Quantum Gravity Based on Mach Principle and its Solar Application

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Abstract The starting point of quantum mechanics is the classical algebraic formula connecting energy to momentum: energy is proportional to the squared momentum. As a result, energy and momentum do not be treated equally. The wave equation of quantum mechanics (a differential equation) results from the replacement of the classical energy quantity with the derivative with time and the replacement of the momentum quantity with the derivative with space. Both replacements have a scale factor that is the Planck constant. Similar to the classical formula, the wave equation does not treat time and space equally, and the Planck constant is not canceled out from both sides of the equation. That is, Planck constant remains which describes the microscopic world. My theory of gravity is the local bending of background space-time based on Mach principle which, as suggested by Einstein, is described by a classical form of second order treating time and space equally. Therefore, the Planck constant is completely canceled out in the wave equation. In other words, the quantization of gravity does not need the Planck constant. This is because gravity obeys Equivalence Principle. But I keep the scale factor which describes the hierarchical structure of local universe as suggested by Laurent Nottale.

keywords: Titius–Bode law - Gravitational Theory – Quantization : Hydrogen Atom

1 My Quantum Gravity

Based on Einstein general relativity, gravity has not been quantized. According to my theory of gravity (local bending of flat spacetime), gravity is successfully quantized. This involves two critical ideas. One is why Planck constant exists and the other is where Mach principle goes. I will study the questions and give the answers.

Einstein's theory does not allow gravity to be quantized. The reason is very simple: Any successful quantization is based on a background causal relation. This causal relation is exactly the background reference frame. In other words, the target which is to be quantized must stay in a background, and be independent of the background. All physical theories except Einstein general relativity claim that there is a background inertial reference frame, and as a result, are successfully quantized. However, Einstein rejected inertial reference frames. According to general relativity, gravity is the bending background space-time itself. My theory of gravity is the local bending of globally flat background space-time. In other words, far away from the local mass and energy, space-time is becoming flat. This requires that the large-scale universe be flat as suggested by my flat-universe model [1]. But Einstein general relativity involves no reference frame not to mention the flat reference frames, and his gravity is the causal relation itself. To quantize Einstein gravity is to directly quantize the causal relation. This is, of course, a failure.

My theory of gravity is also curved space-time. But it is the local bending of background flat space-time. In other words, away from the local bending area, space-time is becoming flat and serves the background causal relation. As a result, my theory of gravity can be quantized classically, and the planetary distribution of the solar system (the Titius-Bode law) can be explained.

The starting point of quantum mechanics is the relationship between energy and momentum: energy is proportional to the squared momentum. As a result, energy and momentum have not been treated equally. The wave equation required by quantization is a differential equation. Quantization means that the energy quantity is replaced by the derivative with time while the momentum quantity is replaced by the derivative with space. Both have a common factor which is exactly the Planck constant. Same to the formula of energy and momentum, the resulting differential equation does not treat time and space equally. As a result, the Planck constant is not canceled out from the two sides of the differential equation. That is, Planck constant (the common factor) remains and is the constant which describes the microscopic world.

My gravity is the local bending of background space-time which, as suggested by Einstein, can be described by a differential form of second order which treats time and space equally. Therefore, the Planck constant is completely canceled out in the resulting wave equation. In other words, the quantization of gravity simply does not need the Planck constant. Quantum gravity is even simpler than the familiar quantization of microscopic world: there is no Planck constant (instead we have a scale factor). This is because gravity obeys Equivalence Principle.

2 Mach Principle Provides my Background Spacetime

Everything is dependent on its environment. Human life depends not only on the gravitational field of Earth but also on the background spacetime set up by the Sun. Similarly, the material distribution of the solar system depends not only on the gravitational field of the Sun but also on the background spacetime set up by the local group of stars. Mach Principle says the same thing. For example, when you stand on the ground and relax, your arms fall down naturally. However, if you rotate your body then your arms are lifted up as the rotation is faster and faster. Mach suggested that there must be some influence or force on your hands exerted by the remote objects, e.g. the Sun. Mach principle is that the matter of the whole universe can affect local dynamical systems. That is, the matter of the whole universe sets up the local absolute reference frames.

However, Einstein general theory of relativity is against Mach Principle. How does relativity coexist with the background absolute reference frame? To keep philosophical aesthetics Einstein chose to abandon the real Mach Principle (the terminology Einstein himself devised).

Therefore, my theory of gravity is ready to be quantized classically. But firstly, let me review Laurent Nottale's quantization description of the material distribution in the solar system [2]. His result is based on the theory of scale relativity. My result is the same but directly from the Mach principle and the classical quantum mechanics which most college students know.

3 The Theories of Newton and Einstein can not Explain the Distribution of Planets in the Solar System

As long as a problem involves three or more free bodies, Newton and Einstein's gravitational theories are powerless. The simplest example is the distribution of planets in the solar system. Is the distribution of planets in the solar system orderly?

The solar system is a planar distribution of planets and asteroids. All planets and asteroids move at almost circular orbits and the orbits center at the Sun. The Titius–Bode law is a hypothesis that the planets and asteroids orbit at the exponential series of radii. The law relates the radius, a, of each planet outward from the sun in the unit such that the Earth's radius is 1, and the formula of the law is

$$a = (n + 4)/10$$

where $n = 0, 3, 6, 12, 24, 48...$, and each value of $n(> 3)$ is twice the previous one: $6 = 2 \times 3, 12 = 2 \times 6, 24 = 2 \times 12, 48 = 2 \times 24$, etc. Here are the distances from the Sun of all planets calculated with the formula and their real ones:

Table 1: Radii of Planetary Orbits in the Solar System

Planet	Real radius	T-B law
Mercury	0.39	0.4 (n = 0)
Venus	0.72	0.7 (n = 3)
Earth	1.00	1.0 (n = 6)
Mars	1.52	1.6 (n = 12)
Asteroid belt	2.90	2.8 (n = 24)
Jupiter	5.20	5.2 (n = 48)
Saturn	9.54	10.0 (n = 96)
Uranus	19.18	19.6 (n = 192)
Neptune	30.06	38.8 (n = 384)

From the Table we see that the distribution of planets and asteroids in the solar system is meaningful and can be expressed by simple formula. However, the theories of Newton and Einstein are no more than the theories of free two-bodies, which can not explain the orderly co-existence of many bodies.

