Indeterminacy in arithmetic, well-known to logicians but missing from quantum theory

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Abstract. This article is one of a series explaining the nature of mathematical undecidability discovered within quantum theory. Crucially, a formula's undecidability certifies its indeterminacy and vice versa. This paper describes the algebraic environment in which the undecidability and indeterminacy originate, provides proof of their existence, and demonstrates the role these play in a 3-valued logic which is free to permeate mathematical physics via this algebra.

The radical ideas applied in this research are taken from well-known results in mathematical logic. All scalars engage in the arithmetic of scalars by way of a single algebra. But in terms of validity, these scalars partition into sets which are logically distinct: those with valid existence with respect to this algebra, and those with indeterminate existence. Failure of mathematical physics to notice this distinction is the reason why quantum theory is logically at odds with quantum experiments.

AMS classification scheme numbers: 81, 03

Note on language. The material here spans both mathematical logic and mathematical physics. These do not share the same language; indeed the language of the former is far smaller. For example, there is no definition for the symbol: 4 and so the proper execution of the logic in physics needs all manner of statements typified by: $4 = 1+1+1+1$ up to those such as: $\exp x = 1+x+\cdots$. In the interest of accessibility, these are omitted and left to the reader's intuitive understanding.

1. Introduction

Inherent within quantum measurement experiments is a decision process which current theory fails to express and does not explain. The act of measurement decides on one resulting value from a spectrum of possibilities. Identical such experiments are characterised by a dependable pattern of statistics for these decisions, not deriving from experimental variation or error. In spite of this definite *spectral distribution*, evidence indicates that prior to any individual measurement, no definite information exists that elects the particular resulting value. Moreover, there is no involvement of any physical influence of which we are in ignorance, encoded in any so called 'hidden variables'. [3][1, 2, 22, 23, 24]. In short, Nature executes the decision but the mechanism is not one of physical influence.

Looking for alternative explanations, we might be persuaded toward a theory rooted in some non-classical logic. Indeed the prospect of such a theory has motivated an extensive history of study scrutinising quantum mathematics for clues. Even so, the absence of any reference in the physics literature indicates that this scrutiny has not extended as far as the non-classical logic inherent within arithmetic beneath quantum theory, upon which the theory rests. Yet most curiously, mathematical logicians are acquainted with elements of this logic, to the extent that they regard them as obvious and self-evident.

The discrepancy between quantum experiments and quantum theory is traceable to a logical detail of arithmetic, generally overlooked, ignored and not encoded in mathematical physics. The arithmetic in question is that of numbers called *scalars*‡. And the logic in question concerns *existence* of scalars: existence afforded (or conferred) by an axiom-set [see Table 1] prescribing this arithmetic. These axioms are not a contrivance for the purpose of satisfying quantum theory; they are natural rules of algebra for our everyday arithmetic. They afford (or confer) existence of scalars via a mechanism of some complexity and due to this, different scalars occur in distinct modes of existence, possessing distinct logical qualities. These modes are: *possible* (afforded) existence and *necessary* (conferred) existence. They are illustrated in the following.

The axiom-set in Table 1 consists of the *Field Axioms* appended with certain further axioms. I call this axiom-set the *Infinite-field Axioms*. They prescribe the usual arithmetic with which we are familiar. The appending axioms merely exclude all modulo addition which the Field Axioms themselves permit, allowing only infinite arithmetic. Mathematical objects satisfying the Infinite-field Axioms are scalars whose existence is consistent with them. These scalars are the numbers in mathematical physics we typically add and multiply, and use as entries in arrays such as vectors or matrices. In this context of consistency with the Infinite-field Axioms, or otherwise, consider the following four mathematical statements written as formulae in *first-order logic*. Each is a proposition asserting the existence of some instance of α, equal to a specified numerical value:

$$\exists\alpha\,(\alpha\times\alpha=4)\,; \tag{1}$$

$$\exists\alpha\,(\alpha\times\alpha=2)\,; \tag{2}$$

$$\exists\alpha\,(\alpha\times\alpha=-1)\,; \tag{3}$$

$$\exists\alpha\,\left(\alpha^{-1}=0\right). \tag{4}$$

Of these propositions, the Infinite-field Axioms disprove only (4) because it is the only one with which these Axioms are inconsistent; in point of fact, (4) is negated by axiom FM2. Other than this, the Infinite-field Axioms prove only (1). In consequence, (2) and (3) are neither proved nor negated, and both, as well as their

‡ In physics, a scalar is a mathematical object representing a physical quantity (such as mass) that is an invariant constant for all inertial frames of reference. More formally, these are zero rank tensors. But in the arithmetical context of this article, the term *scalar* is taken from linear algebra where scalars are mathematical objects whose rules of algebra are the *Field Axioms*.

negations, are consistent with the Infinite-field Axioms. Proof of these claims is given in Section 5.

In accordance, we reject the instance of α in (4) as a *necessarily* non-existent scalar. We accept α in (1) as a scalar proved to *necessarily* exist. And we accept α in (2) and (3) as instances of scalars whose existences are *possible*, in the sense that they can neither be confirmed nor denied. Both (2) and (3) are propositions known as *logically independent* [11, 17, 20] of the Infinite-field Axioms. This independence is virtually synonymous with *mathematical undecidability* and *logical indeterminacy.*

In summary, (1) furnishes a scalar that *necessarily* exists because the Infinite-field Axioms prove it, while (2) and (3) furnish scalars whose existences are *possible* because they satisfy the Infinite-field Axioms. Therefore, there are two distinct modes of existence for scalars; yet strikingly, notwithstanding their logical distinction, the square roots of 4, 2 and -1 are all scalars that engage in the arithmetic without distinction.

