On prime factors in old and new sequences of integers

Marco Ripà

E-mail: marco@marcoripa.net

Abstract The paper shows that the only possible prime terms of the "consecutive sequence" (1,12,123,1234,...) represent $13.\overline{3}\%$ of the total, and their structure is explicited. This outcome is then extended to every permutation of their figures. The previous result is applied to a consistent subset of elements belonging to the circular sequence (resulting from the consecutive one), identifying moreover the 31 first primes. Therefore, a criterion is illustrated (further extendible) that progressively reduces the numerousness of the "candidate prime numbers". §3.3 is devoted to the solution of a similar problem. The last section introduces a new sequence which, although much larger, has the same properties as the previous ones, and it also proposes a few open problems.

§1. Introduction

In the next section I will take on the meaning of the note "unsolved problem" no.15 (that asks to single out, if they exist, which elements of the consecutive sequence - 1,12,123,1234,... - are prime numbers) proposed by Florentin Smarandache in his publication, "Only problems, not solutions!" [5].
In particular, I will be concerned with the problem of the divisibility of a class (small in an absolute sense, but with many variants) of successions of natural numbers (positive whole ones excluding zero), and then go on to confront some of the other questions posed within the above-mentioned text.
Subsequently, I will supply a simple formula to greatly narrow down the search for the possible prime numbers (those that are required to be identified) accompanied by a percentage of "prime number candidates" among all the generic elements of the succession. I will not (unfortunately) give a definitive answer to the 15th question, but will develop a simple criterion, also extendable (with very small modifications) to other types of "similar" successions.

§2. Divisibility by 3 of the elements of a few integer sequences

We know that a natural number is prime when it is only divisible by 1, or itself, and that the unit is not involved in the circle of prime numbers. While a number is divisible by 2 if it ends in an even number (0,2,4,6 or 8), yet divisibility by 3 is assured by the condition that the resultant of the sum of all the figures of the numbers we propose to factorize is in turn divisible by 3. Also, remember that all the numbers that end in 5 will count 5^n (with $n \geq 1$) amongst its divisors [1] [8].

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[1] **I am implicitly referring to the base 10 numeral system (that which we adopt daily). For example, using the binary system for the succession we are analyzing, it would be 1,110,11011,11011100,…; in the octal system, we would have 1,12,123,…, 1234567,123456710,12345671011, 1234567101112,… and so on.**

Definition 2.1.

We indicate with $a_1, a_2, a_3, \ldots, a_{1-i}, a_i, \ldots$ the single elements that comprise the "consecutive sequence" 1,12,123,…, 123456789, 12345678910, 1234567891011,…
Therefore, we know that a_i is simply equal to $a_{i-1}\_g_i$ (with the underscore referring to the postponement of the suffix "i" at the end of the preceding symbol, while with "g_i" we indicate the i-th *insert* – formed by "#Cf" digits -) [2].

The example $a_1=1$ is obviously banal.
a_2 permits us to exploit the criterion of divisibility by 2 and the same consideration is valid for all the other elements of the sequence identified by the subscript i=2*n ($n \in \aleph$).
For sure, studying the divisibility by 3 gives more interesting results. Indicating with g_i the constituent members of the sequence that gradually postpone towards the last of those already present (Definition 2.1) we have $a_i=g_1\_g_2\_g_3\_\ldots\_g_i$.
The generic element g_l (with $1 \le l \le i$) will be composed of an amount of digits (#Cf) equal to #Cf(l) = min $\left\{k: \frac{g_l}{10^k} \le 1\right\}$, with $k \in \aleph \backslash \{0\}$.

Definition 2.2.

We define the singular figure of g_l as $c_1, c_2, c_3, \ldots$: for k_{min}= #Cf(l), it results $g_l = c_1\_c_2\_c_3\_\ldots\_c_{\#Cf(l)}$.

At this point, we notice that the postulant condition for the divisibility by 3 of a_i (that if verified by imposing the non primality of the element a_i) gives a satisfactory result (sufficient condition) if it were for the element a_{i-1} and "i" is divisible by 3 (i=3*n). The same is valid if the element a_{i-2} is divisible by 3 and i-1+i=2*i-1 contains a 3 (with an arbitrary exponent major or is equal to the unit) among its own factors (2*i-1=3*n).
Subsequently, it's shown that the possible cases, which one can verify concretely, are only two of the three that follow [3]:

$$A_1 := \sum_{h=1}^{i} \sum_{l=1}^{\#Cf_h} c_l = 3 * n \qquad (\text{where } A_1 := a_i \equiv \sum_{h=1}^{i} g_h = 3 * n),$$

$$A_2 := \sum_{h=1}^{i-1} \sum_{l=1}^{\#Cf_h} c_l = 3 * n \qquad (\text{where } A_2 := a_{i-1} \equiv \sum_{h=1}^{i-1} g_h = 3 * n),$$

$$A_3 := \sum_{h=1}^{i-2} \sum_{l=1}^{\#Cf_h} c_l = 3 * n \qquad (\text{where } A_3 := a_{i-2} \equiv \sum_{h=1}^{i-2} g_h = 3 * n).$$

It's enough to observe that A_3 is redundant [4], seeing as it is entirely covered by the case A_1 and (alternatively) by that of A_2. The corresponding succession relating to problem no.15, indicating with A_0 the case "sui generis" $a_1=1$, is indeed $A_0,A_1,A_1,A_2,A_1,A_1,A_2,A_1,A_1,A_2,..$; where $\forall i \ge 1$, $\sum_h g_h = A_1$ if i=2+3*m or i=3*(m+1) ($m \in \aleph \ge 0$) and $\sum_h g_h = A_2$ in the remaining cases (i=1+3*m).

In what follows, the sum of the figures of any triplet of consecutive elements (g_{l-1}, g_l, g_{l+1}) forms a number divisible by 3 (where $g_{l-1}+g_l+g_{l+1}=3*g_l$). Operatively speaking, if, for example, we would want to verify the

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[2] It is evident that g_i=i, g_{i-1}=i-1 etc.. The choice of introducing the letter "g" is just for both explanatory and visual simplification.

[3] It is permitted to write the successive equivalences, in as far as they are made legitimate by the congruence relationship 10^n (mod 3)≡1.

[4] Using a pinch of immediate logic, it's clear beforehand that there are only two true and real options: either a number is divisible by another, or it is not. For the first situation, we have A_1, for the other, we have A_2.

divisibility by 3 of a_i, we should go on to study that in $\sum_{h=1}^{i-2}\sum_{l=1}^{\#Cf_h} c_l + \sum_{l=1}^{\#Cf_{(i-1)}} c_l + \sum_{l=1}^{\#Cf_i} c_l$: if $\sum_{h=1}^{i-2}\sum_{l=1}^{\#Cf_h} c_l=3*n$ we will find ourselves in the ambivalent situation A_3, if $\sum_{h=1}^{i-2}\sum_{l=1}^{\#Cf_h} c_l + \sum_{l=1}^{\#Cf_{(i-1)}} c_l=3*n$ we would be in the A_2 condition, while if $\sum_{h=1}^{i-2}\sum_{l=1}^{\#Cf_h} c_l + \sum_{l=1}^{\#Cf_{(i-1)}} c_l + \sum_{l=1}^{\#Cf_i} c_l=3*n$ we will once again have the case called A_1.