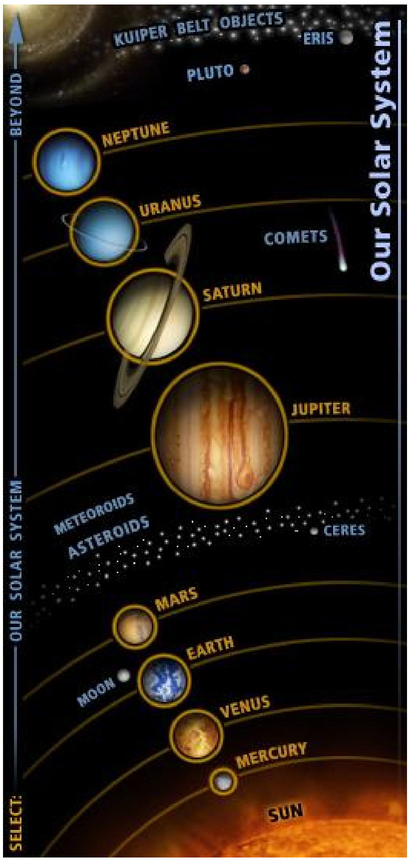

Figure 1: The relative sizes and logarithmic distances of the planets against the Sun in the solar system (image credit [2]).

4 Laurent Nottale's Quantum Explanation of Solar System

Is the solar system an accidental structure? Mainstream astrophysicists' answer is yes because Einstein general relativity can not explain the structure. However, shortly after the quantum wave function of Hydrogen atom was proposed, people found out that the function could be used to explain the structure of the solar system. Of course, the physical parameters contained in the hydrogen atom wave function must be replaced by some parameters describing the macroscopic world. In particular, the Planck constant needs to get rid of.

After Planck proposed the quantum concept in 1900, Bohr proposed in 1914 a concept of discrete energy states of hydrogen atom. Quantum mechanics was established in 1925 to explain the no-orbit motion of microscopic particles with wave functions: the probability wave.

Atomic state is steady state (bound state), and all possible states correspond to a series of wave functions, denoted by a set of integer numbers

$$|n, l>$$

Where n is the energy quantum number and l is the quantum number of angular momentum.

French scientist Laurent Nottale considers the nature not only obeys Einstein's theory of relativity but also the relativity of scales. Nottale used the hydrogen atom wave function to describe the structure of the solar system. Of course, the physical parameters contained in the hydrogen atom wave function must be replaced by some parameters describing the macroscopic world. For example, the Planck constant has been replaced by the scale constant w_0 of scale relativity.

Because the motion of planets in the solar system is approximately the circular one which centers at the Sun, the angular quantum number l takes the maximum value (i.e, $l = n - 1$). Therefore, the remaining quantum number describing the solar system is the energy quantum number n.

According to Nottale,
the quantum number of Mercury is $n = 3$,
the quantum number of Venus is $n = 4$,
the quantum number of Earth is $n = 5$,
the quantum number of Mars is $n = 6$.

In Fig. 2, M stands for Mercury, E for Earth, and so on. Hun, C, Hil, and so on are the asteroids in the asteroid belt. The horizontal axis in Fig. 2 measures the quantum number, and the vertical axis measures the corresponding radii of the planets' orbits (i.e, the distances from the Sun). Nottale suggests that there exist two planets of small masses between the Sun and Mercury corresponding to the quantum number $n = 2$ and $n = 1$ respectively.

Because angular quantum number is $l = n - 1$, each wave function has only one extreme point (maximum point, see Fig. 3). According to Nottale, the solar system is a hierarchical structure. The asteroid belt is between Mars and Jupiter. We note from Fig. 2 that Asteroids and the four planets closest to the Sun have very small masses. According to Nottale's theory of scale relativity, this set of small planets and asteroids forms the first level of the hierarchical structure. Nottale calls it the internal level, denoted by ISS or IS (see Fig. 2). The scale value of the level is w_0.

The solar system has the second level: the external level, denoted as OSS or OS (see Fig. 2). The scale value of the level is $w_0/5$. The scale parameter corresponds to the Planck constant

in microscopic world but varies with the macroscopic levels. The maximum points of the wave functions in the OSS level correspond to the positions of the planets (radii):
the quantum number of ISS is $n = 1$,
the quantum number of Jupiter is $n = 2$,
the quantum number of Saturn is $n = 3$,
the quantum number of Uranus is $n = 4$,
the quantum number of Neptune is $n = 5$,
the quantum number of Pluto is $n = 6$.
Note that the mass center of the first level (i.e, the internal level) is the first "planet" of the second level (i.e, the external level) with the quantum state $n = 1$. This is the hierarchical structure.

The above is the prediction on planets' distances to the Sun. The maximum points correspond to the distances. Because the wave functions of scale relativity are not required to be normalized, the values of the wave amplitudes have the realistic meaning too. They correspond to the masses of the planets (see Fig. 3). Therefore, Nottale's theory has a full account to the solar structure.

Because the sum of amplitudes of all wave functions in the internal level is equal to the amplitude of the first wave function in the external level (see Fig. 3), the mass distribution of the solar system is really a hierarchical structure (see Figures 4 and 5). My quantum gravity has consistently justified Nottale's results.