This existential modality derives from two remarkably intuitive theorems of *model theory*, a branch of mathematical logic. These are the theorems of Soundness and Completeness. I show in this paper, good reason to believe that the logical behaviour of quantum mechanics lies profoundly in these theorems. This claim is corroborated by the separate evidence of the illustrious work of Hans Reichenbach, who very successfully resolved the anomalies of quantum mechanics by inventing

	The Infinite-field Axioms	
	Additive Group	
FA0	$\forall\alpha\forall\beta\exists\gamma\,(\gamma = \alpha + \beta)$	Closure
FA1	$\exists 0\forall\alpha\,(0 + \alpha = \alpha)$	Identity 0
FA2	$\forall\alpha\exists\beta\,(\alpha + \beta = 0)$	Inverses
FA3	$\forall\alpha\forall\beta\forall\gamma\,((\alpha + \beta) + \gamma = \alpha + (\beta + \gamma))$	Associativity
FA4	$\forall\alpha\forall\beta\,(\alpha + \beta = \beta + \alpha)$	Commutativity
	Multiplicative Group	
FM0	$\forall\alpha\forall\beta\exists\gamma\,(\gamma = \alpha \times \beta)$	Closure
FM1	$\exists 1\forall\alpha\,(1 \times \alpha = \alpha \times 1 = \alpha \wedge 0 \neq 1)$	Identity 1
FM2	$\forall\alpha\exists\beta\,(\alpha \times \beta = 1 \wedge \alpha \neq 0)$	Inverses
FM3	$\forall\alpha\forall\beta\forall\gamma\,((\alpha \times \beta) \times \gamma = \alpha \times (\beta \times \gamma))$	Associativity
FM4	$\forall\alpha\forall\beta\,(\alpha \times \beta = \beta \times \alpha)$	Commutativity
FAM	$\forall\alpha\forall\beta\forall\gamma\,(\alpha \times (\beta + \gamma) = (\alpha \times \beta) + (\alpha \times \gamma))$	Distributivity
	Exclusion of all modulo addition	
	$1 + 1 \neq 0;\;\; 1 + 1 + 1 \neq 0;\ldots\ldots\;\; 1 + \cdots + 1 \neq 0$	

Table 1. The the Infinite-field Axioms written as sentences in *first-order logic.* These are an appended version of the Field Axioms which excludes modulo addition. Variables: $\alpha, \beta, \gamma, 0, 1$ represent mathematical objects complying with this axiom-set. The semantic interpretations of these objects are known as *scalars.*

a 3-valued quantum logic [19], with which the logic presented in this article, hidden within arithmetic, happily coincides, isomorphically. And so consequently, arithmetic provides the foundation for Reichenbach's logic, which up to this point has been missing, and substantiates his work.

So what new approach toward mathematical physics is needed in order that quantum theory should manifest this logic? The answer lies in the assumptions made when the concept of scalars is originally adopted into the theory. Conventionally, mathematical physics assumes the *a priori* existence of scalars. In this new approach, apriority is transferred from the scalars to the the Infinite-field Axioms themselves. The crucial point is: the new theory is set up by formally installing *the set of rules* rather than installing *objects obeying those rules*. This singular initiative strategically separates this theory from convention. It promotes mathematical physics from a *semantic theory* to a *logical theory* where validity has greater complexity which encompasses indeterminacy.

In order to grasp the practicalities of recognising scalars whose existences *satisfy* the Infinite-field Axioms and differentiating these from scalars whose existences are *prove*d by these same Axioms, while understanding how these occupy formulae; and furthermore, in order to accept that Soundness and Completeness are profoundly fundamental to quantum physics, it is necessary to provide the reader with a background and minimal working knowledge of model theory.

2. Background

Model theory proves that independent propositions are mathematically undecidable and logically indeterminate. These are complementary features seen in a logical condition present in certain axiomatised mathematical theories [6, 7, 17]. In such theories, indeterminacy describes the state of validity of propositions that are neither valid nor invalid. Undecidability refers to the provability of these indeterminate propositions, being neither provable nor disprovable.

In 1931 Kurt Gödel published his First Incompleteness Theorem. This proves that mathematical undecidability necessarily exists in arithmetic [6, 8, 9, 21]. This is not the kind of undecidability forced upon us through ignorance of information; the distinction is that information necessary for decision does not exist. Chaitin takes this informational approach to Gödel's Theorem. He argues: 'if a theorem [proposition] contains more information than a given set of axioms, then it is impossible for the theorem [proposition] to be derived from the axioms' [8]. Svozil uses Turing's proof of Gödel's Theorem to argue that undecidability exists in Physics [21].

In 1944, Hans Reichenbach proposed a quantum logic consisting of values: true, false and *indeterminate*. This was in response to 'causal anomalies' evident in the results of quantum experiments. His logic is an adaptation of the 3-valued logic of Jan Łukasiewicz [10, 15], which Reichenbach gives certain truth tables, conjunctions, disjunctions, tautology etc,. During its formation, Reichenbach arrived at the particular qualities of his indeterminate middle through detailed, reasoned analysis of results of quantum experiments.