Seeing as A_3, A_2 and A_1 are divisible by 3, we can eliminate all the g_h correspondents and limit ourselves to studying the remaining ends of the sequence (respectively, two, one and zero) [5]. If the sum of the figures of this, or these, inserts, is in turn divisible by 3, it will therefore be the entire a_i. Observing the scheme defined by a_i for $i\rightarrow\infty$, it's clearly apparent which sequence of a_i is divisible by 3.

12 3 45 6 78 9 1011 12 1314 15 1617 18 1920 21 2223 24 2526 27 2829 30 3132 33 3435 36 3738 39 ...
...1767 ...17681769 ...1770 ...17711772 ...1773 ...17741775 ...1776 ...17771778 ...1779 ...17801781 ...

Lemma 2.3.

a_i is <u>not</u> divisible by 3 if and only if $g_i=1+3*n$ (remember that – by definition - $g_i\equiv i$).

Proof

In a consecutive succession, like Sm_N, we find that the increment of the sum total of the figures, passing from a_i to a_{i+1}, is constant in module 3 and in turn forms a purely periodic sequence, if calculated in that module. Indicating with T(i) the sum of the figures of a_i, we have that, initiating T for a certain i, $[T(i-1)](\text{mod } 3)\equiv 0$, $[T(i)](\text{mod } 3)\equiv[T(i-1)](\text{mod } 3)+g_i(\text{mod } 3)\equiv 1$, $[T(i+1)](\text{mod } 3)\equiv[T(i)](\text{mod } 3)+g_{i+1}(\text{mod } 3)\equiv[1+2](\text{mod } 3)\equiv 0$ and $[T(i+2)](\text{mod } 3)\equiv[T(i+1)+3](\text{mod } 3)\equiv[T(i-1)](\text{mod } 3)\equiv 0$.
The congruence in module 3 of T(i-1) and T(i+2) imposes:
$g_i(\text{mod } 3)\equiv[T(i)](\text{mod } 3)\equiv[T(i+\Delta i)](\text{mod } 3)\equiv[T(i)+\Delta i](\text{mod } 3)$ for $\Delta i=3*n$. The assertion automatically follows, since $g_i(\text{mod } 3)\equiv[T(i)+\Delta i](\text{mod } 3)\equiv[T(i)](\text{mod } 3)+\Delta i(\text{mod } 3)\equiv 1$. □

In our case, being that Sm1=1 – equal to 1(mod 3) -, it results that Sm2(mod 3)≡0 and Sm3(mod 3)≡0. In general, $[Sm(1+3*k)](\text{mod } 3)\equiv 1$ and $[Sm(2+3*k)](\text{mod } 3)\equiv[Sm(3*k)](\text{mod } 3)\equiv 0 \quad \forall k \in \aleph_0$.
So occur only cases A_1 and A_2 previously seen and $66.\overline{6}\%$ of the Sm_N results in being divisible by 3. The same is evident for all the possible permutations of the figures that make up $a_i \in$ Sm_N.

Combining the (2.3) with the scheme of $a_i:=2*n$ (described in the opening paragraphs), we can easily strike out the majority of them from the group of prime number candidates. At this point, if we superimpose the scheme a_i that ends in 5 (and is therefore divisible by 5), we obtain a relationship that excludes almost 87% of the a_i from the set (potentially empty) of cardinal numbers only divisible by one or themselves (which we'll conveniently define as a_j) [6].

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[5] In this case, I prefer going back to reason in terms of A_3, to not uselessly weigh down the paper. We know that the insert g_h, that cannot be removed (if it is there) consists of only this one ($i\Rightarrow A_2$) and there is no need to make other evaluations, because A_1 automatically implies divisibility by 3, while A_2 is immediately deduced as the exact opposite.

[6] However much is said for the factors 2, 3 and 5 could also be abstractly repeated for 7, 11, 13 etc..., as long as we take account of an ulterior variable, so it's valid to say that the number of figures in "i": when this element varies, they just mutate the rules to apply. It would be sufficient to implement the known reduction methodology, inherent in divisibility of the above-mentioned factors, to then generalize the results obtained, and, in the end, to

The final resulting formula will therefore be

$$\begin{cases} \mathbf{j = 1 + 6 * n} \\ \mathbf{n \neq 4 + 5 * k} \end{cases} \qquad (n \in \aleph\backslash\{0\}, k \in \aleph).$$

The relationship $\frac{a_j}{a_i} * 100$ indicates the percentage of the "prime number candidates" within our sequence, in relation to their total; consequently, a probability of $a_i \equiv a_j$ is comprised of an absolute maximum of $\frac{3}{19} \approx 0.15385$ (recorded as corresponding to n=3) and the asymptotic value:

$$\lim_{n\to+\infty}\left(\frac{n}{1+6*n} - \frac{n}{24+6*n*5}\right) = \frac{1}{6} - \frac{1}{30} = \frac{1}{7.5} = 0.1\overline{3}\,.$$

I have directly verified (using [1]) that all the "i" terms a_i with i < 217 (the first 28 a_j) are composite numbers, also managing to individualize some patterns within the relative factorizations [2]. Furthermore, the primality of some a_i rests absolutely possible, though this paper will not supply any answer to it.

In virtue of the commutative property of the addition, the inherent rule of the divisibility by 3 of a_i is also valid for the other sequences containing all the terms of $a_{i,}$ though not in a particular order, rather, in a class of an even vaster numeric series.
If a generic sequence S contains only elements s_j established by the same amounts of the figures of some of the a_i, we can limit ourselves to applying the rule of the 3 just seen in this context. It will be enough to explain the s_j in terms of the corresponding a_i, effectuating the substitution and studying these last ones to then transpose the outcome of s_j. Not only will we be able to use what is illustrated to study the divisibility by 3 of all the possible permutations of the "i" members of the consecutive model sequence, but we will also be able to include the permutations of the $\sum_{l=1}^{i} \#\mathrm{Cf}_l$ figures that comprise it. The circular sequence perfectly re-enters this case study, together with the "right-left sequence" (and the "left-right" one) with natural terms and to an infinity of others [7].
The considerations about the divisibility by 2 and by 5 remain the same (those just done while discussing the consecutive sequence set).

§3. Skimming of the 499501 initial terms of the circular sequence and making explicit the smallest 31 prime numbers contained within it

This section contains a study of the primality of the elements of the "circular sequence" (in comparison with what has been done with the consecutive sequence), within the scope of the decimal number system.