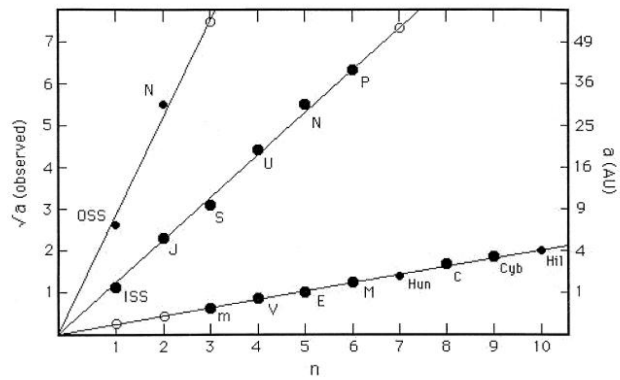

Figure 2: The figure is taken from [3]. Comparison of the observed average distances of planets from the Sun with the theoretical values [3]. On the inner system, one has Mercury (M), Venus (V), the Earth (E), Mars (M), and the main mass peaks of the asteroids belt: Hungarias (Hun), Ceres (C), Hygeia (Hyg) and Hildas (Hil).

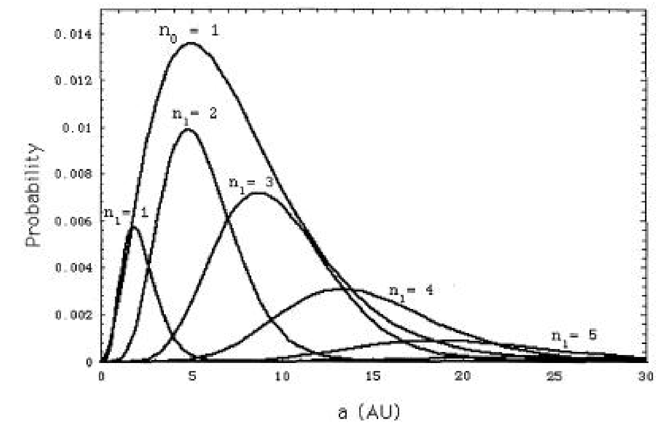

Figure 3: The figure is taken from [3]. Schematic representation of the hierarchical process. The orbital $n_0 = 1$ is divided into sub-orbitals $n_1 = 1$ (inner system), $n_1 = 2$ (Jupiter), $n_1 = 3$ (Saturn). The same is true for the inner system that fragments itself into orbitals $n_2 = 1, 2, 3$ (Mercury), 4 (Venus) ([3]).

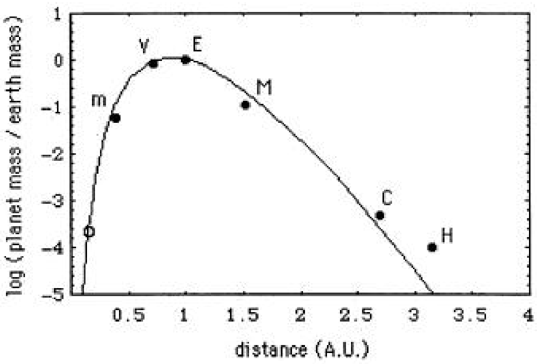

Figure 4: The figure is taken from [3]. Comparison of the predicted and observed masses of planets for the inner solar system. C and H stand for the mass peaks in the asteroid belt (Ceres and Hygeia). The possible additional planet (open circle) is expected to have a mass of about $10^{-4} m_{\rm earth}$ ([3]).

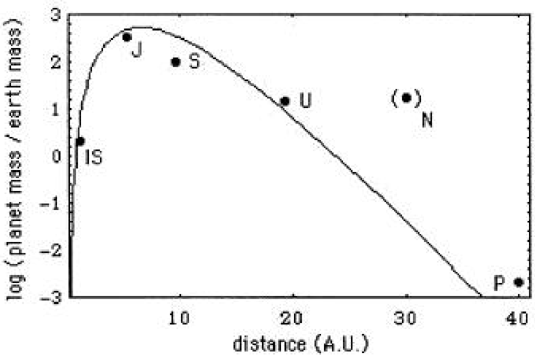

Figure 5: The figure is taken from [3]. Comparison of the predicted and observed masses of planets for the outer solar system. IS stand for the inner system as a whole. Only Neptune is discrepant, which could be the signature of another hierarchic structure at larger scale ([3]).

5 Quantum Gravity of the Sun

Now I introduce my quantum gravity (i.e., the quantum gravitational field of the Sun) which is based on the assumption of local bending of flat spacetime. The local bending of flat spacetime is the modification of general relativity according to Mach Principle.

The general theory of relativity can be spoken of with one sentence: gravity is the curved space-time itself. My theory of gravity is the curved space-time too, but I recognize the Mach principle. I admit that everything is dependent on its material background. That is, I admit the existence of background flat space-time. The Sun with its huge mass and energy warps its flat background. Therefore, my gravity is the local bending of flat background space-time. The space and time on the flat background play the absolute role. But general relativity has no absolute space and time.

It is the mathematician Riemann who first put forward the concept of multi-dimensional bending space and proved his most important theorem: in and only in the flat space does there exist one coordinate system whose coordinates themselves are the global properties of the space: the distances in the space.

You are now aware of the disadvantages of bending spacetime: you can not find a coordinate system in any bending space-time which describes everywhere the global properties: distances or time or angles. The most familiar Cartesian coordinates (distances and time), x, y, z, t, no longer exist in bending spacetime. The coordinates are nothing but mathematical symbols. To calculate distances or time in bending space-time, you have to perform the integration of metrical tensor.

Of course, if confined to the curved space-time itself, you can not find the coordinates which have the direct meaning of distance or time. However, I respect Mach principle and I have the absolute background which provides the absolute Cartesian coordinates of the absolute distance and the absolute time.