He found that his 3-valued logic 'suppresses' the causal anomalies [10, 18, 19]. It furnishes a consistent epistemology for *prepared* and *measured* states: typically the question of what we may know about the state of a photon immediately before measurement. It derives *complimentary* propositions: if statement A is either true or false, statement B is indeterminate, and vice versa. Such statements correspond to measurements of complimentary pairs such as momentum and position. And his logic also overcomes the problem of *action at a distance*, a paradox identified by Einstein, Podolsky & Rosen [14].

Though his results are compelling, Reichenbach's logic is hypothetically based and is not in simple agreement with mainstream quantum logics based on the quantum postulates, originating with Birkhoff and von Neumann [4]. Acceptance of these would tend to imply the unacceptability of Reichenbach's logic. That said, Hardegree argues that these logics are not in opposition but describe different things [13]. While the mainstream logics are based on Hilbert space quantum theory, Reichenbach's logic is a framework for an alternative but yet unknown formulation. This article provides mathematical base for this alternative formulation.

3. Algebraic and logical environment

Model theory is a branch of mathematical logic that evaluates validity of propositions by considering associated mathematical structures. In the context of the Field Axioms these associated structures are *fields* (not to be confused with fields in quantum field theory). Some fields are of finite dimension containing a finite number of elements. Examples are the sets $\{0,1\}$, $\{0,1,2\}$, $\{0,1,2,3,4\}$, all of prime dimension. These satisfy a modulo arithmetic interpretation of the operators $+$ and $\times$. In contrast, the normal, non-modulo interpretation of $+$ and $\times$ excludes the finite fields.

The appended Field Axioms in Table 1 are an adaptation which restricts fields to those which are infinite. I call this appended version the *Infinite-Field Axioms.* The three familiar infinite fields are: the complex plane $\mathbb{C}$, the real line $\mathbb{R}$ and the smallest infinite field, the rational field $\mathbb{Q}$. Each is a closed structure; but jointly they form a field-subfield hierarchical nesting where the most deeply nested (smallest) infinite-field is special because it is a subfield of every infinite field. This fact has critical influence on which propositions are valid and provable, and which are indeterminate and undecidable.

The radical observation of this paper came while noticing the distinction between *necessary* existence, entailing *derivation* from the Infinite-Field Axioms, and *possible* existence that entails *satisfying* these Axioms. This distinction spurns two related logics. One is notionally *causal* where *n*ecessary and possible, together with necessarily-not constitute a *modal logic* [10]. The other is notionally *existential*, consisting of logically valid, logically invalid and logically indeterminate; identifiable with Reichenbach. The environment in which this second logic emerges from the Infinite-field Axioms is now discussed.

From the perspective of applied mathematics, the Infinite-field Axioms are seen as a selection of combination rules for addition and multiplication, to be applied ad hoc, in our most familiar arithmetic. These rules of combination are regarded as properties *belonging* to scalars and so significance, meaning and 'reality' is placed on scalars, with Axioms taking an incidental, appended or background role. In this applied mathematical scenario, scalars are the semantic interpretations of the objects: $\alpha, \beta, \gamma, 0, 1, \ldots$ in Table 1. Such interpretation arises when the mathematician designates $\alpha, \beta, \gamma, 0, 1, \ldots$ to the real line or complex plane, whichever suits the application. This interpretational approach deals in *semantic information.* And the act of such designation irreversibly discards *logical information* imparted by the Infinite-field Axioms.

In contrast to emphasis on the existence of scalars, conventional in applied mathematics, *first-order theory* places precedence on axioms. The first-order theory under the Infinite-field Axioms, I call the *Theory of Infinite-fields* (not to be confused with any connotation in quantum field theory). This poses a quite different scenario in which the Infinite-field Axioms define and generate the objects $\alpha, \beta, \gamma, 0, 1, \ldots$ along with their arithmetical behaviour.

First-order theory is a stricter and stronger system of derivation than applied mathematics. It takes full account of all logical information imparted by axioms, including information that is indeterminate. That said, no indeterminate information, independent of axioms, can be proved to exist, from the axioms themselves. Proof that (1) is a theorem may indeed be established by direct derivation from the Infinite-field Axioms. But direct proof that (2) and (3) *are*, or *are not* theorems is impossible because no information in these Axioms proves or negates them.

Model theory deals in structures and furnishes an environment where the indeterminate information is decidedly identifiable. In order to confirm the existence of any indeterminate information, theorems of model theory are applied generally to the infinite fields. Information underpinning the indeterminacy is held in these