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further restrict the circle of the possible prime number candidates. For example, I studied that for 95≤i≤996, a_i | 7 ⇒ i=95+$\sum_s d_s$, where d_s=0,5,9,5,9,5,9,5,9,… for s=0,1,2,3,…,129. Indeed, the succession of the increments d_s is periodic – of period Δs=2 - starting from s=1 (the increments +5 and +9 alternate as the proposition value grows from the index "s"). Analogically, a_i | 11 ⇒ i=106+$\sum_s d_s$, in which d_s=0,7,15,7,15,7,15,7,15,… for s=0,1,2,3,…,81. Also, here we have that d_s – the succession of the increments - has the period Δs=2 for 1≤s≤81.

[7] Amongst the more illuminating examples is the inverse of the sequence just studied; we obtain it by substituting g_1 with g_i, g_2 with g_{i-1} and so on, until we arrive at the "swap" between g_i and g_1. In this case, the related result of the divisibility by 3 does not change an iota, just in virtue of the commutative property of the sum. It's been verified that among the first 10000 terms of the sequence, only the 82nd is a prime number (82818079...4321) [6].

§3.1. Exclusion Criterion and candidate primes. The generic terms of the circular sequence (ref. unsolved problem no.16) is explicable via the formula proposed by Vassilev-Missana and Atanassov [4-7]: indicating, as always, the first "term" of every $a_i \equiv a(i)$ with g_1, the second with g_2, and so forth up to g_i, we can make the formula in question consistent with the notation that we have used up to now.

Definition 3.1.

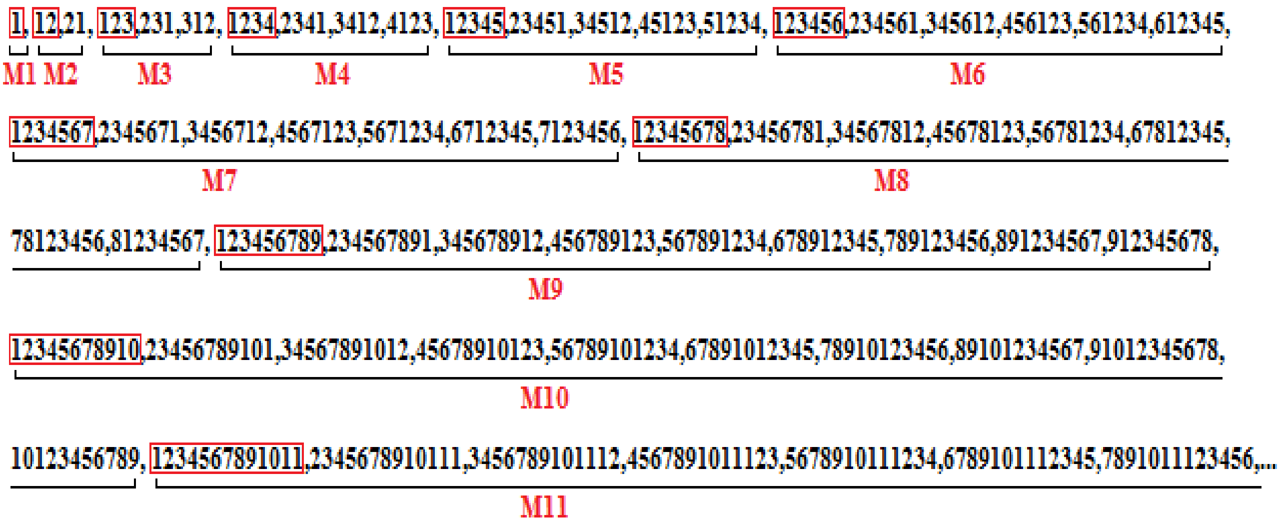
Let a(i) be the i-th term of the circular sequence, for every natural number "i", it results that:

a(i)=s_(s+1)_..._k_1_2_..._(s-2)_(s-1),

where $k \equiv k(i) = \left\lfloor \frac{\sqrt{8*i+1}-1}{2} \right\rfloor$

and $g_1 := s \equiv s(i) = i - \frac{k*(k+1)}{2}$ (in fact, $i = \sum_{l=1}^{r-1} l + g_1 = \frac{k*(k-1)}{2} + g_1$).

The expansion of this is the following:



As shown in the figure above, a specific sub-group of the elements of the circular sequence can be extracted and used to construct a sub-sequence, which we'll call O(r), formed only by the terms that verify $g_1=1, g_2=g_1+1, \ldots, g_i=g_{i-1}+1$. It is evident that the o_r, the prime elements of each re-grouping M(r), coincide with the "old" a_i that we have met while studying the consecutive sequence [8].
Being that all the "r" elements constitute a part of every sub-group M(r) of the particular permutations of the related o_r, we have that the elements of M(r), in virtue of the commutativity of the sum [9], are divisible by 3 if and only if the correspondent o_r is also divisible by 3.

The pattern of the M(r) therefore results in being: $A_2, A_1, A_1, A_2, A_1, A_1, A_2, A_1, A_1, A_2 \ldots$
The M(r) are <u>not</u> divisible by 3 $\Leftrightarrow$ $r := j \equiv 1 + 3*n$ $\quad (\forall n \in \aleph_0)$.

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[8] **Let me rapidly note that $\sum_{r=1}^{m} M(r)$ is always equal to a triangular number; to be precise, summing the primes "m" M(r), we obtain the m-th triangular number.**

[9] **Given $o_r \in M(r)$, a generic term of the consecutive sequence, $\sum_{h=1}^{r} g_h = \frac{r*(r+1)}{2}$.**

At this point, we can be even more selective eliminating all the elements of the M(j) that end in 0,2,4,5,6, or 8, in that they are certainly divisible by 2 and/or 5, and therefore not primes.
We can refer to a specific component of the sequence (in terms of M(r)) or to that (∈M(r)) which has as a prime "constituent term" a given g_1, observing that it finds itself, by construction, in the position $\sum_{t=1}^{r}(t-1)+g_1$ (where with "t" I make reference to the number of terms a_i∈M(t)). Starting from a_2, that is, for r≥2, the previous formula is equivalent to $\sum_{t=1}^{r-1} t + g_1 = \frac{r*(r-1)}{2} + g_1$.

Proposition 3.2.

If we indicate with b_j the terms of M(j), where j:=1+3*n (assuming n is positive as far as it is clear that M(1)≡1 is not a prime number), we can easily calculate the percentage of b_j within the sub-group M(j) associated with the eventual A_2.

Proof

$$\frac{b_j}{b_i} := \frac{b_i(\mathrm{mod}\ 10) \equiv \{1,3,7,9\}}{b_i(\mathrm{mod}\ 10) \equiv \{0,1,2,3,4,5,6,7,8,9\}};$$

$$\lim_{i\to+\infty} \frac{b_j}{b_i} = \frac{4}{10}.$$

As "i" approaches to infinity, the percentage of the terms of the circular sequence not divisible by 2, 3 or 5, reaches – as foreseen – the value $\frac{4}{10} * \frac{1}{3} = \frac{1}{7.5} = 0.1\overline{3}$; equal to the percentage associated with the consecutive sequence.

