REMARKS ON THE FUNCTION $\eta(n)$

BY

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In 1980, F. SMARANDACHE introduced (see [5]) the function $\eta : \mathbb{Z}^* \rightarrow \mathbb{N}$, defined by $\eta(n) = m$, where $m$ is the smallest natural number with the property that $m!$ is divisible by $n$. This function has aroused the interest of several mathematicians because of the simplicity of the definition, because of its interesting properties, important applications as well as a simple algorithm for the computation of its values. This function is now known as Smarandache’s function.

First of all, the $\eta$ function provides us with extremely simple answers to two fundamental problems in the number theory:

1. It helps us formulate a primeness criterion: a number $p \in \mathbb{N}\{0, 1, 4\}$ is prime if and only if it is a fixed point of $\eta$ (i.e., $\eta(p) = p$).

2. Only one formula was known (established by W. Sierpinski in 1953) for the function $\pi : \mathbb{R}_+^* \rightarrow \mathbb{N}$, $\pi(x) =$ the number of prime natural numbers which are less or equal to $x$.

A new formula has been derived from Smarandache’s function, that is:

$$
\pi(x) = \sum_{k=2}^{\lfloor x \rfloor} \eta(k) - 1. \text{ (Here } \pi(x)+1 \text{ means the summation function of } \left\lfloor \frac{\eta(n)}{n} \right\rfloor \text{ in the extended sense and } \lfloor x \rfloor \text{ stands for the greatest integer which is less or equal to the real number } x. \text{)}
$$

Let us remark that the $\eta$ function is uniquely determined by its restriction to the set $\mathbb{N}^*$, because $\eta(-n) = \eta(n)$.

We redefine $\eta(1) = 1$ (according to Smarandache’s definition, $\eta(1) = 0$) hence $\eta : \mathbb{N}^* \rightarrow \mathbb{N}^*$ and $\eta$ becomes invertible with respect to the Dirichlet product.
In [6] and [7] some open problems referring to the $\eta$ function are presented. Here is an interesting problem:

Consider the sequence

$$\eta(1) = 1, \quad \eta(2) = 2, \quad \eta(3) = 3, \quad \eta(4) = 4, \quad \eta(5) = 5, \quad \eta(6) = 3, \quad \eta(7) = 7, \quad \eta(8) = 4, \quad \eta(9) = 6, \quad \eta(10) = 5, \quad \eta(11) = 11, \quad \eta(12) = 4, \ldots$$

and let $\alpha$ be the following decimal number (actually, the reasoning is valid for every basis $g \geq 2$)

$$\alpha = 0.\eta(1)\eta(2)\eta(3)\ldots\eta(n)\ldots$$

that is

$$\alpha = 0.123453746511413\ldots$$

We ask ourselves whether the number $\alpha$ is irrational. Using the following result due to W. Sierpinski [4] we can solve this problem:

**Theorem 1.** For every $m \in \mathbb{N}^*$ and every ciphers $c_1, c_2, \ldots, c_m$ in the decimal basis, $c_i \neq 0$, there exists an infinite number of primes which, written in the decimal basis, have as their first ciphers $c_1, c_2, \ldots, c_m$ (in this order).

**Theorem 2.** The number $\alpha$ given by (1) is irrational.

**Proof.** Since $\eta(p) = p$ for every prime number $p$, after the point of $\alpha$ given by (1) there is a sequence of ciphers which represents the number $p$ written in the decimal basis. For different prime numbers, the corresponding sequences are disjoint. Now, for every natural number $n, n \geq 1$, we consider $m = n + 2$ and take the sequence of ciphers $c_1, c_2, \ldots, c_n, c_{n+1}, c_{n+2}$, where $c_2 = c_3 = \cdots = c_{n+1} = 1$, $c_1 \neq 0$, $c_1 \neq 1$, $c_{n+1} \neq 1$. This sequence appears after the point of (1), even for infinitely many times (according to Theorem 1). It follows that $\alpha$ can be represented as a decimal fraction with an infinite number of significant ciphers which, according to the remark above, is not periodic. Therefore, the number $\alpha$ given by (1) cannot be rational.