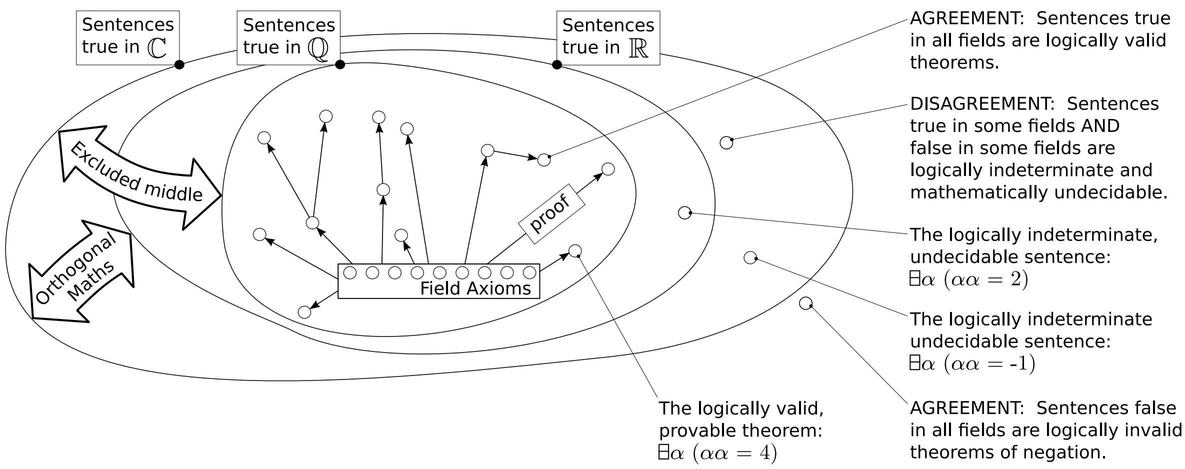

Figure 1. Validity under the Infinite-field Axioms. Due to theorems of Model Theory, sentences (small circles) such as $\exists\alpha\,(\alpha\alpha = 4)$, whose semantic validities agree are *logically valid* and are *theorems.* Sentences such as $\exists\alpha\,(\alpha\alpha = -1)$, whose semantic validities disagree are logically indeterminate and are *mathematically undecidable.* These exhaust all possibilities.

structures, not in the axioms. Every infinite field is a mathematical structure *satisfying* the Infinite-field Axioms. But despite the simplicity of this relationship between fields and their axioms, the *logical* relationship between these is not so simple but involves a certain complexity. This is illustrated in the fact that the smallest, innermost nested, infinite field is the only one that *necessarily* exists while others *possibly* exist.

Model Theory demands that any given proposition derives by proof from the Infinite-field Axioms, if and only if, it is true in every infinite field. All indeterminate propositions, which are therefore independent of the Infinite-field Axioms, always have mixed true/ false values, with disagreement somewhere between the individual infinite fields. For existential propositions such as (1), (2), or (3), the condition of entire agreement on a value of true, is satisfied only for scalars in the innermost nested field: the *rational field*. Therefore, only (1) is a theorem because it is the only case where α is rational. Figure 1 gives a preview of how the nesting of fields brings this about. A consequence is that Infinite-field Axioms prove the existence of all rational scalars; existence of other scalars is undecidable. These are surprising facts considering nothing in the arithmetic distinguishes the rational scalars.

All non-rational scalars are *logically independent* of the Infinite-field Axioms. That is to say: scalars of the non-rational infinite-fields express extraneous information, absent in these Axioms. The rational scalars, whose existence *can* be proved, contain no such extraneous information; they contain only information already in the Infinite-field Axioms. In short, the Infinite-field Axioms are unable to prove or disprove the existence of logically independent scalars.

4. Notes and Concepts

True is a semantical reference, synonymous with **semantically valid**. A proposition modelled by a given mathematical structure is true when interpreted in that structure.

Valid is a logical reference. It is more fully referred to as **logically valid.** A proposition is logically valid if: purely symbolically, independent of interpretation, by following rules of inference, Axioms imply the proposition.

Connectives: $\wedge$ $\vee$ $\neg$ $\Rightarrow$ $\Leftrightarrow$ (conjunction, disjunction, negation, implication, bi-implication)

Quantifiers: $\forall$ $\exists$ (for all, there exists)

Turnstile symbols: $\vdash$ $\models$ (derives, models)

First-order theories comprise formulae written as propositions in *first-order* t is not reference to approximation. Any first-order theory is specified in a set of axiom sentences, drawn up for the purpose. A crucial feature that distinguishes first-order theories from applied mathematics is their strict accounting of logical information. Variables satisfy all axiom sentences but are attributed with nothing more. They are purely abstract and meaningless. If this is misunderstood, the integrity of any derivation is at risk. In particular, the mathematician may not introduce new information, logically independent of

Axioms, without recording the fact in an account of *assumed* dependencies. She may not, for example, assign a variable to the real line, simply by saying so, as is done in applied mathematics. Effectively, a first-order theory is a computational machine that runs according to a programme of axiom sentences. Output from this machine exhibits richer conceptualisations of theorem and validity than does applied mathematics. In absence of any *logically independent* input, output of the machine consists solely of theorems. In cases when there is logically independent input, output relying on that independency is always undecidable and indeterminate.

Bound variable: when we write the equation:

$$\alpha + \beta = \beta + \alpha \,, \tag{5}$$

this is an informal use of *bound variables.* Notice this relation *specifies* something about the algebraic behaviour of the objects α and β rather than suggesting the performance of some arithmetic. Bound variables occur where there is specification. When writing the formal version of (5), quantifiers $\forall$ are shown. These explicitly state the logic but also do the job of highlighting the fact that *specification* is intended rather than arithmetic. Thus:

$$\forall\alpha\forall\beta\,(\alpha + \beta = \beta + \alpha) \ . \tag{6}$$

The format of parenthesisation is typical of formulae in first-order logic. Quantifiers $\forall\alpha$ and $\forall\beta$ apply to every occurrence of α and β within the brackets.

Sentence: formulae such as (6), where every variable is bound, are known as *sentences.* (6) happens to be the sentence adopted as axiom FA4 in Table 1. An example of a formula which is not a sentence is the formula:

$$\forall\beta\exists\alpha\,(\alpha = \beta + \vartheta) \ . \tag{7}$$

In this ϑ is not bound.

Free variable: In (7), ϑ is a *free variable* as opposed to a bound variable. It is free to be substituted by a particular value; thus inviting the performance of some arithmetic rather than specification.