□

Going on to analyze these terms (I have personally studied the possible candidates for M<211 – M(211) is formed of numbers of 525 figures –), it's evident that the smallest value of the subscript, such that a_i is a prime number, results in being i=8 (the second component of M(4)), or a_8≡**2341**. The next 30 terms, not divisible by any other numbers other than one or themselves, are (in order):

a_{53}≡**89101234567**,

a_{82}≡**45678910111213123**,

a_{302}≡**23456789101112131415161718192021222324251** (This takes stock of the prime element of the sequence M, for a fixed j, as being a prime number – and net after the removal of the numbers ending in 0,2,4,5,6 o 8 -),

a_{591}≡**303132333412345678910111213141516171819202122232425262728 29**,

a_{1055}≡**2021222324252627282930313233343536373839404142434445461234567891011121314151617181 9**,

a_{1077}≡**4243444546123456789101112131415161718192021222324252627282930313233343536373839404 1** (It is an example of 2 primes enclosed within M(j)→M(46) itself),

a_{1340}≡**141516171819202122232425262728293031323334353637383940414243444546474849505152123 4 5678910111213**,

a_{1499}≡**1415161718192021222324252627282930313233343536373839404142434445464748495051525354 551234567891011121 3**,

a_{1890}≡**60611234567891011121314151617181920212223242526272829303132333435363738394041424344 4546474849505152535455565758 59** (This time, the prime number is represented by the last number of the possible "candidates" of the sequence M(j)),

a_{2231}≡**20212223242526272829303132333435363738394041424344454647484950515253545556575859606 1626364656667123456789101112131415161718 19**,

a_{3109}≡**282930313233343536373839404142434445464748495051525354555657585960616263646566676 8697071727374757677787912345678910111213141516171819202122232425262 7**,

a_{3145}≡**646566676869707172737475767778791234567891011121314151617181920212223242526272829 30313233343536373839404142434445464748495051525354555657585960616263**,

a_{3620}≡**505152535455565758596061626364656667686970717273747576777879808182838485123456789 1011121314151617181920212223242526272829303132333435363738394041424344454647484 9**,

a_{3878}≡**505152535455565758596061626364656667686970717273747576777879808182838485868788123 4567891011121314151617181920212223242526272829303132333435363738394041424344454647484 9** (Note that this term is congruent - in 10^{89} – with the preceding prime number of the sequence),

a_{4405}≡**343536373839404142434445464748495051525354555657585960616263646566676869707172737 4757677787980818283848586878889909192939412345678910111213141516171819202122232425262 728 2930313233**,

a_{6248}≡**323334353637383940414243444546474849505152535455565758596061626364656667686970717 2737475767778798081828384858687888990919293949596979899100101102103104105106107108109 110 1111121234567891011121314151617181920212223242526272829303 1**,

a_{8878}≡**100101102103104105106107108109110111112113114115116117118119120121122123124125126 1 271281291301311321331234567891011121314151617181920212223242526272829303132333435363 738 394041424344454647484950515253545556575859606162636465666768697071727374757677787980 818 28384858687888990919293949596979899** (The particular value taken on by the first term of this number – g_1=100 – gives an indisputable harmony),

a_{8888}≡**110111112113114115116117118119120121122123124125126127128129130131132133123456789 1 011121314151617181920212223242526272829303132333435363738394041424344454647484950515 253 545556575859606162636465666768697071727374757677787980818283848586878889909192939495 969 79899100101102103104105106107108109**,

a_{11329}≡**4567891011121314151617181920212223242526272829303132333435363738394041424344454 64 748495051525354555657585960616263646566676869707172737475767778798081828384858687888990 919293949596979899100101102103104105106107108109110111112113114115116117118119120121 122 123124125126127128129130131132133134135136137138139140141142143144145146147148149150151 123**,

a_{11439}≡**114115116117118119120121122123124125126127128129130131132133134135136137138139140 141142143144145146147148149150151123456789101112131415161718192021222324252627282930 313 233343536373839404142434445464748495051525354555657585960616263646566676869707172737475 767778798081828384858687888990919293949596979899100101102103104105106107108109110111112 113**,

a_{12310}≡**646566676869707172737475767778798081828384858687888990919293949596979899100101102 103104105106107108109110111112113114115116117118119120121122123124125126127128129130131 132133134135136137138139140141142143144145146147148149150151152153154155156157123456789**

10111213141516171819202122232425262728293031323334353637383940414243444546474849505152 5
3545556575859606162 63,

a_{12344}≡**98991001011021031041051061071081091101111121131141151161171181191201211221231241 2**
512612712812913013113213313413513613713813914014114214314414514614714814915015115215315
415515615712345678910111213141516171819202122232425262728293031323334353637383940414243
444546474849505152535455565758596061626364656667686970717273747576777879808182838485868
788899091929394959697,

a_{13323}≡**120121122123124125126127128129130131132133134135136137138139140141142143144145146**
147148149150151152153154155156157158159160161162163123456789101112131415161718192021222
324252627282930313233343536373839404142434445464748495051525354555657585960616263646566
676869707172737475767778798081828384858687888990919293949596979899100101102103104105106
107108109110111112113114115116117118119,

a_{13747}≡**525354555657585960616263646566676869707172737475767778798081828384858687888990919**
293949596979899100101102103104105106107108109110111112113114115116117118119120121122123
124125126127128129130131132133134135136137138139140141142143144145146147148149150151152
153154155156157158159160161162163164165166123456789101112131415161718192021222324252627
282930313233343536373839404142434445464748495051,

a_{15883}≡**130131132133134135136137138139140141142143144145146147148149150151152153154155156**
157158159160161162163164165166167168169170171172173174175176177178123456789101112131415
161718192021222324252627282930313233343536373839404142434445464748495051525354555657585
960616263646566676869707172737475767778798081828384858687888990919293949596979899100101
102103104105106107108109110111112113114115116117118119120121122123124125126127128129,

a_{17471}≡**808182838485868788899091929394959697989910010110210310410510610710810911011111211**
311411511611711811912012112212312412512612712812913013113213313413513613713813914014114
214314414514614714814915015115215315415515615715815916016116216316416516616716816917017
117217317417517617717817918018118218318418518618712345678910111213141516171819202122232
425262728293031323334353637383940414243444546474849505152535455565758596061626364656667
686970717273747576777879,

a_{17985}≡**303132333435363738394041424344454647484950515253545556575859606162636465666768697**
071727374757677787980818283848586878889909192939495969798991001011021031041051061071081
091101111121131141151161171181191201211221231241251261271281291301311321331341351361371
381391401411421431441451461471481491501511521531541551561571581591601611621631641651661
671681691701711721731741751761771781791801811821831841851861871881891901234567891011121
314151617181920212223242526272829,

a_{19815}≡**114115116117118119120121122123124125126127128129130131132133134135136137138139140**
141142143144145146147148149150151152153154155156157158159160161162163164165166167168169
170171172173174175176177178179180181182183184185186187188189190191192193194195196197198
199123456789101112131415161718192021222324252627282930313233343536373839404142434445464
748495051525354555657585960616263646566676869707172737475767778798081828384858687888990
919293949596979899100101102103104105106107108109110111112113,

a_{20335}≡**343536373839404142434445464748495051525354555657585960616263646566676869707172737**
475767778798081828384858687888990919293949596979899100101102103104105106107108109110111
112113114115116117118119120121122123124125126127128129130131132133134135136137138139140
141142143144145146147148149150151152153154155156157158159160161162163164165166167168169
170171172173174175176177178179180181182183184185186187188189190191192193194195196197198
199200201202123456789101112131415161718192021222324252627282930313233,

a_{21676}≡**148149150151152153154155156157158159160161162163164165166167168169170171172173174**
175176177178179180181182183184185186187188189190191192193194195196197198199200201202203
204205206207208123456789101112131415161718192021222324252627282930313233343536373839404
142434445464748495051525354555657585960616263646566676869707172737475767778798081828384
858687888990919293949596979899100101102103104105106107108109110111112113114115116117118
119120121122123124125126127128129130131132133134135136137138139140141142143144145146147.

