem 1.** By Lemma 1 and the comments in the Section 1, we have
\[
diam(G(n,p)) \leq rc(G(n,p)) < 20n/\delta(G(n,p))
\]  
(7)
if $G(n,p)$ is connected.

To get the concentration result, we need to estimate the diameter and minimum degree of random graph $G(n,p)$. It follows from the assumptions (2) and (3) that $diam(G(n,p)) = d$ almost surely (see [2] or [3] pp.259). By the assumption (1), we get $\delta(G(n,p)) = 20n/(d+1)$ (see [1] or [3] pp.65). Now we almost conclude our proof by (7).

There are nevertheless two things remain to check: (i) The assumptions (1)-(3) are reasonable, that is, there really exist such $p$ and $d$. (ii) $G(n,p)$ is almost surely connected.

Define $c = c(n) \to 0$ by the equation
\[
\ln \ln \left(\frac{n^2}{c}\right) = (\ln n) \cdot \ln \ln n
\]  
(8)
and let $\omega(n) \to -\infty$ sufficiently slowly. By the assumption (1), we define a function of $d$
\[
f(d) := (np)^d = \left(\ln n + \frac{20n\ln \ln n}{d+1} - \omega\right)^d.
\]  
(9)
Take \( d = \ln n \), and we obtain
\[
\ln f(d) = (\ln n) \cdot \ln \left( \ln n + \frac{20n\ln n}{1 + \ln n} - \omega \right)
\geq (\ln n) \cdot \ln \left( \frac{n\ln n}{\ln n} \right)
\geq \ln n + (\ln n) \cdot \ln \ln n
= \ln \left( n \cdot \ln \left( \frac{n^2}{c} \right) \right)
\] (10)
where the last equality holds by the definition (8).

Take \( d = \ln \ln n \), and we have
\[
\ln f(d) = (\ln \ln n) \cdot \ln(n + 20 - \omega)
\leq (\ln \ln n) \cdot \ln(21n)
\leq \ln n + (\ln n) \cdot \ln \ln n
= \ln \left( n \cdot \ln \left( \frac{n^2}{c} \right) \right)
\] (11)
where the last equality holds by the definition (8).

From (10), (11) and the fact that \( f(d) \) is continuous, we derive that there exists some \( d \in [\ln \ln n, \ln n] \) such that \( \ln f(d) = \ln(n \ln(n^2/c)) \) holds. Consequently, the assumption (2) holds. For such \( d \), by (9), we have
\[
np = \Omega \left( \frac{n \ln \ln n}{\ln n} \right),
\] (12)
which clearly satisfies the assumption (3), and \( G(n, p) \) is connected almost surely (c.f. [3] pp.164).

Hence, both (i) and (ii) have been checked and the proof is finally completed. \( \square \)

**Proof of Theorem 2.** It can be proved similarly by noting the fact
\[
diam(G(n, p)) - 1 \leq rvc(G(n, p)) < 11n/\delta(G(n, p)). \] (13)

We leave the details to the interested readers. \( \square \)

**References**


[10] M. Krivelevich, R. Yuster, The rainbow connection of a graph is (at most) reciprocal to its minimum degree. *J. Graph Theory* 63(2010), 185–191