I use the external environment of the Sun, i.e., the flat background spacetime to describe the Sun's gravity. The Cartesian coordinates of the external environment are x, y, z, t. Therefore, my theory of gravity, whether it is the Sun's gravitational field or its quantization, is based on the Cartesian coordinates x, y, z, t. That is, *from now on, the coordinates involved are all the background Cartesian coordinates* x, y, z, t! This is against the spirit of Einstein but it is real in this paper as well as in my flat universe model [1]. It is also true in all physical non-gravity theories. The mathematical details are given in the following Section. Here is its summary.

Firstly let us see the Lagrange and Hamilton description of the gravity of a mass M. The formula (13) (see the following Section) is the metric tensor of Einstein curved space-time

$$\begin{aligned} &-\tfrac{1}{2}\grave{s}^2 = L(x^i, x^0, \grave{x}^i, \grave{x}^0) \\ &= \tfrac{1}{2} g_{\alpha\beta}\grave{x}^\alpha \grave{x}^\beta. \end{aligned}$$

However, the coordinates x, y, z, t is the absolute coordinates of the flat background which have the direct meaning of distance and time. You may notice the uppercase letter L and H in the Section. They are the Lagrangian and Hamiltonian of the gravitational field respectively. I use the coordinates of the flat background space-time to describe the curved space-time. Therefore, the use of Lagrangian and Hamiltonian is legal. If the gravitational field is weak, the Lagrangian and Hamiltonian do return to the classical Lagrangian and energy.

The formula (16) is the well-known vacuum solution of Einstein field equation, i.e., the

Schwarzschild metric

$$-\tfrac{1}{2}\grave{s}^2 = -\tfrac{1}{2}(1 - 2r_g/r)c^2\grave{t}^2$$
$$+\tfrac{1}{2}\grave{r}^2/(1 - 2r_g/r) + \tfrac{1}{2}r^2(\grave{\theta}^2 + \sin^2\theta\grave{\phi}^2).$$

Its only parameter is r_g in (15)

$$r_g(= GM/c^2)$$

This parameter is called the Schwarzschild radius. For the Earth, this radius is equal to about 0.001 meters, or the magnitude of millimeters. Because the Earth's physical radius is 6.378 million meters and humans live on the Earth radius, the Schwarzschild radius divided by the Earth's radius is about 0.000,000,000,01. Therefore, the above formula is approximately the metric of flat space-time, i.e., the indefinite Euclidean metric. In other words, the bending degree of the space-time where humans live is 0.000,000,000,01.

From now on, we divide the Schwarzschild radius of the involved mass (e.g., the Earth or the Sun) by the physical radius of the mass, and the result is called the bending parameter of curved spacetime. Because the bending parameters of Earth and Sun are both very small, current astronomical observations can only confirm Einstein theory in its first-order approximation with respect to the parameter. Therefore, any declaration that Einstein general theory of relativity is fully proved is not true. If you provide a theory and its first-order approximation shares the same result as general relativity then you are as great as Einstein.

All my calculation is not based on the Schwarzschild metric. It is based on the quantum-solvable metric (14). Its first-order prediction is the same as Schwarzschild metric but the prediction of time delay effect with respect to Mercury is 260 microseconds while the prediction of Schwarzschild metric is 240 microseconds.

Why should I find a quantum-solvable metric? The reason is that curved space-time is not a simple effect. It is a highly non-linear effect. Non-linear differential equations generally have no analytic solutions. Einstein field equation is highly nonlinear. Currently only three or four solutions are found. I want to quantize gravity and I need analytic solution. That is why I provide the quantum-solvable metric. The following quantization of solar gravity is based on the metric.

Because the coordinates in the metric are the Cartesian coordinates on flat background, we can follow the common way of classical mechanics to calculate the corresponding Hamiltonian. Then we follow the classical method of quantum mechanics to calculate the canonical coordinates and the corresponding canonical momentum. Replacing the canonical coordinate and momentum with the corresponding operators, we get the wave equation of quantum gravity.

The formulas (22) through (26) are the general quantization description for the quantum-solvable metric. The symbol with the "goat's horn"

$$\breve{h}$$

is the proportion constant corresponding to the Planck constant. However, the space-time is covariant and the quantities of time and space have been treated equally. The proportion constant cancels out completely from the wave equation of quantum gravity. This means that quantum gravity has no need of the Planck constant. Quantum gravity is simpler than the quantum mechanics of microscopic world. However, this constant is taken to be the scale parameter w_0 of Nottale's scale relativity.

The formulas (36) through (42) are the result of the general formula applied to the gravity of the Sun. The resulting quantum wave function turns out to be the wave function of hydrogen atom with the Planck constant being replaced by the scale parameter w_0. This confirms the prediction of more than 80 years ago that solar planetary distribution obeys the quantum states of hydrogen atom. This is absolutely not a coincidence.

6 Mathematical Details: Quantum Gravity of the Sun

6.1 Covariant Description of Vanishing Gravity

(i) Lagrangian.

This paper deals with gravitational interaction only, no other interaction being involved. Newton's first law of motion that a particle in a vanishing gravitational field must move in straight direction with constant (or zero) velocity with respect to any inertial frame $txyz$, can be described by the language of relativity by introducing Minkowski metric $\eta_{\alpha\beta}$ and proper distance s to the frames,

$$\begin{aligned} -\tfrac{1}{2}\grave{s}^2 &= L(x^i, x^0, \grave{x}^i, \grave{x}^0) \\ &= \tfrac{1}{2}(-c^2\grave{t}^2 + \grave{x}^2 + \grave{y}^2 + \grave{z}^2) \\ &= \tfrac{1}{2}\eta_{\alpha\beta}\grave{x}^\alpha\grave{x}^\beta \end{aligned} \tag{1}$$

where

$$\grave{x}^i = \frac{dx^i}{dp}, \text{ etc.,} \tag{2}$$

p is the curve parameter in the 4-dimentional flat spacetime which can be chosen $\propto s$ (an invariant parameter, or $\propto \bar{s}$, another invariant parameter in the following), $x^0 = ct, x^1 = x, x^2 = y, x^3 = z$, c is light speed, and $\eta_{00} = -1, \eta_{11} = \eta_{22} = \eta_{33} = 1, \eta_{\alpha\beta} = 0 (\alpha \neq \beta)$. Note that, from now on, all letters with grave accent (e. g., $\grave{x}$) denotes the derivative of the quantity (e. g. x) with the curve parameter p. This is to be distinguished from the dot accent (e. g., $\dot{x}$) which is the derivative of the quantity (e. g. x) with time t (a common notation). The array $\eta_{\alpha\beta}$ is the Minkowski metric which is the basis of special relativity. I call the distance s along the curves of spacetime by real distance. The distance is generally called proper distance which can be negative because the matrix $\eta_{\alpha\beta}$ is indefinite. The indefinite quadratic form (1) is the generalization of Pythagoras theorem to Minkowski spactime.