All the preceding results have been verified utilizing the elliptic curve method so have not undergone any risk of error characteristic of a probabilistic approach.

Therefore, of the first 22155 terms of the circular sequence, 31 are prime numbers [10].
For sure it's not surprising to notice that no initial term, among the first 210 M(r), is a prime. That was easily deducible *a priori*, because these elements coincide with those of the consecutive sequence set, that I have personally verified to not contain any prime numbers amongst the 216 initial components. Nonetheless, they have been found to be terms, for which, the prime element of the circular sequence, for a given j, (just considering the elements not to be skipped over beforehand – because they end in 1,3,7 or 9 -) is, effectively, a prime number. The smallest amongst them is a_{302}∈M(25).

An interesting question (open problem) is the following.

Because the probability of a certain number being a prime, within a comprehending interval 'lot' of numbers to test, reduces as the growth of the dimensions of such numbers increases, is it plausible that 2 is the maximum quantitative of prime numbers that we can trace within M(r), or M(j), for a fixed value of r (and therefore of j) [11]? Furthermore, were that not true, is it possible, establishing an arbitrary value "*l*", to find a whole j such that M(j) contains at least "*l*" prime numbers? Also, if this affirmation is seen to be erroneous, what is the maximum value of "*l*" for which the previous result is verifiable (it's evident that $l \geq 2$)?

I will now enunciate, in a synthetic way, what I have called (with a great deal of fantasy) "**Exclusion Criterion**". We banally state an iterative method to reach, within the range of well-defined terms, some "acceptable" percentages of "prime number candidates" (considering the correct relationship of a prefixed quantitative of prime factors – that divide the elements of the consecutive/circular sequence -). In this circumstance, of the 2 sequences that we are interested in, I'm going to consider just the elements composed of variable figures between 192 and 2899.
The procedure bases itself on cancellation, from the list of numbers that could be primes, of the elements of the sequence that we know to be divisible by a certain factor. It's something very similar to what we've seen in the case of 3, but with the substantial difference that the rules which we will apply are not valid for all the infinite terms to study. Another cardinal principle is represented by a property that joins determinate

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[10] **Rather, 31 prime numbers out of 22156 terms, if we also add o_{211} – which we already know to be composed -.**

[11] **In reality, we have to also take account the fact that the pure numerousness of the b_j∈M(j) – the elements to test – augments when j grows, so the result appears anything but taken for granted – even in a probability sense.**

elements of the circular sequence: in precise situations, it's found that all the M(r) have (at least) one common factor. Once these fundamental relationships are identified, we can foresee when such events will verify themselves again and we will avoid "worrying" about the corresponding terms of the sequence. On this premise, Sm_N≡O(r) being strictly contained within the circular sequence, we will certainly have a prime number that divides both, but it's not obvious that the opposite is true: the divisibility by the prime "pr" of a generic o_r, represents a necessary but <u>not</u> sufficient condition, in that pr also divides all the other "r-l" elements of M(r).

I'm going to refrain from rigorously showing every rule proposed in the following pages, contrary to what I've done for the divisibility by 3 of Sm_N. Provided that we keep in mind that the values of the variable r have to be composed of 3 figures (that is 100≤r<1000), the known properties of the algebraic sum, together with the fact that we are just considering the uneven terms within a specific M(r) – for a fixed value of r - should be sufficient enough to lead the diligent reader to the end of the associated process that's required. Another discourse is to verify that the property can be generalized for an arbitrary set of prime factors, even if it is still to be proven that they divide the o_r (and the M(r)) with a strict periodic cadence (function of r) – in particular the period is a multiple of pr itself -.

We could ask the following question: "Is this exclusion criterion efficient (at least for the elements formed by the same number of figures)?"
Answer: No, at least not in any absolute sense. In fact, it does not provide any information about what happens within a given M(r), as soon as "r" has overcome all the barriers (and is therefore considered a reserve of prime number candidates – r:=j -). Considering the consecutive sequence, the discourse does not change very much; to realize it, it's enough to observe the factorization, for example, of the 133rd term: its smallest prime number is, in fact, composed of 19 figures!
Stubbornly persevering in the previous direction, we vainly ask ourselves if, superimposing all these (innumerable) terms, it were possible to cover the entire Sm_N set.
In the case of the circular sequence, we could ask if it were true, or not, from a certain point onwards (seen that the periodicity is not strictly pure). At this juncture, however, I'll gladly leave the burden of the answer to whoever is stubborn enough to go on with the subject.
Aware that this iterative method will not be able to lead us to the definitive answer, I reserve the right to confide in you that it can at least be a starting point for more valid future arguments, based on a larger amount of data than I've been able to provide here.

Exclusion Criterion, valid for $i \in [100,1000); r \in [100,1000)$:
An operative algorithm for the research on prime numbers in the gamut of the terms of Sm_N≡a_i (if they exist) and exhaustive sampling of the M(r) which may contain prime numbers, for some selected values of the variables $i, r \in [100, 999)$:

Laws of exclusion, valid for every Sm_N (a_i with 100≤i<1000):

$a_i \mid 2 \Rightarrow i=100+2*k$

$a_i \mid 3 \Rightarrow i=101+3*k$ or $i=102+3*k$

$a_i \mid 5 \Rightarrow i=100+5*k$

$a_i \mid 7 \Rightarrow i=100+\sum_s d_s$, where d_s=0,9,5,9,5,9,5,… for s=0,1,2,3,…,129

$a_i \mid 11 \Rightarrow i=106+\sum_s d_s$, where d_s=0,7,15,7,15,7,15,… for s=0,1,2,3,…,81

$a_i \mid 13 \Rightarrow i=113+\sum_s d_s$, where d_s=0,7,19,7,19,7,19,… for s=0,1,2,3,…,68

And we could continue in this way, extending the analysis to the following prime numbers (over 13).