Computing $\eta(n)$ for even greater values of $n$, we can notice a very irregular repartition of the values of $\eta$.

We remark that for every $n, \eta(n) \leq n$, and every $m \in \mathbb{N}^*$ is a value of $\eta(\eta(m!)) = m$. Every $m \in \mathbb{N}^*, m \geq 3$, appears several times, but a finite numbers of times, because $\eta$ is generally nondecreasing function: for every $a \in \mathbb{N}^*$ there exists $b \in \mathbb{N}^*$ so that for every $c \in \mathbb{N}^*, c \geq b$, we have
η(c) > η(a). For every positive real number M, there exist prime numbers p satisfying p = η(p) > M. It follows that

$$\lim_{n \to \infty} \frac{\eta(n)}{n} = +\infty.$$ 

Now, let us formulate the following result referring to the Gauss–Dirichlet mean:

**Theorem 3.** \(\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \eta(k) = +\infty.\)

The next three propositions show the nonregularity of the distribution of the values of η.

**Proposition 1.** There exist subsequences \((n_k)_{k \in \mathbb{N}^*}\) of \(\mathbb{N}^*\) so that

$$\lim_{k \to \infty} \frac{\eta(n_k)}{n_k} = 0.$$ 

**Proof.** We take \(n_k = k!\). Since \(\eta(k!) = k\), we have

$$\lim_{k \to \infty} \frac{\eta(n_k)}{n_k} = \lim_{k \to \infty} \frac{1}{(k-1)!} = 0.$$ 

**Proposition 2.** There exist subsequences \((n_k)_{k \in \mathbb{N}^*}\) of \(\mathbb{N}^*\) so that

$$\lim_{k \to \infty} \frac{\eta(n_k)}{n_k} = 1.$$ 

**Proof.** It suffices to consider \((n_k)_{k \in \mathbb{N}^*}\) as the sequence of prime natural numbers, because in this case \(\eta(n_k) = n_k\).

**Proposition 3.** For every \(m \in \mathbb{N}^*\), there exists a subsequence \((n_k)_{k \in \mathbb{N}^*}\) of \(\mathbb{N}^*\) so that

$$\lim_{k \to \infty} \frac{\eta(n_k)}{n_k} = \frac{1}{m}.$$ 

**Proof.** For \(m = 1\) the result is true (see Proposition 2). Let \(m > 1\) with the canonical decomposition \(m = p_1^{\alpha_1}p_2^{\alpha_2}...p_s^{\alpha_s}\). There exist infinitely many prime numbers p,

(2) \[ p > \max_{1 \leq i \leq s} p_i^{\alpha_i} \]
We consider the sequence \((n_p)_p, n_p = m \cdot p\), where \(p\) ranges the set of prime numbers which satisfies (2). We have \(\eta(n_p) = p\), hence

\[
\lim_{p \to \infty} \frac{\eta(n_p)}{n_p} = \lim_{p \to \infty} \frac{p}{m \cdot p} = \frac{1}{m}.
\]

Although there exists a simple algorithm, based on the decomposition of natural numbers in prime factors (which allows the generation of values of \(\eta\) on the computer), we have no formula to give \(\eta(n)\) by means of the prime factors of \(n\). The \(\eta\) function is neither multiplicative nor additive. We cannot put the summation function of \(\eta\) in a convenient form. Also, we cannot put in a convenient form neither the function whose the summation function is \(\eta\) nor the inverse function of \(\eta\) in the Dirichlet sense.

In order to study \(\eta(n)\) in an easier manner, it would be useful for us to find an invertible arithmetic function so that the Dirichlet product of \(\eta\) and this function should be known. This would allow the determination of the generating function and the Dirichlet series of \(\eta\) as well as the obtaining of some interesting identities, referring to the \(\eta\) function.

REFERENCES