It is straightforward to show that the first Newton law of motion (vanishing gravity) is the result of variation principle applied to the Lagrangian L (the formula (1)). That is, I need to prove that the straight motion with constant velocity from p_A to p_B is such that the line integral

$$I = \int_{p_A}^{p_B} L dp, \tag{3}$$

is an extremum for the path of motion. The resulting Lagrange's equation is

$$\frac{d}{dp}\left(\frac{\partial L}{\partial \grave{x}^\alpha}\right) - \frac{\partial L}{\partial x^\alpha} = \pm\frac{d^2x^\alpha}{dp^2} = 0. \tag{4}$$

This is exactly the straight motion with constant velocity,

$$\begin{aligned} &\tfrac{dx^0}{dp} = \text{constant}, \quad \tfrac{dx^i}{dp} = \text{constant}, \\ &i = 1, 2, 3. \end{aligned} \tag{5}$$

We can choose $dt/dp = 1$. Then

$$\frac{dx^i}{dt} = \text{constant} = v^i, \; i = 1, 2, 3. \tag{6}$$

I have recovered the first Newton's law by introducing the Lagrangian and Minkowski metric (1). In fact, choosing a new Lagrangian which is any monotonous function of the original Lagrangian results the same Lagrange's equation. For example, choose the Lagrangian

$$L_1(x^i, x^0, \grave{x}^i, \grave{x}^0) = \sqrt{-2L} = \grave{s}. \tag{7}$$

The resulting Lagrange's equation is the same formula (5). The resulting line integral,

$$\int_{p_A}^{p_B} L_1 dp = s, \tag{8}$$

is the proper distance along the line. Therefore, the Lagrange's equation describes the path of motion which has shortest proper distance between the two spacetime points corresponding to parameters p_A and p_B respectively.

(ii) Hamiltonian. For quantization according to common procedure, however, we need a Lerendre transformation to transform the Lagrangian to the Hamiltonian which represents classical total energy. In the present section we deal with the motion of free particle (vanishing gravity). Now we derive the Hamiltonian of unit mass based on the Lagrangian (1). The canonical momentums to $x^\alpha, \alpha = 0, 1, 2, 3$ are the following,

$$\begin{array}{l} P_0 = \frac{\partial}{\partial \grave{x}^0} L = -\frac{cdt}{dp} \\ P_i = \frac{\partial}{\partial \grave{x}^i} L = \frac{dx^i}{dp}, \;\; i = 1, 2, 3 \end{array} \tag{9}$$

Therefore, the Hamiltonian of the free particle is

$$\begin{array}{ll} H & = \grave{x}^0 P_0 + \grave{x}^i P_i - L \\ & = \frac{1}{2}\sum_{i=1}^{3}\left(\frac{dx^i}{dp}\right)^2 - \frac{1}{2}c^2\left(\frac{dt}{dp}\right)^2 \\ & = \frac{1}{2}\sum_{i=1}^{3} P_i^2 - \frac{1}{2}P_0^2. \end{array} \tag{10}$$

If we choose

$$\frac{dt}{dp} = 1 \tag{11}$$

then $P_0^2 = c^2$ and finally the Hamiltonian (total energy) is

$$H = \frac{1}{2}\sum_{i=1}^{3}\left(\frac{dx^i}{dt}\right)^2 - c^2. \tag{12}$$

We see that the spatial part of the Hamiltonian corresponds to kinetic energy while the temporal part corresponds to potential energy. Both energies are constants. The potential energy is $-c^2$ which is chosen to be zero in non-covariant theory.

6.2 Einstein's Metric Form Considered to be the Lagrangian on Flat Spacetime and a Solvable Lagrangian

(i) Geometrization of Gravity (GR). It is more important to consider test particle's motion in an inertial frame in which the particle does experience gravitational force. In the frame, the particle no longer moves in straight direction with a constant (or zero) velocity. The motion is described in good approximation by the Newton's universal law of gravitation which is, however, a non-relativistic theory and needs to be generalized to give account for the solar observations which deviate from Newton laws' calculation. Einstein's general relativity (GR) is the most important try. The basic assumption of GR is that gravity is spacetime itself but the spacetime is curved. Therefore, the gravity of GR is the simple replacement of the above matrix $\eta_{\alpha\beta}$ by a tensor field $g_{\alpha\beta}$ whose components are position functions on spacetime, instead of the constants ± 1,

$$\begin{aligned} -\tfrac{1}{2}\grave{s}^2 &= L(x^i, x^0, \grave{x}^i, \grave{x}^0) \\ &= \tfrac{1}{2} g_{\alpha\beta}\grave{x}^\alpha \grave{x}^\beta. \end{aligned} \tag{13}$$

Similar to the description in the case of vanishing gravity, the motion of the particle from p_A to p_B in the gravitational field is such that the line integral (3) is an extremum for the path of motion. Therefore, the equation of motion is the corresponding Lagrange's equation which is exactly the known geodesic equation given in the general theory of relativity (GR). However, when we call the Lagrange's equation by geodesic equation, we mean that the spacetime is curved and the real distance on the curved spacetime is $\sqrt{-2L} = s$. This kind of explanation of the Lagrangian is called Einstein's geometrization of gravity and the coordinates x, y, z, t has no direct meaning of spatial distance or time interval. The tensor field $g_{\alpha\beta}$ in (13) is called the metric of the curved spacetime.