Laws of exclusion, valid for every M(100≤r<1000):

M(r) | 3 $\Rightarrow$ r=101+3*k or r=102+3*k

M(r) | 7 $\Rightarrow$ r=100+14*k

M(r) | 11 $\Rightarrow$ r=106+22*k

M(r) | 13 $\Rightarrow$ r=120+26*k

Etc…

Reminder: *<u>not</u> all the remaining terms are prime number candidates, in that, prizing out the elements within every M(r≡j) remaining, we'll have to thoroughly examine only the values ending in 1,3,7 or 9.*

Taking into consideration just the laws of exclusion of 3,7,11 and 13, we would register a relationship $\frac{a_j}{a_i}$ (for i(r) which varies within the range 4951-499500) equal to 28.29845314*0.40042158≈**0.11331** [12].

If we were not already satisfied, we could further improve the result obtained, reducing the value $\frac{a_j}{a_i}$ (for both Sm_N and M(r)), observing, for example, that:

M(r) | 37 $\Leftrightarrow$ a_i | 37 $\Rightarrow$ r (and i)=123+$\sum_s d_s$, where d_s=0,12,25,12,25,12,25,… for s=0,1,2,3,…,47
And so forth (these relationships are formalized in section 3.2).
Curiously: for r=172 it verifies 3 of the preceding conditions; in fact M(r=172) results in being divisible by 11, 13 and 37 at the same time. This means that <u>all</u> the 172 M(172) are divisible by 5291 (and obviously none of them will be a prime).
Taking up again for a moment the classic consecutive sequence in consideration, we can try to apply, all together, the laws related to 2,3,5,7,11,13 – and also add the rule relative to 37 just illustrated. By so doing, for 100≤a_i≤1002, we get to a relationship $\frac{a_j}{a_i}$ expressible in a percentage inferior to **8.54%** (of the 902 elements comprised within Sm 100 and Sm 1002, extremes included, only 77 elements are not excludable by means of the conditions that we have imposed).

<u>The new rules that I have formulated, about the divisibility of Sm_N/M(r), are valid when the independent variable assumes values with 3 figures, but they are not valid anymore for 10^3≤i<10^4 (respectively 10^3≤r<10^4), because the divisibility criteria (on which I have based my analysis) are strictly linked to the amount of figures that compose the number to factorize. Therefore, if we wanted to apply the exclusion criterion to M(r≥1000), we'd need to identify the new patterns, relative to fixed prime numbers >5, that lead us to the formulation of rules similar to those already seen and that would be applicable to 10^4≤r<10^5</u>.

To whoever would like to venture into the research of the larger prime numbers, I would advise starting from the elements of the sequence Sm_N≡O(r) (if they know the factorization of it) in which the smallest factor – in the form a^b - has a very large base "a". In other words, numbers that, when factorized, are explainable via the product of prime numbers, of which the smallest is, in turn, a large prime!

This is a summary of the 241 macro-candidates M(100≤r<1000) within which, based on the previously mentioned rule for 3,7,11,13 and 37, the presence of prime numbers is not excludable (I have already tested the values associated with r<211 and the results are those reported in this article):

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[12] **In this case, the value 40.042158...% =$\frac{56039}{139950} * 100$ is an exact number; it represents the percentage of possible prime terms within the M(r≡j), with 100≤r≤999. In section 3.3 we will resolve the general case – which appears on page 25 of [3].**

103,109,112,115,118,121,124,127,130,133,139,145,151,154,157,163,166,169,175,178,181,187,190,193,196,199,202,205,208,**211,214,217,220,223,229,232,235,241,244,247,256,259,262,265,274,277,280,283,286,289,292,295,298,301,307,313,316,319,322,325,331,334,337,340,343,346,349,355,358,361,364,367,373,376,379,385,388,391,397,400,403,409,412,415,418,421,424,427,430,433,439,442,445,448,451,454,457,460,463,466,469,472,475,481,487,490,496,499,508,511,514,517,523,526,529,532,535,538,541,544,547,550,553,556,559,565,571,574,577,580,583,586,589,592,595,598,601,607,610,613,619,622,625,628,631,637,643,649,652,655,658,661,664,667,670,673,676,679,682,685,694,697,703,706,709,712,721,724,733,736,739,742,745,748,751,754,757,760,763,769,775,778,781,784,787,790,793,799,802,805,808,811,817,820,823,829,835,841,844,847,850,853,859,862,865,868,871,877,880,883,886,889,892,895,901,904,907,910,913,916,919,922,925,928,931,934,943,946,955,958,961,967,970,973,976,979,985,988,991,994,997.**

(If we have the proof that the o_r - for $r\leq k$ – are all composed, it's permissible to exclude every first element of $M(r\leq k)$ from the research – or the r terms $a_{\sum_{t=1}^{r}(t-1)+1}$ of the sequence -).

N.B.
The number of the M(j), in which the presence of terms only divisible by one and by themselves is not excludable, depends on how many and from which prime factors we have employed to extend the exclusion criterion (for r within the fixed interval): for example, proposing r=118, we have that 83 is a fixed factor for all the terms of the circular sequence formed of 246 digits, but there's no hope of revealing such a property just basing itself on the few relationships that we have previously chosen to consider.

§3.2. A few linear exclusive conditions. Formalization of the exclusion rules, that a prime number candidate of the consecutive sequence must respect, for $100\leq i<1000$ – just taking into account the divisors 2,3,5,7,11,13 and 37 -:

$$\begin{cases} i = 103 + 6 * k \\ i \neq 100 + 3 * k \\ i \neq 100 + 5 * k \\ i \neq 100 + 14 * k \\ i \neq 109 + 14 * k \\ i \neq 106 + 22 * k \\ i \neq 113 + 22 * k \\ i \neq 113 + 26 * k \\ i \neq 120 + 26 * k \\ i \neq 123 + 37 * k \\ i \neq 135 + 37 * k \end{cases}$$

The corresponding rules, such that $a_i \in M(100\leq r<1000)$ is a prime candidate of the circular sequence – considering the divisors 3,7,11,13 and 37 and implying that $a_i(\text{base } 10) \equiv \{1,3,7,9\}$ - are instead:

$$\begin{cases} r = 100 + 3 * k \\ r \neq 100 + 14 * k \\ r \neq 106 + 22 * k \\ r \neq 120 + 26 * k \\ r \neq 123 + 37 * k \\ r \neq 135 + 37 * k \end{cases}$$

§3.3. General solution of the circular sequence final digits probabilistic problem. Kenichiro Kashihara [3] also cites the circular sequence, posing two additional questions regarding the series in consideration. While the second question is not at all pertinent to the current article, a variant of the first question has been briefly confronted, when I had to calculate the percentage of the terms of M(j) (with

100≤j≤999) that can be made up of one prime number. In this circumstance I have calculated the sum of the probability associated with c ∈ {1,3,7,9} for the terms of the sequence, comprising those between the 4951st and the 499500th, such that r≡j.
So, the question I will answer is linked to the extension of that calculation: "What is the probability that a generic element of the circular sequence ends with a given final figure - c ∈ {0,1,2,3,4,5,6,7,8,9} -?"