The Field Axioms comprise a set of axiom sentences formed by the union of axioms for the Additive Group and the Multiplicative Group. In addition to these, there is one axiom for distributivity. In these Axioms, different possibilities of interpretation exist for the symbols $+$ and $\times$. For example, modulo arithmetics are options, but these are not under consideration here. In this paper, $+$ and $\times$ are interpreted in the usual way, as symbols of an unbounded (infinite) arithmetic.

Model: This is a *mathematical structure* that satisfies a sentence. It is usual to say that such a structure *models* the sentence. As an illustration, consider the axiom sentence FA4 from Table 1, specifying additive commutativity. This is modelled by any of the sets: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\{1\}$, $\{1, -1\}$, $\{1, 2, 3\}$, $\{\text{all } 4 \times 3$ matrices$\}$, etc.. As well as individual sentences, sets of sentences also have models. To illustrate, take two sentences. As before, take axiom FA4 from Table 1, but now model axiom FM4 also. Together these two sentences specify both additive *and* multiplicative commutativity. The addition of this second sentence eliminates the former inclusion of 4×3 matrices from the set of models.

Semantic interpretation: Bound variables, such as the objects $\alpha, \beta, \gamma, 0, 1, \ldots$ complying with Axioms in Table 1, convey no more meaning than the properties bestowed upon them by those Axioms. That said, they may be *interpreted* as elements of a particular model of the Axioms. For instance, these objects might be interpreted as members of the real line $\mathbb{R}$. This would be a *semantic interpretation* of $\alpha, \beta, \gamma, 0, 1, \ldots$, and would involve an injection of information originating not from the Axioms but from elsewhere.

Infinite-field: This is the general name for mathematical structures that model the Infinite-field Axioms. There are at least three infinite-fields. These are the complex plane $\mathbb{C}$, the real line $\mathbb{R}$ and the rational field $\mathbb{Q}$. The term *field* is likely to cause confusion. In quantum field theory, fields are entities associated with the mechanics of elementary particles. This meaning is not intended here. In this paper, definition is taken from Linear Algebra.

Scalar: An element of a field. Semantic interpretation of the objects $\alpha, \beta, \gamma, 0, 1, \ldots$ in Table 1 are *scalars*: either complex scalars, real scalars or rational scalars, depending on the field elected. The term *scalar* is likely to cause confusion. In relativity, a scalar is a zero rank tensor: under change of inertial reference frame, an object that transforms as a constant number. In this paper, definition is taken from Linear Algebra.

5. Model Theory

Our specific interest in Model Theory is the Soundness Theorem and its converse, the Completeness Theorem. These are two standard theorems in model theory which apply to all first-order theories [6, 7]. We shall see that jointly, they isolate an excluded middle of mathematically undecidable sentences, from the set of all other sentences which are theorems.

5.1. Standard theorems

The Soundness Theorem:

$$\Sigma \vdash \mathcal{S} \Rightarrow \forall \mathcal{M}^{\Sigma} \left(\mathcal{M}^{\Sigma} \models \mathcal{S} \right). \tag{8}$$

If structure $\mathcal{M}^{\Sigma}$ models axiom-set Σ, and Σ derives sentence $\mathcal{S}$, then every structure $\mathcal{M}^{\Sigma}$models $\mathcal{S}$.

Alternatively**:** *If a sentence is a theorem, provable under an axiom-set, then that sentence is true for every model of that axiom-set.*

The Completeness Theorem:

$$\Sigma \vdash \mathcal{S} \Leftarrow \forall \mathcal{M}^{\Sigma} \left(\mathcal{M}^{\Sigma} \models \mathcal{S} \right). \tag{9}$$

If structure $\mathcal{M}^{\Sigma}$ models axiom-set Σ, and every structure $\mathcal{M}^{\Sigma}$ models sentence $\mathcal{S}$, then Σ *derives sentence $\mathcal{S}$.*

Alternatively**:** *If a sentence is true for every model of an axiom-set, then that sentence is a theorem, provable under that axiom-set.*

5.2. Proofs

We now proceed to prove further theorems of model theory. Jointly, (8) and (9) result in the 2-way implication:

Validity Theorem:

$$\Sigma \vdash \mathcal{S} \Leftrightarrow \forall \mathcal{M}^{\Sigma} \left(\mathcal{M}^{\Sigma} \models \mathcal{S} \right). \tag{10}$$

If structure $\mathcal{M}^{\Sigma}$ models axiom-set Σ, then axiom-set Σ derives sentence $\mathcal{S}$, if and only if, all structures $\mathcal{M}^{\Sigma}$ model sentence $\mathcal{S}$.
Alternatively**:** *A sentence is provable under an axiom-set, if and only if, that sentence is true for all models of that axiom-set.*

Furthermore, for every sentence $\mathcal{S}$ there is a sentence $\neg\mathcal{S}$; hence, jointly, (8) and (9) also guarantee a second 2-way implication:

Invalidity Theorem:

$$\Sigma \vdash \neg\mathcal{S} \Leftrightarrow \forall \mathcal{M}^{\Sigma} \left(\mathcal{M}^{\Sigma} \models \neg\mathcal{S} \right). \tag{11}$$

If structure $\mathcal{M}^{\Sigma}$ models axiom-set Σ, then axiom-set Σ derives the negation of sentence $\mathcal{S}$, if and only if, all structures $\mathcal{M}^{\Sigma}$ model the negation of $\mathcal{S}$.
Alternatively**:** *A sentence is disprovable under an axiom-set, if and only if, that sentence is false for all models of that axiom-set.*