My theory of gravity is the curved space-time too, but I recognize the Mach principle. I admit that everything is dependent on its material background. That is, I admit the existence of background flat space-time. The Sun with its huge mass and energy warps its flat background. Therefore, my gravity is the local bending of flat background space-time. The space and time on the flat background play the absolute role. Therefore, all coordinates in the following formulas have the absolute meaning: they are the Cartesian or polar coordinates of the flat background spacetime. Therefore, all the formulas have the absolute meaning and the use of Lagrangian and Hamiltonian is legal. If the gravitational field is weak, the Lagrangian and Hamiltonian do return to the classical Lagrangian and energy.

(ii) A solvable Lagrangian. Now I give you the quantum-solvable metric:

$$\begin{aligned} -\tfrac{1}{2}\grave{s}^2 &= L(x^i, x^0, \grave{x}^i, \grave{x}^0) \\ &= -\tfrac{1}{2}(\grave{x}^0)^2/D(r) + \tfrac{1}{2}D(r)\textstyle\sum_{i=1}^3 (\grave{x}^i)^2 \\ &= -\tfrac{1}{2}c^2\grave{t}^2/D(r) \\ &\quad +\tfrac{1}{2}D(r)(\grave{r}^2 + r^2(\grave{\theta}^2 + \sin^2\theta\grave{\phi}^2)) \end{aligned} \tag{14}$$

where

$$D(r) = 1 + 2r_g/r, \tag{15}$$

$r_g (= GM/c^2)$ is the Schwarzschild radius, M is the central point mass, and G is the gravitational constant. The Lagrandian is called solvable Lagrangian because its quantization admits exact analytic solution. Note that curved space-time is not a simple effect. It is a highly non-linear effect. Non-linear differential equations generally have no analytic solutions. Einstein field equation is highly nonlinear. Currently only three or four solutions are found. I want to quantize gravity and I need analytic solution. That is why I provided the quantum-solvable metric. The following quantization of solar gravity is based on the metric.

The Lagrangian is a little different from the Schwarzschild metric form:

$$\begin{aligned} -\tfrac{1}{2}\grave{s}^2 &= -\tfrac{1}{2}(1 - 2r_g/r)c^2\grave{t}^2 \\ &\quad +\tfrac{1}{2}\grave{r}^2/(1 - 2r_g/r) + \tfrac{1}{2}r^2(\grave{\theta}^2 + \sin^2\theta\grave{\phi}^2). \end{aligned} \tag{16}$$

For solar system, $2r_g/r \ll 1$ and only the first-order approximations (in $2r_g/r$) of the metric coefficients are testified. Therefore, we are interested in the approximations. These approximations of the Schwarzschild coefficients are $-1 \times 2r_g/r$ to the term $-\frac{1}{2}c^2\grave{t}^2$, $1 \times 2r_g/r$ to the term

$\frac{1}{2}\grave{r}^2$, and $0 \times 2r_g/r$ to the term $\frac{1}{2}r^2(\grave{\theta}^2 + \sin^2\theta\grave{\phi}^2)$. Simply say, the first-order approximations are (-1, +1, 0). The same approximations for the solvable Lagrangian coefficients are $(-1, +1, j)$ where $j = +1$. Therefore, only the approximation to the third term is different between the Schwarzschild form and the solvable Lagrangian form ($j = 0$ for the Schwarzschild form).

(iii) Integration of the Lagrange's equations of the solvable Lagrangian. Due to the spatially isotopic metric, the test particle's spatial motions are planar. Therefore, we can choose $\theta = \pi/2$ in (14). Particle's motion follows the corresponding Lagrange's equation. The integration of the temporal Lagrange's equation is

$$\frac{dt}{dp} = aD(r) \tag{17}$$

where a is an arbitrary constant. From now on we choose $a = 1$,

$$\frac{dt}{dp} = D(r). \tag{18}$$

The integration of the Lagrange's equation which corresponds to ϕ is

$$D(r)r^2\frac{d\phi}{dp} = J. \tag{19}$$

Furthermore, we have the integration of the Lagrange's equation which corresponds to radius r, with the help of the other solutions,

$$D(r)\left(\frac{dr}{dp}\right)^2 + \frac{J^2}{r^2D(r)} - D(r)c^2 = E \text{ (constant)}. \tag{20}$$

(iv) The first-order solar predictions of the solvable Lagrangian. While geometrization requires spacetime curvature to determine gravitational metric (i.e., the Einstein field equation), the Lagrangian on flat-spacetime has no such constraint. Without the constraint, we are free to see which kinds of Lagrangians give the similar predictions as Schwarzschild metric. Because all solar tests of the Schwarzschild metric are made on its first order approximation in $2r_g/r$, we consider the above-said first-order approximations (f, g, j). My paper [4] shows that any diagonal effective metric whose first-order approximation is $(-1, +1, j)$, where j is any real numbers, has the same predictions as Schwarschild metric, on the deflection of light by the sun and the precession of perihelia.