To be brief, I will omit describing in detail how I have identified the successive relationships, seeing that the test of their validity is relatively simple and quick. With p(c=k) I will indicate the probability that $a_i \equiv a(r) \in M(r)$ – a generic element of the circular sequence that is between a_1 and $a_{\sum_{t=1}^{r} t}$ – has k as a final figure (that is to say a_i(base 10)≡k). I have chosen to consider "r" as a parameter (rather than "i") to not excessively complicate the exposition. Remember, however, that to recognize the exact probability linked to the first a_h (with $\sum_{t=1}^{r-1} t = \frac{r*(r-1)}{2} < h < \frac{r*(r+1)}{2} = \sum_{t=1}^{r} t$) terms of the sequence, it is sufficient to insert in the numerator (in the relationship that defines the probability that we propose to calculate) the "successful cases" among the elements [13] contained in the M(t≤r-1), then to add to these the other successes, associated with the remaining terms of the sequence (of the numerousness surely inferior to r) and finally to divide all this by $h \equiv \frac{r*(r-1)}{2} + g_1$ (the complex of the terms considered). The substance of the procedure remaining intact, I will refer, as already said, just to $a_i \leq a_{\sum_{t=1}^{r} t}$.

Specifically, we have that:

$$p(c=0) = \begin{cases} 0 \quad \text{if } r \leq 9 \\ \dfrac{10 * \sum_{l=0}^{m_0} l + (m_0+1) * [r - (m_0 * 10 + 9)]}{\sum_{i=1}^{r} i} \quad \text{if } r \geq 10 \end{cases}$$

in which $m_0 \equiv \left\lfloor \frac{r-10}{10} \right\rfloor$

$$p(c=9) = \begin{cases} 0 \quad \text{if } r \leq 8 \\ \dfrac{10 * \sum_{l=0}^{m_9} l + (m_9+1) * [r - (m_9 * 10 + 8)]}{\sum_{i=1}^{r} i} \quad \text{if } r \geq 9 \end{cases}$$

in which $m_9 \equiv \left\lfloor \frac{r-9}{10} \right\rfloor$

$$p(c=8) = \begin{cases} 0 \quad \text{if } r \leq 7 \\ \dfrac{10 * \sum_{l=0}^{m_8} l + (m_8+1) * [r - (m_8 * 10 + 7)]}{\sum_{i=1}^{r} i} \quad \text{if } r \geq 8 \end{cases}$$

in which $m_8 \equiv \left\lfloor \frac{r-8}{10} \right\rfloor$

$$p(c=7) = \begin{cases} 0 \quad \text{if } r \leq 6 \\ \dfrac{10 * \sum_{l=0}^{m_7} l + (m_7+1) * [r - (m_7 * 10 + 6)]}{\sum_{i=1}^{r} i} \quad \text{if } r \geq 7 \end{cases}$$

in which $m_7 \equiv \left\lfloor \frac{r-7}{10} \right\rfloor$

---

[13] **Such elements are exactly $\frac{r*(r-1)}{2}$.**

We can say the same for the other 6 remaining cases. In general, if we indicate with "k" the value taken up by "c", the following relationship is valid:

$$\mathbf{p(c=k)}=\begin{cases}\mathbf{0} & \mathbf{if\ \ r \le k-1}\\ \mathbf{2*\dfrac{10*\sum_{\mathit{l}=0}^{m_k}\mathit{l}+(m_k+1)*\left[r-\left(m_k*10+(k-1)\right)\right]}{r*(r+1)}} & \mathbf{if\ \ r\ge k}\end{cases}$$

where $\mathbf{m_k} \equiv \left\lfloor\frac{\mathbf{r-k}}{\mathbf{10}}\right\rfloor$

($\lfloor x \rfloor$ indicates the operator "floor" or the smallest integer regarding $x \in \mathcal{R}^{+}$).

Doing the calculations, we can rewrite the entirety more compactly as:

$$\mathbf{p(c=k)}=\begin{cases}\mathbf{0} & \mathbf{if\ \ r \le k-1}\\ \mathbf{2*\dfrac{\left\lfloor\frac{r-k}{10}\right\rfloor*\left(-5*\left\lfloor\frac{r-k}{10}\right\rfloor+r-k-4\right)+r-k+1}{r*(r+1)}} & \mathbf{if\ \ r\ge k}\end{cases}$$

Remembering that, by definition, r≥1; from the preceding we deduce that the last of the 6 cases, that I avoided explaining, is that to which the highest probability of success is associated, $\forall r \in \aleph\backslash\{0\}$. In fact, for k=1, it results that:

$$p(c=1)=\frac{10*\sum_{l=0}^{m_1} l+(m_1+1)*[r-(m_1*10)]}{\sum_{i=1}^{r} i}=2*\frac{m_1*(-5*m_1+r-5)+r}{r*(r+1)}$$

with $m_1 \equiv \left\lfloor\frac{r-1}{10}\right\rfloor$

Logically, ∀r, p(c=1)+p(c=2)+p(c=3)+…+p(c=9)+p(c=0)=1 and furthermore $\lim_{r\to+\infty} p(c=0)=\lim_{r\to+\infty} p(c=9)=\lim_{r\to+\infty} p(c=8)=\cdots=\lim_{r\to+\infty} p(c=1)$, because, as the independent variable draws out towards infinity, the chain of equivalence results in being valid: $\lim_{r\to+\infty} m_0=\lim_{r\to+\infty} m_9=\lim_{r\to+\infty} m_8=\cdots=\lim_{r\to+\infty} m_1$.

Out of curiosity, we can analyze the case in which p(c)>0, ∀ c∈ {0,1,2,3,4,5,6,7,8,9}, is characterized by the maximum discrepancy of values (under our initial condition). It is easy to understand that I am referring to the event r=10:

$p(c=0)=\frac{1}{\sum_{i=1}^{10} i}=0.0\overline{18}$,

$p(c=9)=\frac{2}{\sum_{i=1}^{10} i}=2*p(c=0)=0.0\overline{36}$,

$p(c=8)=\frac{3}{\sum_{i=1}^{10} i}=3*p(c=0)=0.0\overline{54}$,

…

$p(c=1)=\frac{10}{\sum_{i=1}^{10} i}=10*p(c=0)=0.\overline{18}$.

Definitively, for every finite number of terms of the succession, we have that

$p(c=0) < p(c=9) < p(c=8) < p(c=7) < p(c=6) < p(c=5) < p(c=4) < p(c=3) < p(c=2) < p(c=1)$ [14].

§4. The consecutive-permutational integer sequence and the primality of its terms

As I've explained in §2, the rule of the divisibility by 3 is commutative and this property let us apply a more extensive criterion, focused on groups of figures rather than their sum. Therefore, combining this consideration with what has been illustrated in the opening (about Sm_N), we can create an even "larger" sequence with the same characteristic. This sequence (chosen a base, for example, the decimal one) is formed by all the permutations of the figures of Sm_N, arranged in ascending order [15]: fixed the number of terms in the previous sequence (that is, by definition, unlimited), we have to take each term and write down the permutations of the elements of it. Then we have to order these items from the smallest to the biggest. The first terms of this new sequence, which we'll define as P(i(r)), - taking into account the growth of "r" – coincide with the o(r) of the circular one, but only for r<10, because, for r=10 (and above), the first term of the group is p_{409114}≡01123456789, which we consider to be equal to 1123456789. So, for r≥10, a few terms of the same group have a different amount of digits, but the total of the elements in that group still remains $\#Cf_{o(r)}!$ (the factorial of the number of the digits of o(r)).