Each of (10) and (11) excludes the sentences of the other. And moreover, together they isolate sentences excluded by both. In the left hand sides of (10) and (11), there is no indication of other sentences existing which satisfy neither, that is: sentences that are neither provable nor disprovable. And so, it is of particular interest that the right hand sides of (10) and (11) do indeed imply the existence of sentences that correspond precisely to this condition. These are the sentences excluded by the right hand sides of (10) and (11) and so satisfy the following condition on modelling:

$$\neg\forall \mathcal{M}^{\Sigma} \left(\mathcal{M}^{\Sigma} \models \mathcal{S} \right) \wedge \neg\forall \mathcal{M}^{\Sigma} \left(\mathcal{M}^{\Sigma} \models \neg\mathcal{S} \right). \tag{12}$$

The aim now is to find the status of provability for sentences excluded by (12). We firstly deduce (13) and (14), the negations of (10) and (11):

$$\neg \left(\Sigma \vdash \mathcal{S} \right) \Leftrightarrow \neg\forall \mathcal{M}^{\Sigma} \left(\mathcal{M}^{\Sigma} \models \mathcal{S} \right); \tag{13}$$

$$\neg \left(\Sigma \vdash \neg\mathcal{S} \right) \Leftrightarrow \neg\forall \mathcal{M}^{\Sigma} \left(\mathcal{M}^{\Sigma} \models \neg\mathcal{S} \right); \tag{14}$$

and combine these, so as to construct:

$$\neg \left(\Sigma \vdash \mathcal{S} \right) \wedge \neg \left(\Sigma \vdash \neg\mathcal{S} \right) \Leftrightarrow \neg\forall \mathcal{M}^{\Sigma} \left(\mathcal{M}^{\Sigma} \models \mathcal{S} \right) \wedge \neg\forall \mathcal{M}^{\Sigma} \left(\mathcal{M}^{\Sigma} \models \neg\mathcal{S} \right). \tag{15}$$

This limits sentences that are neither provable nor negatable, to those that are neither true nor false across all structures that model the Axioms. For theories whose axioms are modelled by more than one single structure, where $\mathcal{M}_1^{\Sigma}$ and $\mathcal{M}_2^{\Sigma}$ are distinct, we can assert (16):

Indeterminacy Theorem:

$$\neg \left(\Sigma \vdash \mathcal{S} \right) \wedge \neg \left(\Sigma \vdash \neg\mathcal{S} \right) \Leftrightarrow \exists \mathcal{M}_1^{\Sigma} \left(\mathcal{M}_1^{\Sigma} \models \mathcal{S} \right) \wedge \exists \mathcal{M}_2^{\Sigma} \left(\mathcal{M}_2^{\Sigma} \models \neg\mathcal{S} \right). \tag{16}$$

Axiom-set Σ derives neither sentence $\mathcal{S}$ nor its negation, if and only if, there exist structures $\mathcal{M}_1^\Sigma$ and *$\mathcal{M}_2^\Sigma$ which each model axiom-set Σ, such that $\mathcal{M}_1^\Sigma$ models $\mathcal{S}$, and $\mathcal{M}_2^\Sigma$ models the negation of $\mathcal{S}$.*

Alternatively**:** *A sentence is true for some but not all models of an axiom-set, if and only if, that sentence is undecidable under that axiom-set.*

6. Application

The above theorems apply to first-order theories generally; and the Theory of Infinite-fields is a first-order theory. We may therefore adapt those theorems to the specific context of the Theory of Infinite-fields and construct practical tests that establish whether any given first-order sentence is indeterminate, or is valid, with respect to this Theory. All that is necessary for this is the appointment of the infinite-fields to the models mentioned in the theorems. The following test procedure results.

Indeterminacy Test: *For any given sentence: interpret its variables as scalars of the complex plane $\mathbb{C}$, the real line $\mathbb{R}$, the rational field $\mathbb{Q}$, in turn. Then, the Infinite-field Axioms neither prove nor disprove this sentence, if and only if, it is true in at least one infinite-field and false in at least one infinite-field.* This reduces to a simple check for *disagreement* within truth-tables.

This particular test confirms certain indeterminate sentences in the Theory of Infinite-fields. But note that it is not exhaustively comprehensive because it samples only three infinite-fields and may not find *every* indeterminacy. An analogous adaptation of the Validity Theorem, nevertheless, yields a validity test that is a totally impractical prospect since using it would require building a truth table that samples every infinite-field, no matter how obscure.

For a realistic test of validity we resort to *direct derivation* from the Infinite-field Axioms and embrace the model theory that characterises this. When a formula asserts the existence of some particular number, and is provable directly from the Infinite-field Axioms, that number will be rational. This follows because the span of all numbers deriving from the Infinite-field Axioms, is restricted to arithmetical combinations of the numbers 0 and 1, originating in sentences FA1 and FM1 of Table 1. Therefore, numbers are limited in form to p/q, where p and q are integers.