However, the prediction on the excess radar echo delay depends on the choice of j. The prediction on the maximum excess delay of the round-trip to Mercury when it is at superior conjunction with respect to Sun is

$$(\Delta t)_{\max} \simeq 19.7(1 + j + 11.2)\,\mu\,\text{sec}. \tag{21}$$

Our solvable Lagrangian corresponds to $j = 1$ and its prediction is $(\Delta t)_{\max} \simeq 260\mu\,$sec while the corresponding result of GR ($j = 0$) is 240 $\mu\,$sec. The difference is less than 8 percents. However, there is difficulty in the test. We can transmit radar signals to Mercury at its series of orbital positions around the event of superior conjunction. The time for single round-trip is many minutes and an accuracy of the order of 0.1 $\mu\,$sec can be achieved [5]. In order to compute an excess time delay, we have to know the time t_0 that the radar signal would have taken in the absence of the sun's gravitation to that accuracy. This accuracy of time corresponds to an accuracy of 15 meters in distance. This presents the fundamental difficulty in the above test. In order to have a theoretical value of t_0, Shapiro's group proposed to use GR itself to calculate the orbits of Mercury as well as the earth[6-8]. The data of time for the above series of real round-trips minus the corresponding theoretical values of t_0 presents a pattern of excess time delay against observational date and was fitted to the theoretical calculations with a fitting parameter γ. The group and the following researchers declared that, among other similar geometrical theories of gravity represented by γ, GR fit the pattern best.

6.3 Hamiltonian of the Solvable Lagrangian and its Quantization

To quantize gravity according to common procedure, however, we need a Lerendre transformation to transform the solvable Lagrangian to the Hamiltonian which represents the total energy per unit mass. The canonical momentums to $x^\alpha, \alpha = 0, 1, 2, 3$ are the following,

$$\begin{array}{l} P_0 = \frac{\partial}{\partial \grave{x}^0} L = -\frac{cdt}{dp} / D(r) \\ P_i = \frac{\partial}{\partial \grave{x}^i} L = D(r) \frac{dx^i}{dp}, \ i = 1, 2, 3 \end{array} \tag{22}$$

Therefore, the Hamiltonian of the system is

$$\begin{array}{l} H = \grave{x}^0 P_0 + \grave{x}^i P_i - L \\ = -\frac{1}{2} c^2 \left(\frac{dt}{dp}\right)^2 / D(r) + \frac{1}{2} \sum_{i=1}^3 D(r) \left(\frac{dx^i}{dp}\right)^2 \\ = -\frac{1}{2} D(r) P_0^2 + \frac{1}{2} \sum_{i=1}^3 P_i^2 / D(r). \end{array} \tag{23}$$

We see that the Hamiltonian H equals to the solvable Lagrangian L, $H = L$. We also see that the temporal part of the Hamiltonian is potential energy while the spatial part is kinetic energy.

Application of the common procedure,

$$\begin{array}{l} H \to i \breve{h} \frac{\partial}{\partial p} \\ P^0 \to i \breve{h} \frac{\partial}{\partial \tilde{t}} \\ P^i \to \ - i \breve{h} \nabla, \end{array} \tag{24}$$

produces the quantization of the system, where

$$\tilde{t} = ct = x^0. \tag{25}$$

The resulting wave equation is

$$\begin{array}{l} i \breve{h} \frac{\partial}{\partial p} \Psi(x, y, z, \tilde{t}, p) = \\ \frac{\breve{h}^2 D(r)}{2} \frac{\partial^2}{\partial \tilde{t}^2} \Psi - \frac{\breve{h}^2}{2D(r)} \nabla^2 \Psi. \end{array} \tag{26}$$

We consider only the eigenstates of energy,

$$\Psi(x, y, z, \tilde{t}, p) = \Phi(x, y, z, \tilde{t}) e^{-iEp}. \tag{27}$$

The eigenstates $\Phi(x, y, z, \tilde{t})$ satisfy the equation

$$\begin{array}{l} \frac{2E}{-\breve{h}} \Phi(x, y, z, \tilde{t}) = \\ -D(r) \frac{\partial^2}{\partial \tilde{t}^2} \Phi + \frac{1}{D(r)} \nabla^2 \Phi. \end{array} \tag{28}$$

I absorb the quantization number $\breve{h}$ into the energy constant E. We can not measure the energy of solar system which depends on the specific theory used. The observable quantities are the planetary distances and orbital motions. We will not distinguish energy distributions with constant factors and constant terms. We denote $2E/\breve{h}$ by $\tilde{E}$. Our wave equation is

$$\begin{array}{l} -\tilde{E}\Phi(r,\theta,\phi,\tilde{t}) = -D(r)\frac{\partial^2}{\partial\tilde{t}^2}\Phi \\ +\frac{1}{D(r)}\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} - \frac{l(l+1)}{r^2}\right)\Phi. \end{array} \tag{29}$$

We consider only the eigenstates of temporal operator,

$$\Phi(r,\theta,\phi,\tilde{t}) = \Phi(r,\theta,\phi)e^{-i\omega\tilde{t}}. \tag{30}$$

The eigenstates $\Phi(r,\theta,\phi)$ satisfy the equation

$$\begin{array}{l} -\tilde{E}\Phi(r,\theta,\phi) = \omega^2 D(r)\Phi \\ +\frac{1}{D(r)}\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} - \frac{l(l+1)}{r^2}\right)\Phi. \end{array} \tag{31}$$

The solution of the equation is

$$\begin{array}{l} \Phi(r,\theta,\phi) = \chi_{nl}Y_{lm}(\theta,\phi)/r, \\ \chi_{nl}(r) = \mathrm{WhittackerM}(\frac{r_g(2\omega^2+\tilde{E})}{\sqrt{-\tilde{E}-\omega^2}}, \\ \frac{1}{2}\sqrt{(2l+1)^2 - 16\omega^2 r_g^2}, 2\sqrt{-\tilde{E}-\omega^2}r) \end{array} \tag{32}$$

The Whittacker function $\mathrm{WhittackerM}(\kappa,\mu,z)$ is connected to Kummer's function, i. e. confluent hypergeometric function $_1F_1(\alpha,\gamma,z)$, by the following formula

$$\begin{array}{l} \mathrm{WhittackerM}(\kappa,\mu,z) = \\ e^{-z/2}z^{\mu+1/2}\ {}_1F_1(\mu+1/2-\kappa, 1+2\mu, z). \end{array} \tag{33}$$