Applying the rule of 3, $p_{i(r)}$ is not divisible by 3 iff r:=j=1+3*n ($\forall n \in \aleph\backslash\{0\}$). We only have to study the groups (formed by $\#Cf_{o(r)}!$ terms) such that they are congruent modulo 10 to {1,3,7,9}, also satisfying the condition r=j.

At this point, we could ask ourselves the following questions:

- What is the general term for the new sequence?
- How many primes are there (is it possible to find a formula to identify them)?
- Are there terms that can be written as a power of a prime number?
- What is the probability that the trail digit of a (general) term of the sequence is (0,1,2,3,4,5,6,7,8 or 9) – we need a universal formula -

About the first and the last questions, remember that, for a fixed value "k", the number of terms $p_i:=p_{(r\leq k)}$ is $p_{\sum_{r=1}^{k}[\#Cf_{o(r)}!]}$ (that is ≥ of $p_{\sum_{r=1}^{k} r!}$).
The above result could easily be extended to find the formula for the numerousness of terms of the sequence P formed by, at most, "k" digits, just calculating $p_{\sum_h h!}$ taking as "h" the r values of $h(r)\equiv(r+1)*\lfloor\log_{10}r\rfloor+1-\frac{10^{\lfloor\log_{10}r\rfloor+1}-1}{9}$ such that h(r)≤k [9].

---

[14] **Therefore, to estimate the probability that a generic element of the sequence, between the first and the h-th one, ends with the figure k, it will be sufficient to calculate – with the formulae just illustrated – what the extreme values of the interval are in which (with absolute certainty) p(c=k) will be found:**
seeing that $\sum_{t=1}^{r-1} t\leq h\leq\sum_{t=1}^{r} t$, $p_h(c=k)$ will be between the probability associated with r'≡(r-1) and r''≡r. Given that p(r')<p(r'') for k=(1,2,3,4,5) and p(r')>p(r'') for k=(6,7,8,9,0), if 1≤k≤5 the interval will be p(r')≤p(c=k)≤p(r''), otherwise we will have p(r'')≤p(c=k)≤p(r').

[15] **This means that the generic term "p_i" is strictly less than "p_{i+1}", $\forall i \in \aleph\backslash\{0\}$.**

To answer the second problem, we could try to calculate the first primes of this sequence. The result appears quite interesting.
Among the first 1000 terms ($p_i \leq 1325467$), we have to test only these 62 elements:

1243, **<u>1423</u>**, **<u>2143</u>**, **<u>2341</u>**(three consecutive prime terms - for p(j=4) -), 2413, 2431, 3241, 3421, 4123, 4213, **<u>4231</u>**, 4321,
1234567≡o(7), **<u>1234657</u>**, 1235467, 1235647, 1236457, 1236547, 1243567, 1243657, 1245367, 1245637, 1245673, **<u>1245763</u>**, 1246357, **<u>1246537</u>**, **<u>1246573</u>**, 1246753, **<u>1247563</u>**, 1247653, 1253467, 1253647, **<u>1254367</u>**, **<u>1254637</u>**, 1254673, 1254763, **<u>1256347</u>**, 1256437, 1256473, 1256743, **<u>1257463</u>**, 1257643, 1263457, **<u>1263547</u>**, 1264357, **<u>1264537</u>**, **<u>1264573</u>**, 1264753, **<u>1265347</u>**, 1265437, 1265473, 1265743,1267453, 1267543, 1274563, 1274653, 1275463, **<u>1275643</u>**, **<u>1276543</u>**, **<u>1324567</u>**, 1324657, 1325467.

I have underlined only the terms that are prime numbers, so the $\frac{\text{prime terms}}{p_{i(j)}}$ ratio could seem "high" ($\frac{20}{62} \approx 0.3226$), but this is "untrue". Even if this ratio, calculated taking into account the terms between 1234567 and 1325467, is roughly the same ($\frac{16}{50} \approx 0.32$), according to the prime number theorem, we could expect a result close to $\frac{1}{\ln(1234567)} \approx 0.071$ (or $\frac{1}{\ln(1325467)}$) [16].

If we consider that we are not taking into account the even terms, the elements with 5 as the trail digit, plus all the terms divisible by 3, we could conclude that the general rule is also sound to describe the percentage of prime terms in the sequence P. Just keeping attention on my extended criterion (for prime factors above 5) is not applicable to all the permutations of the digits, even if it remains valid for every "rotation" (*circular permutations*) of them (performed by cutting any amount of the initial digits of a term of the consecutive sequence and pasting them at the end of the number [17], to form a new one that still remains divisible by the given factor).

This new sequence, encompassing a lot of well-known integer sequences (like the consecutive one, the reversed, the circular, the left-right, the right-left, etc…). This means that the study of its properties directly involves them and vice versa the known features of these sequences (like the primality of its terms) lest we know more about the consecutive-permutational sequence!
In my opinion, P is the most "normal" sequence dealing with the divisibility criteria, because they are based on the sum-difference of figures and on their own positions. So, the study of P(r) and the divisibility rules result in being strictly connected with each other.

References:

[1] D. A. Alpern, "Factorization using the Elliptic curve method", 2009, http://www.alpertron.com.ar/ECM.HTM

[16] **I've exclude from the "candidate primes" list the terms of o(r), because it's well known that there are no primes among the first terms of this sequence.**

[17] **The procedure is iterative, so you can achieve "m" distinct numbers from a single element of the consecutive sequence composed of "m" digits.**

[2] M. Fleuren, “Smarandache factors and reverse factors”, 1998, http://www.asahi-net.or.jp/~kc2h-msm/mathland/matha1/micha.txt

[3] K. Kashihara, “Comments and topics on Smarandache’s notions and problems”, Erhus Univ. Press, 1996, 25

[4] “Scientia Magna” Vol. 1, No. 2, Nothwest University Xi ‘an, Shaanxi, P. R. China, 2005, 1-2

[5] F. Smarandache, “Only problems, not solutions!”, Xiquan Publ. House, Phoenix-Chicago, 1993 (fourth edition)

[6] R. W. Stephan, "Factors and Primes in Two Smarandache Sequences", Smarandache Notions Journal, Vol. 9, No. 1-2. 1998, 4-10

[7] M. Vassilev-Missana and K. Atanassov, “Some Smarandache problems”, Hexis, 2004

[8] N. N. Vorob'ev's Criteria for Divisibility, University Of Chicago Press, 1980

[9] E. W. Weisstein, "Consecutive number sequences" from MathWorld (a Wolfram web resource)