Crucially, the set of numbers whose existences are provable from the Infinite-field Axioms is identical with the set of all rational numbers. Hence, every formula asserting existence of a rational number, is provable. And so consequently, by the Soundness Theorem, every such formula is true in every infinite-field. In summary, any provable formulae is true independent of interpretation, and is thereby valid, by definition. In short:

Validity Test: *Any existential proposition is valid, if and only if, it is true in the rational field $\mathbb{Q}$.*

6.1. Examples

	$\alpha \in \mathbb{C}$	$\alpha \in \mathbb{R}$	$\alpha \in \mathbb{Q}$
$\exists\alpha\,(\alpha \times \alpha = 4)$	T	T	T

	$\alpha \in \mathbb{C}$	$\alpha \in \mathbb{R}$	$\alpha \in \mathbb{Q}$
$\exists\alpha\,(\alpha \times \alpha = 2)$	T	T	F

	$\alpha \in \mathbb{C}$	$\alpha \in \mathbb{R}$	$\alpha \in \mathbb{Q}$
$\exists\alpha\,(\alpha \times \alpha = -1)$	T	F	F

	$\alpha \in \mathbb{C}$	$\alpha \in \mathbb{R}$	$\alpha \in \mathbb{Q}$
$\exists\alpha\,(\alpha^{-1} = 0)$	F	F	F

Table 2. Truth-tables for propositions: $\exists\alpha\,(\alpha \times \alpha = 4)$, $\exists\alpha\,(\alpha \times \alpha = 2)$, $\exists\alpha\,(\alpha \times \alpha = -1)$ and $\exists\alpha\,\left(\alpha^{-1} = 0\right)$. In these T and F denote true and false.

Existence of scalars Formulae (1), (2), (3) and (4) on page 2, each proposes the existence of a particular scalar. In the context of the Theory of Infinite-fields, each of these propositions poses the question: do the Infinite-field Axioms derive this formula? These questions are answered in the four truth-tables of Table 2. In the first, proposition (1) is seen to be true in the rational field, so by the Validity Test, (1) is a theorem. The second two truth-tables show disagreeing truth values; hence, by the Indeterminacy Test, (2) and (3) are undecidable. In the last of these truth tables, proposition (4) is seen to be false in the rational field and so its negation is true in the rational field, and by the Validity Test, its negation is a theorem.

	$\alpha \in \mathbb{C}$	$\alpha \in \mathbb{R}$	$\alpha \in \mathbb{Q}$
$\exists\alpha\,\left(\alpha = \xi^{\mathbb{Q}}\right)$	T	T	T

	$\alpha \in \mathbb{C}$	$\alpha \in \mathbb{R}$	$\alpha \in \mathbb{Q}$
$\exists\alpha\,\left(\alpha = \zeta^{\mathbb{R}}\right)$	T	T	F

	$\alpha \in \mathbb{C}$	$\alpha \in \mathbb{R}$	$\alpha \in \mathbb{Q}$
$\exists\alpha\,\left(\alpha = \eta^{\mathbb{C}}\right)$	T	F	F

Table 3. Truth-tables for propositions $\exists\alpha\,\left(\alpha = \xi^{\mathbb{Q}}\right)$, $\exists\alpha\,\left(\alpha = \zeta^{\mathbb{R}}\right)$ and $\exists\alpha\,\left(\alpha = \eta^{\mathbb{C}}\right)$.

	$x, y \in \mathbb{C}$	$x, y \in \mathbb{R}$	$x, y \in \mathbb{Q}$
$\forall x \exists y \, (y = x^2)$	T	T	T

	$x, y \in \mathbb{C}$	$x, y \in \mathbb{R}$	$x, y \in \mathbb{Q}$
$\forall x \exists y \, (y = x^2)$	T	F	F

Table 4. Truth-tables concerning the function $y = x^2$.

Existence of rational scalars The rational field is a subfield of all infinite-fields. Consequently, propositions of existence that are true in this, innermost nested, smallest infinite-field are necessarily true in all infinite-fields. See Figure 1. This means that *rational* scalars exist by theorem. Table 3 illustrates the provability of the general rational scalar $\xi^{\mathbb{Q}}$, the undecidability of the general real scalar $\zeta^{\mathbb{R}}$ and the undecidability of the general complex scalar $\eta^{\mathbb{C}}$.

Existence of functions A function in applied mathematics can spurn different first-order propositions; some of which might be theorems and some which might be undecidable. Propositions: $\forall x \exists y \, (y = x^2)$ and $\forall y \exists x \, (y = x^2)$ have quantifiers reversed. This makes an important logical difference. Table 4 shows the first of these two propositions is a theorem, yet the second is undecidable.

Existence of finite polynomials versus transcendental functions Table 5 compares formulae proposing the existence of a finite polynomial with an example of transcendental function: the exponential. The first truth-table in Table 5 is for the proposition: $\forall x \exists y \, (y = p\,(x))$. In this, p is a finite polynomial, so if x is rational then so also is any finite sum of terms $p\,(x)$. Corresponding reasoning applies to real or complex x. In contrast, in the proposition $\forall x \exists y \, (y = \exp\,(x))$ where

$$\exp\,(x) \equiv \lim_{n \to \infty} \left[1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} \right],$$

rational x is not necessarily mapped to a rational point by the exponential function. Hence, $p\,(x)$ exists by theorem but $\exp\,(x)$ exists undecidably.

	$x, y \in \mathbb{C}$	$x, y \in \mathbb{R}$	$x, y \in \mathbb{Q}$
$\forall x \exists y \, (y = p\,(x))$	T	T	T

	$x, y \in \mathbb{C}$	$x, y \in \mathbb{R}$	$x, y \in \mathbb{Q}$
$\forall x \exists y \, (y = \exp\,(x))$	T	T	F

Table 5. Truth-table for finite polynomial: $\forall x \exists y \, (y = p\,(x))$ and the transcendental function: $\forall x \exists y \, (y = \exp\,(x))$.