For the wave function to be finite at infinity $r = +\infty$, the confluent function must be a polynomial, that is,

$$\begin{array}{l} \frac{1}{2}\sqrt{(2l+1)^2 - 16\omega^2 r_g^2} + \frac{1}{2} \\ -\frac{r_g(2\omega^2+\tilde{E})}{\sqrt{-\tilde{E}-\omega^2}} = -n_r \end{array} \tag{34}$$

where n_r is any non-negative integer. This is the quantization condition for our system of macrophysics. The solution of the condition is

$$\begin{array}{l} \sqrt{-\tilde{E}-\omega^2} = \\ \frac{1}{2r_g}(-\left(n_r + \frac{1}{2} + \frac{1}{2}\sqrt{(2l+1)^2 - 16\omega^2 r_g^2}\right) \\ +\sqrt{4\omega^2 r_g^2 + \left(n_r + \frac{1}{2} + \frac{1}{2}\sqrt{(2l+1)^2 - 16\omega^2 r_g^2}\right)^2}). \end{array} \tag{35}$$

Note that our wave function depends on the spatial quantum numbers n_r, l, temporal quantum number ω, and the gravitational strength r_g. The quantization number $\breve{h}$ is not involved, because our Lagrangian is homogeneous of the generalized velocity components, dx^α/dp (see (13) and (29)).

6.4 Application to Solar System

(i) Energy levels in the first order approximation. From equation (32), the peaks of probability density start at the "Bohr radius"

$$a_0 \propto 1/(2\sqrt{-\tilde{E} - \omega^2}). \tag{36}$$

The closer planets to the Sun, e. g. Mercury, have approximately such a radius. Because $2l + 1 \geq 1$ and $r_g \approx 1.5 \times 10^3\,\mathrm{m}$, the quantity $\omega^2 r_g^2$ must be very small, $\omega^2 r_g^2 \ll 1$, so that the reciprocal of (34) approaches the required radius. This requirement leads to

$$\sqrt{-\tilde{E} - \omega^2} = \frac{\omega^2 r_g}{4(n_r + l + 1)} \tag{37}$$

in the first order approximation of $\omega^2 r_g^2$. This is the energy level formula for solar system. As usual, we use the quantum number $n = n_r + l + 1$,

$$\begin{array}{l} n = n_r + l + 1, \\ \tilde{E} + \omega^2 = -\frac{\omega^4 r_g^2}{16 n^2} = -\frac{1}{a_\omega^2 n^2} \end{array} \tag{38}$$

where we have the definition of the "Bohr radius"

$$a_\omega = \frac{4}{\omega^2 r_g}. \tag{39}$$

Note that real Bohr radius depends on Planck constant and does not depend on any quantum number. However, the "Bohr radius" defined here depends on the temporal quantum number ω and the gravitational strength r_g. The quantization number $\breve{h}$ is not involved. In the mean time, the WhittackerM function reduces to the radial wave function of Hydrogen atom,

$$\begin{array}{l} \mathrm{WhittackerM}(\kappa, \mu, z) = \\ e^{-\frac{r}{n a_\omega}} r^{l+1} \; {}_1F_1(-n + l + 1, 2l + 2, \frac{r}{a_\omega}) \\ \propto r R_{nl}(r). \end{array} \tag{40}$$

(ii) Fitting the planetary distances. By using the wave function of Hydrogen atom, Nottale, Schumacher, and Gay [2] gave an excellent fitting of the distributions of planetary distances and planetary masses (See the above Section 4). The distances of the inner planets (Mercury, Venus, the Earth, Mars, and the main mass peaks of the asteroids belt) can be fitted by the formula

$$a_{nl} = \left(\frac{3}{2} n^2 - \frac{1}{2} l(l+1)\right) a_\omega \tag{41}$$

where $l = n - 1$. The probability density peaks of Hydrogen atom wave functions do identify with the formula if we choose

$$\begin{array}{l} \omega_1 = 6 \times 10^{-7}\,\mathrm{m}^{-1} \\ a_{\omega_1} = 6.3 \times 10^9\,\mathrm{m} \end{array} \tag{42}$$

As revealed in Nottale, Schumacher, and Gay [2], the distances of the outer planets (Jupiter, Saturn, Uranus, Neptune, Pluto) and the one of the center mass of the inner planetary system can be fitted by the formula too if we choose

$$\begin{array}{l} \omega_2 = \sqrt{5}\omega_1 \\ a_{\omega_2} = a_{\omega_1}/5 \end{array} \tag{43}$$

This constitutes the hierarchical explanation of the quantized planetary distances. Because the coefficients of our solvable Lagrangian (14) do not involve time (static description), the temporal operator has continual quantum number ω (see (30)). However, the formulas (42) and (43) indicate that ω is quantized too. The hierarchical structure of solar system is consistently explained by the quantized values of ω of the temporal operator. In the solar quantization theory of Nottale, Schumacher, and Gay [2], however, not only the wave function but also the "Bohr radius" involve a so-called universal quantization constant w_0 and the hierarchical structure has to be an additional assumption. Nottale, Schumacher, and Gay [2] also gave an excellent fragmentation explanation of solar planetary masses based on the same wave function (See the above Section 4).

References

[1] He, J. 2006, http://www.arxiv.org/abs/astro-ph/0605213

[2] Nottale, L., Schumacher, G., and Gay, J.: 1997,*A&A*, 322, 1018

[3] http://solarsystem.nasa.gov/planets/index.cfm

[4] He, J. 2005, http://www.arxiv.org/abs/astro-ph/0512614

[5] Anderson, J. D. *et al.*: 1975, *ApJ*, **200**, 221

[6] Shapiro, I. I.: 1964, *Phys. Rev. Lett.*, **13**, 789

[7] Shapiro, I. I. *et al.*: 1968, *Phys. Rev. Lett.*, **20**, 1265

[8] Shapiro, I. I. *et al.*: 1971, *Phys. Rev. Lett.*, **26**, 1132