6.2. *Theorems from undecidability*

Arithmetical combination Scalars that exist undecidably can be combined to yield scalars that exist by theorem. Consider the two propositions: $\exists\alpha\,(\alpha = 3 + 4\mathrm{i})$ and $\exists\alpha^*\,(\alpha^* = 3 - 4\mathrm{i})$. These are undecidable, but the product of these scalars is the rational scalar: 25, which is logically valid and so exists by theorem. See Table 6.

	$\alpha \in \mathbb{C}$	$\alpha \in \mathbb{R}$	$\alpha \in \mathbb{Q}$
$\exists\alpha\,(\alpha = 3 + 4\mathrm{i})$	T	F	F

	$\alpha^* \in \mathbb{C}$	$\alpha^* \in \mathbb{R}$	$\alpha^* \in \mathbb{Q}$
$\exists\alpha^*\,(\alpha^* = 3 - 4\mathrm{i})$	T	F	F

	$\beta \in \mathbb{C}$	$\beta \in \mathbb{R}$	$\beta \in \mathbb{Q}$
$\exists\beta\,(\beta = \alpha\alpha^*)$	T	T	T

Table 6. Truth-tables for the proposition $\exists\alpha\,(\alpha = 3 + 4\mathrm{i})$, $\exists\alpha^*\,(\alpha^* = 3 - 4\mathrm{i})$ and $\exists\beta\,(\beta = \alpha\alpha^*)$.

Limiting Theorems The limit of an undecidable scalar can exist by theorem. The proposition $\exists y\,(y^2 = -x^2)$ is undecidable. Nevertheless, it has a limiting case: $\exists y\,(\lim_{x\to 0}[y^2 = -x^2])$ which is a theorem. See Table 7.

	$y \in \mathbb{C}$	$y \in \mathbb{R}$	$y \in \mathbb{Q}$
$\exists y\,(y^2 = -x^2)$	T	F	F

	$y \in \mathbb{C}$	$y \in \mathbb{R}$	$y \in \mathbb{Q}$
$\exists y\,(\lim_{x\to 0}[y^2 = -x^2])$	T	T	T

Table 7. Truth-table for proposition $\exists y\left(y^2 = -x^2\right)$ and its limiting case: $\exists y\left(\lim_{x\to 0}\left[y^2 = -x^2\right]\right)$.

Conclusion

The arithmetic of scalars embodies a well-known non-classical logic. This paper examines its logical intricacies and discusses them in the context of quantum theory. Study is motivated by the elemental position this arithmetic occupies in mathematical physics along with the doubtless implication that the logic must enter quantum theory. Findings show, via the work of Hans Reichenbach, this logic is isomorphic to logic exhibited in quantum experiments. Reasons why the logic goes unnoticed in classical physics stem from an absolute scale existing in Nature, against which indeterminate information is sensitive. When related to macroscopic reference systems, indeterminate information vanishes and manifests paradoxically. In classical experiments, logic of the quantum world is not related to the macroscopic; no vanishing occurs and no paradox is manifest.

The logic is revealed when the concept of scalars is installed unconventionally, as follows. Rather than installing scalars by way of adopting *mathematical objects* whose algebraic rules are those of scalars; instead we install: *the algebraic rules of scalars,* themselves. The logical theory that results implies foundations for quantum theory in the Infinite-field Axioms, and the Soundness and Completeness Theorems of mathematical logic.

We commonly understand formulae in algebra to be either true or false, depending on whether they are derived correctly or erroneously and we expect no alternative to these possibilities. But the Soundness and Completeness Theorems show there does exist another alternative: that of *indeterminate* or *mathematically undecidable.* Such logical information is not picked up by the standard algebraic formalism but it does perpetuate unnoticed throughout applied mathematics and into quantum mechanics where finally its absence becomes conspicuous by paradox. And unfortunately, mathematical physics in its current formalism, denies physical theory any possibility of linking these theoretical indeterminacies from the arithmetic with indeterminacies we observe in Nature.

The said non-classical logic of scalars is founded within their *existence.* Scalars exist in two distinct qualities or *modes.* By definition, all scalars *satisfy* the Infinite-field Axioms and so, in the context of these Axioms, existence of all scalars is *possible.* On top of this, the existence of a subset of scalars, the rationals, is proved by these Axioms, hence the rational scalars *necessarily* exist. Satisfying and being proved are seen as causally distinct: a distinction not noted in applied mathematics.

The above claims rest centrally on a proof, given in this paper, deriving from Soundness and Completeness. This confirms the existence of indeterminacy under the Infinite-field Axioms: indeterminacy that cannot be derived directly. Application of this and other closely related theorems in model theory furnish two simple tests identifying those formulae which Axioms render logically indeterminate and those they render logically valid theorems in the Theory of Infinite-fields. The said theorems in model theory strictly identify undecidable propositions as those with truth values that do not concur across all semantic interpretations, but disagree. That is: they are not consistently true, or false, when interpreted in turn as members of the complex plane $\mathbb{C}$, the real line $\mathbb{R}$, and the rational field $\mathbb{Q}$. This result is used in various examples of interest, checking truth-tables for agreement or disagreement. Rational scalars are shown to exist by theorem while strictly imaginary or irrational scalars are undecidable. This ultimately follows from the fact that only the rational field is a subfield of all infinite-fields.

An important finding of this research is this: generally, undecidable formulae, convey with them their undecidability during the performance of algebraic operations. This seems reasonable and to be expected. However, certain algebraic operations spurn formulae that are not undecidable but are *theorems.* This has ramifications for our understanding of the mechanism for measurement in quantum mechanics and will be explored in greater detail in a subsequent paper.

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